METHOD OF AUTOMORPHIC FUNCTIONS IN THE STUDY OF FLOW AROUND A STACK OF POROUS CYLINDERS

by Y. A. ANTIPOV†

(*Department of Mathematics, Louisiana State University, Baton Rouge,* LA 70803, *USA*)

and V. V. SILVESTROV‡

(*Department of Mathematics, Gubkin Russian State University of Oil and Gas, Moscow* 119991, *Russia*)

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Summary

This paper studies the ideal flow around a stack of stationary cylinders with porous walls. The boundary conditions on the surfaces of the cylinders are nonlinear. For small values of the porosity parameters, by applying the asymptotic method, the boundary conditions are linearized. The use of Möbius transformations generating a symmetric Schottky group reduces the problem to a Riemann–Hilbert boundary-value problem for symmetric automorphic functions. Its solution is found in a series form in terms of a quasiautomorphic analogue of the Cauchy integral. The absolute and uniform convergence of the series is guaranteed when the associated flow domain symmetric Schottky group is a first class group. An example of a symmetric Schottky group of divergent type (not of the first class) is given. Formulae for the drag and lift forces acting on the cylinders are derived, and the dependence of the porosity parameters on the forces is studied. In particular, the drag force is zero for a single solid cylinder (d'Alambert's paradox), while for a cylinder with a porous surface this is not true.

1. Introduction

The problem of potential flow past a group of stationary and moving circular cylinders (discs) with rigid non-penetrable walls is well understood. A detailed discussion of the problem for a single cylinder can be found, for example, in (**1**). The interaction between two parallel cylinders has been studied (**2** to **5**) since the work by Hicks (**6**) who found the solution in terms of elliptic functions. The problem for n stationary cylinders was reduced (**7**) to a Fredholm integral equation of the second kind and then to a linear system of algebraic equations. By the method of images, the complex potential of flow was expressed (**8**) in terms of a series of doublets, and the hydrodynamic forces were found numerically by approximate integrating of the pressure on the boundaries of the cylinders. The integration of the pressure required in the method of images was implemented analytically in (**9**). An exact series form solution for the problem on n stationary and

†⟨antipov@math.lsu.edu⟩
‡⟨v_silvestrov@mail.ru⟩

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moving cylinders was first presented in (**10**, **11**). The method uses the theory of the Riemann–Hilbert problem of symmetric automorphic functions (**12** to **14**). The solution was found in terms of absolutely and uniformly convergent series for arbitrarily located three cylinders of arbitrary radii and for more than three cylinders, provided the associated Schottky group (**15**, **16**) of symmetry transformations was a first class group (**17**) (a group of the convergent type). Numerical results were reported for the case of three cylinders. The conformal mapping technique was recently proposed (**18**) (the conformal map (**19**) could also be used) to derive another form of the solution to the problem on a stack of stationary cylinders. The conformal mapping was given in terms of the Schottky–Klein prime function expressible in terms of infinite products. It can be shown, after taking the logarithm, that the convergence of the products is guaranteed for the first class group as it is in (**10**).

If the surfaces of the cylinders have small gaps or perforations, then they can be considered as porous surfaces. The normal component of the velocity vector does not vanish on the surface, and in general, the boundaries of the discs are not streamlines. Woods (**20**) suggested to use the following effective boundary condition:

$$\mathbf{u}\cdot\boldsymbol{\nu} = \lambda_* u_a \left(\frac{\Delta p}{\rho_a u_a^2}\right)^m, \tag{1.1}$$

where $\mathbf{u}\cdot\boldsymbol{\nu}$ is the normal component of the velocity, Δp is the pressure jump through the surface, ρ_a and u_a are characteristic values of the density and speed, respectively, and $\lambda_* > 0$ and $m \geqslant 0$ are dimensionless parameters which reflect the properties of the porous wall. The particular case $m = 0$ of the boundary condition was used in (**21**, **22**) for modelling the impact of a jet on a porous wall. When the perforations are small in size and distributed uniformly, then $m = 1$ (**23**). This paper addresses the problem for a group of porous cylinders whose surfaces are modelled by the boundary condition $\mathbf{u}\cdot\boldsymbol{\nu} = \lambda_j^0 + \lambda_j \Delta p$, when the dimensionless porosity parameters $\kappa_j = \frac{1}{2}\rho V_\infty \lambda_j$ for the surface of the jth cylinder are small. Here, V_∞ is the speed at infinity and ρ is the density of the fluid. Another model for the flow around and through a circular porous cylinder was proposed in (**24**). For a single cylinder, the problem was reduced to a nonlinear integral equation and solved numerically.

The mathematical background of the paper comprises two different methods. The first one is the asymptotic method which linearizes the boundary condition for small values of the porosity parameters and leads to a sequence of auxiliary linear boundary-value problems. The second method reduces the auxiliary problems to a Hilbert problem for a multiply connected circular domain (**25** to **27**) and then to a Riemann–Hilbert problem of the theory of symmetric automorphic functions (**12** to **14**).

In section 2, we formulate the problem (Problem 2.1) for $n+1$ porous discs D_j $(j = 0, 1, \dots, n)$ whose boundaries are circles $L_j = \partial D_j$ centred at z_j, radius r_j. For small values of the porosity parameters κ_j, it reduces to an auxiliary problem (Problem 2.2) for a multi-valued complex potential analytic in the flow domain and satisfying a certain boundary condition. For an arbitrary $(n+1)$-connected region, this problem can be reduced to a system of integral equations (**28**). In section 3, for an $(n+1)$-connected circular domain bounded by the line $L = \bigcup_{j=0}^{n} L_j$, the auxiliary problem maps to a Riemann–Hilbert problem (Problem 3.1) for G-automorphic symmetric functions. Here, G is the Schottky group of the line $L =$ generated by the Möbius transformations $\sigma_j = T_j T_0(z)$ $(j = 1, 2, \dots, n)$, and $T_j(z) = z_j + r_j^2(\bar{z} - \bar{z}_j)^{-1}$ $(j = 0, 1, \dots, n)$ is the symmetry transformation with respect to the circle L_j. Its solution is derived in terms of a series whose coefficients are expressed through a quasiautomorphic analogue of the Cauchy integral. It is shown that the found solution is a G-automorphic symmetric function. Section 4 simplifies the final form of the solution (formulae (4.18) and (4.19)). In section 5, we compute the pressure, the drag and lift forces and report numerical results. The series form solution always converges for $n+1 \leqslant 3$ cylinders. If $n+1 \geqslant 4$, then it is required that the associated symmetric Schottky group is a first class group which is not always the case. Appendix A gives an example (to the best of our knowledge, the first one published in the scientific literature) of a symmetric Schottky group which is not a first class group.

2. Formulation

PROBLEM 2.1. (Main problem) Let $n+1$ parallel circular cylinders D_j be placed in an inviscid incompressible fluid which is in steady irrotational motion (Fig. 1). The cylinders do not touch each other. Far away from the cylinders, the flow is uniform with velocity V_∞ (without loss of generality V_∞ is real) and pressure p_∞. The circulation around the jth cylinder is Γ_j ($j = 0, 1, \ldots, n$). The surfaces of the cylinders are assumed to be porous, and the normal component of the velocity vector obeys the law

$$\mathbf{u} \cdot \boldsymbol{\nu} = F_j(\Delta p_j) \quad \text{on } L_j = \{z \in \mathbb{C}\colon |z - z_j| = r_j\}, \quad j = 0, 1, \ldots, n. \tag{2.1}$$

Here, $\mathbf{u}$ is the velocity vector of the flow, $\boldsymbol{\nu}$ is a unit normal to L_j in the direction away from the flow domain $\mathcal{D}$, $\Delta p_j = p_j^- - p_j^+$ is the pressure jump through the surface L_j, and p_j^- and p_j^+ are the pressure on the external and internal surfaces of the cylinder D_j. The internal pressure p_j^+ is prescribed, while the external pressure p_j^- and the velocity $\mathbf{u}$ are to be determined.

The results of tests on a number of porous materials (**23**) some of which are presented in Fig. 2 demonstrate the linear dependence of the pressure difference on the velocity V, where V is the mean of the outlet and inlet velocities. In these tests, the maximum particle to pass a pore is 0·0001″, 0·0002″ and 0·001″ for grade A, B and D bronze, respectively. These graphs show that,

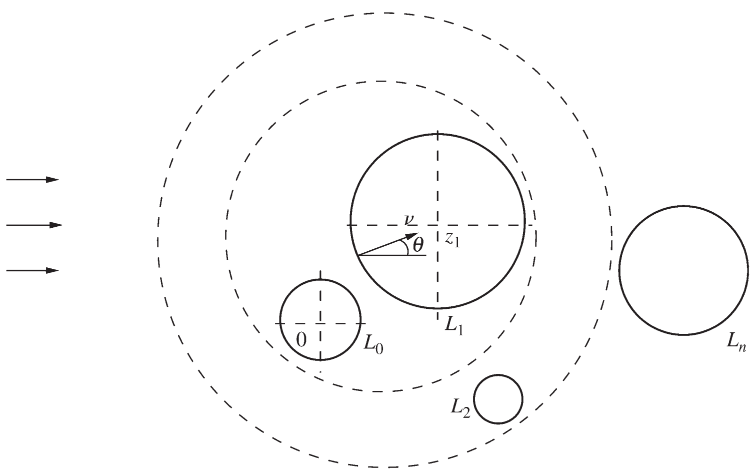

Fig. 1 Flow domain $\mathcal{D}$

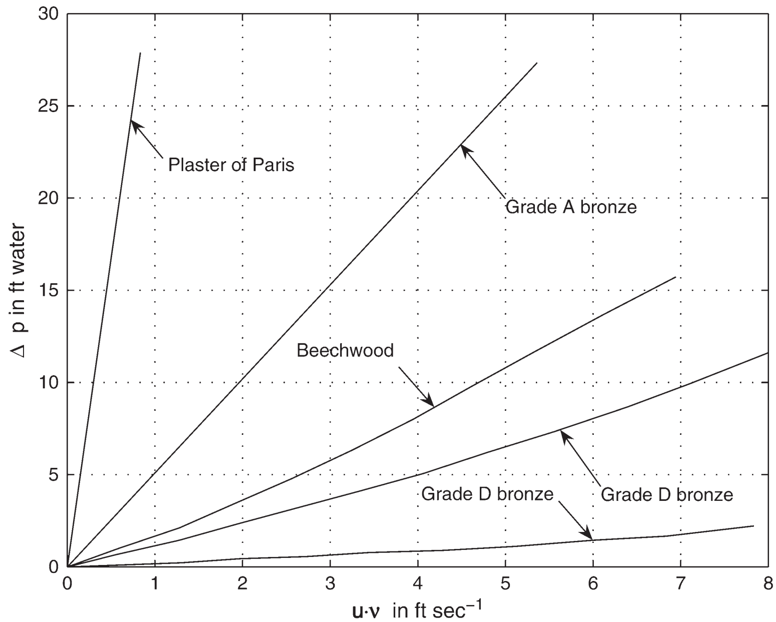

Fig. 2 Resistance of various porous materials (**23**). Thickness of the wall is $1/8''$

approximately, $\mathbf{u}\cdot\boldsymbol{\nu} = \lambda\Delta p$, where $\lambda = 0{\cdot}030, 0{\cdot}20, 0{\cdot}44, 0{\cdot}69$ and $3{\cdot}53$ for plaster of Paris, grade A bronze, beechwood, grade B and D bronze, respectively. On the other hand, the models (**21**, **22**) and the manufacturing data (**23**) show that the curves do not necessarily pass through the origin. Therefore, in the present study,

$$F_j(\Delta p_j) = \lambda_j^0 + \lambda_j \Delta p_j, \tag{2.2}$$

where λ_j^0 and λ_j are prescribed parameters depending on the character of the porous surface. From the Bernoulli theorem, everywhere in the flow domain $\mathcal{D}$,

$$\frac{V_\infty^2}{2}(u^2+v^2) + \frac{p-p_\infty}{\rho} - \frac{V_\infty^2}{2} = 0, \tag{2.3}$$

where $V_\infty^{-1}\mathbf{u} = (u, v)$, ρ is the fluid density and p_∞ is the pressure at infinity. On using the relation

$$\mathbf{u}\cdot\boldsymbol{\nu} = V_\infty(u\cos\theta + v\sin\theta) \tag{2.4}$$

and combining this with (2.1), we arrive at the following porous boundary condition:

$$u\cos\theta + v\sin\theta + \kappa_j(u^2+v^2) = \beta_j + \kappa_j\mu_j \quad \text{on } L_j. \tag{2.5}$$

Here, θ is the angle between $\boldsymbol{\nu}$ and the x-axis, and

$$\kappa_j = \frac{\rho V_\infty \lambda_j}{2}, \quad \beta_j = \frac{\lambda_j^0}{V_\infty}, \quad \mu_j = 1 + \frac{2}{\rho V_\infty^2}(p_\infty - p_j^+) \tag{2.6}$$

are dimensionless parameters. In what follows, the constants κ_j are assumed to be small: $\kappa_j = \kappa\alpha_j$, $\alpha_j = O(1)$ as $\kappa \to 0$. This may occur even for large parameters λ_j provided the quantity ρV_∞ is small. If $\alpha_j = 0$, then the wall of the cylinder D_j is impenetrable.

Expand next the velocity vector $\mathbf{u} = V_\infty \mathbf{U}$

$$\mathbf{U} \sim \mathbf{U}_0 + \kappa\mathbf{U}_1 + \kappa^2\mathbf{U}_2 + \cdots, \tag{2.7}$$

where $\mathbf{U} = (u, v)$ and $\mathbf{U}_k = (U_k, V_k)$, $k = 0, 1, \ldots$. The above relation is the Poincaré asymptotic expansion (**29**) of the vector $\mathbf{U}$ for $\kappa \to 0$ and $(x, y) \in \mathcal{D}$,

$$\mathbf{U}(x, y) = \mathbf{U}_0(x, y) + \kappa\mathbf{U}_1(x, y) + \kappa^2\mathbf{U}_2(x, y) + \cdots + \kappa^N\mathbf{U}_N(x, y) + R_{N+1}(x, y; \kappa), \tag{2.8}$$

where

$$R_{N+1}(x, y; \kappa) = O(\kappa^{N+1}), \quad \kappa \to 0, \ \ (x, y) \in \mathcal{D}. \tag{2.9}$$

Substituting these expansions into (2.5) replaces the nonlinear boundary condition by

$$\begin{gathered} u_k(\xi)\cos\theta(\xi) + v_k(\xi)\sin\theta(\xi) = f_{kj}(\xi), \quad \xi = (x, y) \in L_j, \\ k = 0, 1, \ldots, \quad j = 0, 1, \ldots, n, \end{gathered} \tag{2.10}$$

where

$$\begin{aligned} & u_k(\xi) = U_k(x, y), \quad v_k(\xi) = V_k(x, y), \quad (x, y) \in L = \cup_{j=0}^{n} L_j, \\ & f_{0j}(\xi) = \beta_j, \\ & f_{1j}(\xi) = \alpha_j(\mu_j - u_0^2(\xi) - v_0^2(\xi)), \\ & f_{kj}(\xi) = -\alpha_j \sum_{s=0}^{k-1}(u_{k-1-s}u_s + v_{k-1-s}v_s), \quad k = 2, 3, \ldots. \end{aligned} \tag{2.11}$$

The functions U_k and V_k can be considered as dimensionless velocity components of the kth ‘flow’ with a complex potential $w_k(z) = \phi_k(x, y) + i\psi_k(x, y)$: $\overline{w_k'(z)} = U_k + iV_k$. On the boundary of each cylinder,

$$u_k(\xi)\cos\theta + v_k(\xi)\sin\theta = \frac{\partial\phi_k}{\partial\nu} = \frac{\partial\psi_k}{\partial s}, \quad \xi \in L_j, \tag{2.12}$$

where $\partial\phi_k/\partial\nu$ and $\partial\psi_k/\partial s$ are the normal and tangential derivatives. The use of (2.10) and integration of (2.12) yield

$$\psi_k(\xi) = \int_{\xi_j}^{\xi} f_{kj}(\tau)ds + a_{kj}, \quad \xi \in L_j. \tag{2.13}$$

Here, ξ_j is an arbitrary fixed point on L_j, and a_{kj} are some real constants.

The complex potentials $w_k(z)$ are analytic multi-valued functions in the flow domain $\mathcal{D}$ with some cyclic periods

$$\int_{L_j} dw_k(z) = \gamma_{kj} + i\eta_{kj}, \quad j = 0, 1, \ldots, n, \tag{2.14}$$

where γ_{kj} is the circulation of the velocity vector associated with the kth flow around the contour L_j, and η_{kj} is the flux of the flow through L_j

$$\eta_{kj} = \int_{L_j} \frac{\partial \phi_k}{\partial \nu} ds = \int_{L_j} f_{kj}(\xi) ds. \tag{2.15}$$

At infinity, the complex potentials $w_k(z)$ admit the representations

$$w_k(z) = \bar{V}_{k\infty} z + \frac{\gamma_k + i\eta_k}{2\pi i} \log z + w_{k\infty}(z), \quad z \to \infty, \tag{2.16}$$

where $w_{k\infty}(z)$ are single-valued analytic functions in a neighbourhood of the infinite point, $V_{k\infty}$ is the velocity at infinity of the kth flow and

$$\gamma_k + i\eta_k = \sum_{j=0}^{n} (\gamma_{kj} + i\eta_{kj}). \tag{2.17}$$

Thus, the study of flow of a fluid around $n+1$ cylinders with porous walls requires the solution of the following problem.

PROBLEM 2.2. Find all multi-valued functions $w_k(z)$ analytic in the domain $\mathcal{D} = \mathbb{C} \backslash \bigcup_{j=0}^{n} D_j$, having the cyclic periods (2.14), representable at infinity in the form (2.16) and satisfying the boundary condition

$$\operatorname{Im} w_k(\xi) = \int_{\xi_j}^{\xi} f_{kj}(\tau) ds + a_{kj}, \quad \xi \in L_j = \partial D_j, \quad j = 0, 1, \ldots, n. \tag{2.18}$$

It will later be shown that the actual values of the parameters γ_{kj} and $\bar{V}_{k\infty}$ do not affect the solvability of Problem 2.2. Therefore, it will be convenient to take

$$\begin{gathered} \gamma_{0j} = \gamma_j = \frac{\Gamma_j}{V_\infty}, \quad V_{0\infty} = 1, \\ \gamma_{kj} = 0, \quad V_{k\infty} = 0, \quad k = 1, 2, \ldots. \end{gathered} \tag{2.19}$$

The parameters η_{kj} cannot be chosen arbitrary, they are defined uniquely for each flow by (2.15).

The total complex potential associated with Problem 2.1 under the assumption (2.2) has the form

$$w(z) = V_\infty [w_0(z) + \kappa w_1(z) + \kappa^2 w_2(z) + \cdots]. \tag{2.20}$$

It defines the flow with prescribed circulation Γ_j around the contour L_j and the velocity V_∞ at infinity.

REMARK 2.1 The more general case of (1.1) for integer $m \geqslant 2$ can be treated similarly. The asymptotic procedure leads to Problem 2.2 with appropriately chosen functions $f_{kj}(\xi)$.

3. Riemann–Hilbert problem of the theory of automorphic functions

In this section, Problem 2.2 will be converted into a Riemann–Hilbert problem for symmetric automorphic functions. Its solution will be constructed in terms of quasiautomorphic analogues of the Cauchy singular integrals.

3.1 *Symmetry group G*

Let G be the symmetry group of the line $L = L_0 \cup L_1 \cup \cdots \cup L_n$ generated by the linear transformations

$$\sigma_j(z) = T_j T_0(z), \quad j = 1, 2, \ldots, n, \tag{3.1}$$

where

$$T_j(z) = z_j + \frac{r_j^2}{\bar{z} - \bar{z}_j}, \quad j = 0, 1, \ldots, n, \tag{3.2}$$

is the symmetry transformation with respect to the circle L_j. Here as in (2.1) z_j and r_j are the centres and radii of the circles L_j $(j = 0, 1, \ldots, n)$, respectively. The transformation $\sigma_j(z)$ $(j = 1, 2, \ldots, n)$ maps the interior (exterior) of the circle $L'_j = T_0(L_j)$ onto the exterior (interior) of the circle L_j. The exterior of all the circles L_j, L'_j $(j = 1, 2, \ldots, n)$ is a fundamental region, say $\mathcal{F}_G$, of the group G. It will be convenient to add the circles L_j $(j = 1, 2, \ldots, n)$ to the region $\mathcal{F}_G$ so that $\mathcal{F}_G = \bar{\mathcal{D}} \cup T_0(\mathcal{D})$. The group G is a symmetry Schottky group (**16**), and consists of the identical map $\sigma_0(z) = z$ and all possible compositions of the generators $\sigma_j = T_j T_0$ and the inverse maps $\sigma_j^{-1} = T_0 T_j$ $(j = 1, 2, \ldots, n)$. Therefore, each element of the group G is a composition of an even number of the symmetry maps $T_j(z)$ $(j = 0, 1, \ldots, n)$

$$\begin{gathered} \sigma = T_{k_1} T_{k_2} \cdots T_{k_{2\mu-1}} T_{k_{2\mu}}, \quad \mu = 1, 2, \ldots, \\ k_1, k_2, \ldots, k_{2\mu} = 0, 1, \ldots, n, \quad k_2 \neq k_1, \ \ k_3 \neq k_2, \ldots, k_{2\mu} \neq k_{2\mu-1}. \end{gathered} \tag{3.3}$$

The region $\Omega = \bigcup_{\sigma \in G} \sigma(\mathcal{F}_G)$ is invariant with respect to the group G: $\sigma(\Omega) = \Omega$ for all $\sigma \in G$. This region is symmetric with respect to all the circles L_j $(j = 0, 1, \ldots, n)$ and $\Omega = \bar{\mathbb{C}} \setminus \Lambda$, where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and Λ is the set of the limit points of the group G.

All maps of the group G admit the representation

$$\sigma(z) = \frac{a_\sigma z + b_\sigma}{c_\sigma z + d_\sigma}, \quad a_\sigma d_\sigma - b_\sigma c_\sigma \neq 0, \tag{3.4}$$

and $c_\sigma \neq 0$ if $\sigma \neq \sigma_0$. In what follows, it is assumed that the numerical series

$$\sum_{\sigma \in G \setminus \sigma_0} \frac{|a_\sigma d_\sigma - b_\sigma c_\sigma|}{|c_\sigma|^2} \tag{3.5}$$

is convergent. According to the Burnside (**17**) classification, if a discrete group of maps (3.4) obeys this condition, then it belongs to the first class of groups. For such groups, it is possible to represent a G-automorphic function as a series whose elements are simple fractions. These series which are Poincaré theta series of dimension 2 converge uniformly (**30**) but not necessarily absolutely (the series (3.5) may diverge). By the sufficient Schottky condition (**15**), G is a first class group if the domain $\mathcal{D}$ can be split into a union of triple or double connected domains by circles which do not intersect each other and the circles L_j $(j = 0, 1, \ldots, n)$. Examples of such domains include double and triple domains themselves, circular multiply connected domains for which the centres z_j of the discs D_j, radii r_j, lie on the same straight line and the one presented in Fig. 1. Also, G is a first class Schottky group if the domain $\mathcal{D}$ meets the condition (**25**) $R_j > r_j\sqrt{n}$, where $j = 0, 1, \ldots, n$, and R_j is the distance between the point z_j and the circle L_i $(i \neq j)$ which is the nearest one to the circle L_j. The known examples (**31**, **32**) of groups for which the series (3.5) is divergent relate to (2) dimensional Poincaré theta series with respect to general Schottky groups and Kleinian groups and which are not symmetry groups. The Appendix gives an example of the domain $\mathcal{D}$ for which the symmetry Schottky group is a second class group, the series (3.5) is divergent and the corresponding Poincaré series of dimension 2 is not absolutely convergent. In what follows, it is assumed that the flow domain obeys the sufficient conditions which guarantee the convergence of the series (3.5). This justifies the change of order of summation used in the representation of the quasiautomorphic analogue of the Cauchy kernel (3.31) and relation (3.37). The absolute convergence will also be needed for the derivation of the complex potentials (4.18) and the physical quantities presented in section 5.2.

3.2 *Reduction to a Riemann–Hilbert problem*

Introduce a new function $\Phi(z)$ defined in the flow domain $\mathcal{D}$ by

$$\Phi(z) = w_k(z) - \bar{V}_{k\infty} z - \sum_{\nu=0}^{n} \frac{\gamma_{k\nu} + i\eta_{k\nu}}{2\pi i} \log(z - z_\nu), \quad z \in \mathcal{D}, \tag{3.6}$$

and extend its definition for the whole domain $\boldsymbol{\Omega}$ by

$$\begin{aligned} \Phi(z) &= \overline{\Phi(T_0(z))}, \quad z \in T_0(\mathcal{D}), \\ \Phi(z) &= \Phi(\sigma^{-1}(z)), \quad z \in \sigma(\mathcal{D} \cup T_0(\mathcal{D})), \ \ \sigma \in G. \end{aligned} \tag{3.7}$$

Then $\Phi(z)$ is a piecewise meromorphic and G-automorphic function and satisfies the symmetry condition

$$\overline{\Phi(T_j(z))} = \Phi(z), \quad z \in T_j(\mathcal{D}) = \sigma_j(T_0(\mathcal{D})), \ \ j = 1, \dots, n. \tag{3.8}$$

All circles $\sigma(L)$ including L are discontinuity lines of the function $\Phi(z)$. Let $\Phi^+(\xi)$ and $\Phi^-(\xi)$ be the boundary values of the function $\Phi(z)$ from the interior and exterior of the circles $\sigma(L)$, $\sigma \in G$, respectively. It follows from (3.6), (3.7) and (3.8) that the function $\Phi(z)$ solves the following Riemann–Hilbert problem.

PROBLEM 3.1. Find all piecewise meromorphic and G-automorphic functions bounded at infinity that meet the symmetry condition (3.8) and satisfy the linear relation

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\xi), \quad \xi \in L, \tag{3.9}$$

where

$$g(\xi) = -2i \left[\int_{\xi_j}^{\xi} f_{kj}(\tau) ds + a_{kj} \right] + 2i \operatorname{Im} \left[\bar{V}_{k\infty} \xi + \sum_{\nu=0}^{n} \frac{\gamma_{k\nu} + i\eta_{k\nu}}{2\pi i} \log(\xi - z_\nu) \right], \quad \xi \in L_j. \tag{3.10}$$

Notice that because the function $\Phi(z)$ is G-automorphic, it is discontinuous through the circles $\sigma(L)$, $\sigma \in G \backslash \sigma_0$,

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\sigma^{-1}(\xi)), \quad \xi \in \sigma(L), \ \ \sigma \in G \setminus \sigma_0. \tag{3.11}$$

The expression (3.10) for the jump function $g(\xi)$ follows from (3.6), (3.8) and (2.13). It will be convenient to deal with another form of the function (3.10). Notice that on the contours L_j,

$$\bar{\xi} - \bar{z}_j = \frac{r_j^2}{\xi - z_j},$$

$$2i\,\mathrm{Im}\frac{\gamma_{k\nu} + i\eta_{k\nu}}{2\pi i}\log(\xi - z_\nu) = \frac{\gamma_{k\nu} + i\eta_{k\nu}}{2\pi i}\log(\xi - z_\nu) + \frac{\gamma_{k\nu} - i\eta_{k\nu}}{2\pi i}\log\left(\bar{z}_j - \bar{z}_\nu + \frac{r_j^2}{\xi - z_j}\right), \quad \nu \neq j,$$

$$2i\,\mathrm{Im}(\bar{V}_{k\infty}\xi) = \bar{V}_{k\infty}(\xi - z_j) - \frac{V_{k\infty} r_j^2}{\xi - z_j} + 2i\,\mathrm{Im}(\bar{V}_{k\infty} z_j). \tag{3.12}$$

The function

$$\hat{h}_{kj}(\xi) = \int_{\xi_j}^{\xi} f_{kj}(\tau)\,ds \tag{3.13}$$

is discontinuous on the contour L_j: $\hat{h}_{kj}(\xi_j^-) - \hat{h}_{kj}(\xi_j^+) = \eta_{kj}$, whilst the function

$$h_{kj}^*(\xi) = \hat{h}_{kj}(\xi) - \frac{\eta_{kj}}{2\pi i}\log(\xi - z_j) \tag{3.14}$$

is continuous. Here, $\log(z - z_j)$ is the single branch of the logarithmic function in the z-plane cut along a line joining the points z_j and the infinite point and passing through the point ξ_j. The argument of the function is fixed by the condition $0 \leqslant \arg(\xi - z_j) < 2\pi$. Utilizing the relation

$$\log(\xi - z_j) = \log r_j + i\int_{\xi_j}^{\xi} \frac{ds}{r_j}, \tag{3.15}$$

it is possible to write the function $h_{kj}^*(\xi)$ in the alternative form

$$h_{kj}^*(\xi) = -\int_{\xi_j}^{\xi}\left[\frac{\eta_{kj}}{2\pi r_j} - f_{kj}(\tau)\right]ds - \frac{\eta_{kj}}{2\pi i}\log r_j. \tag{3.16}$$

Finally, from (3.12) and (3.16) it follows that

$$g(\xi) = 2i h_{kj}(\xi) + g_j^+(\xi) + g_j^-(\xi) + i b_j, \tag{3.17}$$

where

$$h_{kj}(\xi) = \int_{\xi_j}^{\xi}\left[\frac{\eta_{kj}}{2\pi r_j} - f_{kj}(\tau)\right]ds,$$

$$g_j^+(\xi) = \bar{V}_{k\infty}(\xi - z_j) + \sum_{\nu=0,\nu\neq j}^{n} \gamma_{k\nu}^+ \log(\xi - z_\nu),$$

$$g_j^-(\xi) = -\frac{V_{k\infty} r_j^2}{\xi - z_j} + \sum_{\nu=0,\nu\neq j}^{n} \gamma_{k\nu}^- \log\left(\bar{z}_j - \bar{z}_\nu + \frac{r_j^2}{\xi - z_j}\right),$$

$$\gamma_{k\nu}^{\pm} = \frac{\gamma_{k\nu} \pm i\eta_{k\nu}}{2\pi i}, \quad b_j = -2a_{kj} + 2\mathrm{Im}(\bar{V}_{k\infty} z_j) - \frac{\gamma_{kj}}{\pi}\log r_j. \tag{3.18}$$

Notice that for $k = 0$, $f_{kj} = \beta_j =$ constant. By (2.15), $\eta_{0j} = 2\pi r_j f_{0j}$ and therefore $h_{0j} = 0$. The functions $g_j^+(z)$ and $g_j^-(z)$ are analytic in the interior and exterior of the circles L_j ($j = 0, 1, \ldots, n$).

3.3 *Quasiautomorphic analogue of the Cauchy integral*

Consider the following series:

$$\Psi(z) = \sum_{\sigma\in G} \frac{1}{2\pi i} \int_{\sigma(L)} g(\sigma^{-1}(\xi)) \left(\frac{1}{\xi - z} - \frac{1}{\xi - z_*}\right) d\xi, \tag{3.19}$$

where z_* is an arbitrary fixed point of the domain $\mathcal{D}$. Analyse first the convergence of the series (3.19). By making the substitution $\xi = \sigma(\tau)$, $\tau \in L$, we get

$$\Psi(z) = \sum_{\sigma\in G} \frac{1}{2\pi i} \int_L g(\tau) \left(\frac{1}{\sigma(\tau) - z} - \frac{1}{\sigma(\tau) - z_*}\right) \sigma'(\tau) d\tau. \tag{3.20}$$

Since for $\tau \in L$ the point $\sigma(\tau)$ ($\sigma \in G \setminus \sigma_0$) is inside one of the circles L_j ($j = 0, 1, \ldots, n$), it follows that for all $z \in \mathcal{D} \setminus L$ and $\sigma \in G \setminus \sigma_0$,

$$|\sigma(\tau) - z| \geqslant M_0, \quad |\sigma(\tau) - z_*| \geqslant M_0, \quad M_0 = \text{constant} > 0. \tag{3.21}$$

Therefore, the convergence of the series (3.20) is guaranteed if the series

$$\sum_{\sigma\in G} \sigma'(\tau), \quad \tau \in L, \quad \sigma'(\tau) = \frac{a_\sigma d_\sigma - b_\sigma c_\sigma}{(c_\sigma \tau + d_\sigma)^2}, \tag{3.22}$$

is convergent. If, in addition, the above series converges absolutely, then the series (3.20) converges absolutely and uniformly with respect to z in the flow domain $\mathcal{D}$ and with respect to τ on the line L. Since $-d_\sigma/c_\sigma \notin \mathcal{F}_G$, there exists a positive constant M_1 such that $|\tau + d_\sigma/c_\sigma| > M_1$. Thus the absolute convergence of the series (3.22) is guaranteed by the convergence of the series (3.5) or, equivalently, by the fact that the group G is of the first class. This justifies the change of order of integration and summation in (3.20)

$$\Psi(z) = \frac{1}{2\pi i} \int_L K(z, \tau) g(\tau) d\tau, \tag{3.23}$$

where

$$K(z, \tau) = \sum_{\sigma\in G} \left(\frac{1}{\sigma(\tau) - z} - \frac{1}{\sigma(\tau) - z_*}\right) \sigma'(\tau). \tag{3.24}$$

The kernel $K(z, \tau)$ is a quasiautomorphic analogue of the Cauchy kernel and has the following properties (**12**, **13**).

(i) Since the group G contains the identical transformation σ_0,

$$K(z,\tau)=\frac{1}{z-\tau}+B(z,\tau), \tag{3.25}$$

where $B(z,\tau)$ is an analytic function of z in Ω.

(ii) $K(z_*,\tau)=0$ for all $\tau\in L$.

(iii) As a function of τ, the kernel $K(z,\tau)$ is a Poincaré series of dimension 2,

$$\sigma'(\tau)K(z,\sigma(\tau))=K(z,\tau), \quad \sigma\in G. \tag{3.26}$$

(iv) Under the transformation $\sigma(z)\in G$, the kernel $K(z,\tau)$ takes on an extra term depending on τ only

$$K(\sigma(z),\tau)=K(z,\tau)+\eta_\sigma(\tau), \quad \sigma\in G, \tag{3.27}$$

where $\eta_\sigma(\tau)=K(\sigma(z_*),\tau)$.

Since the last property is not obvious by inspection, it is worthwhile to show it. Note first that

$$\frac{\sigma'(\tau)}{\sigma(\tau)-z}=\frac{1}{\tau-\sigma^{-1}(z)}-\frac{1}{\tau-\sigma^{-1}(\infty)}, \tag{3.28}$$

which follows directly from formulae (3.4) and (3.22) for $\sigma(\tau)$ and $\sigma'(\tau)$ and

$$\sigma^{-1}(\tau)=\frac{b_\sigma-d_\sigma\tau}{c_\sigma\tau-a_\sigma}, \quad \sigma^{-1}(\infty)=-\frac{d_\sigma}{c_\sigma}. \tag{3.29}$$

By using the identity (3.28) rewrite next formula (3.24) as

$$K(z,\tau)=\sum_{\sigma\in G}\left(\frac{1}{\tau-\sigma^{-1}(z)}-\frac{1}{\tau-\sigma^{-1}(z_*)}\right). \tag{3.30}$$

Clearly, if G consists of all linear transformations σ given by (3.3) including the identity map, then $\omega=\sigma^{-1}\in G$ and none of the transformations (3.3) is repeated twice. Therefore, because of the absolute convergence of the above series ($|\tau-\omega(z)|\geqslant M_1$, $|\tau-\omega(z_*)|\geqslant M_1$, $M_1=\text{constant}>0$),

$$K(z,\tau)=\sum_{\omega\in G}\left(\frac{1}{\tau-\omega(z)}-\frac{1}{\tau-\omega(z_*)}\right). \tag{3.31}$$

Making the transformation $z\to\sigma(z)$ in the last formula gives

$$K(\sigma(z),\tau)=\sum_{\omega\in G}\left(\frac{1}{\tau-\omega\sigma(z)}-\frac{1}{\tau-\omega\sigma(z_*)}\right)+\sum_{\omega\in G}\left(\frac{1}{\tau-\omega\sigma(z_*)}-\frac{1}{\tau-\sigma(z_*)}\right). \tag{3.32}$$

Since $\omega\sigma\in G$ when $\omega\in G$ and $\sigma\in G$, the above relation implies the property (3.27).

(v) Utilizing formulae (3.25) and (3.27), it is possible to conclude that first the integral (3.20) is a quasiautomorphic analogue of the Cauchy integral

$$\Psi(\sigma(z))=\Psi(z)+\zeta_\sigma, \quad \sigma\in G, \;\; \zeta_\sigma=\Psi(\sigma(z_*)), \tag{3.33}$$

and secondly, the term ζ_σ forms an additive group of complex numbers isomorphic to the group G, namely, if $\sigma = \hat{\sigma}\tilde{\sigma}$, then $\zeta_\sigma = \zeta_{\hat{\sigma}} + \zeta_{\tilde{\sigma}}$.

From here it immediately follows that if $\zeta_\sigma = 0$ for the generators of the group G, then $\zeta_\sigma = 0$ for all transformations of the group G, and therefore the function $\Psi(z)$ is G-automorphic.

Notice that the properties (i) to (v) of the kernel $K(z, \tau)$ and the function $\Psi(z)$ are established for any discontinuous group G of the first class, not necessarily the Schottky group and not necessarily the symmetry group of the line L.

Use next the fact that the group G is the symmetry group of the line $L = L_1 \cup L_2 \cup \cdots \cup L_n$, and as was stated in Problem 1.1, the circles L_j $(j = 0, 1, \ldots, n)$ do not touch each other and none of them lies inside another. In what follows, we aim to prove the following identity:

$$\overline{K(T_\nu(z), \tau)d\tau} = K(z, \tau)d\tau - \frac{r_j^2}{(\tau - z_j)^2}\overline{K(T_\nu(z_*), T_j(\tau))d\tau}, \quad \tau \in L_j, \quad \nu, j = 0, 1, \ldots, n, \tag{3.34}$$

significant for future derivations.

Let $\tau \in L_j (j = 0, 1, \ldots, n)$. Then $\tau = T_j(\tau)$ and

$$\overline{K(T_\nu(z), \tau)d\tau} = -\overline{K(T_\nu(z), T_j(\tau))}\frac{r_j^2 d\tau}{(\tau - z_j)^2}. \tag{3.35}$$

On the other hand, because of (3.31) and $T_j T_j(z) \equiv z$,

$$K(T_\nu(z), T_j(\tau)) = \sum_{\omega \in G}\left(\frac{1}{T_j(\tau) - T_j T_j \omega T_\nu(z)} - \frac{1}{T_j(\tau) - T_j T_j \omega T_\nu(z_*)}\right) + \sum_{\omega \in G}\left(\frac{1}{T_j(\tau) - \omega T_\nu(z_*)} - \frac{1}{T_j(\tau) - \omega(z_*)}\right). \tag{3.36}$$

By making the substitution $\sigma = T_j \omega T_\nu$, write (3.36) in the form

$$K(T_\nu(z), T_j(\tau)) = -\frac{(\bar{\tau} - \bar{z}_j)^2}{r_j^2}\sum_{\sigma \in G}\left(\frac{1}{\bar{\tau} - \overline{\sigma(z)}} - \frac{1}{\bar{\tau} - \overline{\sigma(z_*)}}\right) + K(T_\nu(z_*), T_j(\tau)). \tag{3.37}$$

Its complex conjugate has the form

$$\overline{K(T_\nu(z), T_j(\tau))} = -\frac{(\tau - z_j)^2}{r_j^2}K(z, \tau) + \overline{K(T_\nu(z_*), T_j(\tau))}. \tag{3.38}$$

In combination, formulae (3.35) and (3.38) yield the identity (3.34).

3.4 *Solution to the Riemann–Hilbert problem*

The general solution to Problem 2.2 can be expressed in terms of the quasiautomorphic integrals by ((**14**), see also (**27**))

$$\Phi(z) = \Psi(z) + \overline{\Psi(T_0(z))} + \text{constant}, \tag{3.39}$$

where

$$\Psi(z) = \frac{1}{4\pi i}\sum_{m=0}^{n}\int_{L_m}(g_m^*(\tau) + ib_m)K(z,\tau)d\tau,$$
$$g_m^*(\tau) = 2ih_{km}(\tau) + g_m^+(\tau) + g_m^-(\tau). \tag{3.40}$$

The real coefficients b_m are expressed through the real constants a_{km} by (3.18). These coefficients can be found from the condition $\Phi(\sigma_j(z)) = \Phi(z)$, $j = 1, 2, \ldots, m$, which guarantees the property of the function $\Phi(z)$ being G-automorphic. It follows from the property (3.34) of the kernel $K(z,\tau)$ that

$$\Phi(\sigma_j(z)) = \Phi(z) + t_j - \bar{t}_j, \quad j = 1, 2, \ldots, n, \tag{3.41}$$

where

$$t_j = \frac{1}{4\pi i}\sum_{m=0}^{n}\int_{L_m}(g_m^*(\tau) + ib_m)\eta_{\sigma_j}(\tau)d\tau,$$
$$\eta_{\sigma_j}(\tau) = K(\sigma_j(z_*),\tau). \tag{3.42}$$

Therefore, the function $\Phi(z)$ is automorphic if and only if

$$\operatorname{Im} t_j = 0, \quad j = 1, 2, \ldots, n. \tag{3.43}$$

By the Cauchy theorem, evaluate the integrals

$$\int_{L_m}\eta_{\sigma_j}(\tau)d\tau = \begin{cases} -2\pi i, & m = 0, \\ 2\pi i, & m = j, m = 1, 2, \ldots, n, \\ 0, & m \neq j, m = 1, 2, \ldots, n. \end{cases} \tag{3.44}$$

Hence, the conditions (3.43) are satisfied if the constants b_j ($j = 1, 2, \ldots, n$) are chosen to be

$$b_j = b_0 + \frac{1}{2\pi}\operatorname{Re}\sum_{m=0}^{n}\int_{L_m}g_m^*(\tau)\eta_{\sigma_j}(\tau)d\tau, \quad j = 1, 2, \ldots, n. \tag{3.45}$$

Then the function $\Phi(z)$ is G-automorphic and it depends on one arbitrary constant b_0 or, equivalently, on the arbitrary constant a_{k0}. This means that the automorphic solution of Problem 2.2 exists for any set of the constants γ_{kj} and $V_{k\infty}$, provided the total circulation around the contour L_j is Γ_j and the total velocity at infinity is V_∞. This justifies the choice (2.19) of the constants γ_{kj} and $V_{k\infty}$ made in section 2.

4. Complex potentials $w_k(z)$

The complex potentials $w_k(z)$ of the kth flow can now be expressed through the solution to the Riemann–Hilbert problem by

$$w_k(z) = \bar{V}_{k\infty}z + \sum_{\nu=0}^{n}\frac{\gamma_{k\nu} + i\eta_{k\nu}}{2\pi i}\log(z - z_\nu) + \Phi_0(z) + \Phi_1(z) + \text{constant}, \tag{4.1}$$

where

$$\Phi_m(z) = \Psi_m(z) + \overline{\Psi_m(T_0(z))}, \quad m = 0, 1,$$

$$\Psi_0(z) = \frac{1}{4\pi i}\sum_{j=0}^{n}\int_{L_j}[g_j^+(\tau) + g_j^-(\tau) + ib_j]K(z,\tau)d\tau, \tag{4.2}$$

$$\Psi_1(z) = \frac{1}{2\pi}\sum_{j=0}^{n}\int_{L_j} h_{kj}(\tau)K(z,\tau)d\tau.$$

By using the property of the kernel (3.34),

$$\Phi_1(z) = \frac{1}{\pi}\sum_{j=0}^{n}\int_{L_j} h_{kj}(\tau)K(z,\tau)d\tau + \text{constant}. \tag{4.3}$$

To simplify the formula for the function $\Phi_0(z)$, consider the integrals

$$I_j(\sigma(z)) = \frac{1}{2\pi i}\int_{L_j}[g_j^+(\tau) + g_j^-(\tau) + ib_j]\left(\frac{1}{\tau - \sigma(z)} - \frac{1}{\tau - \sigma(z_*)}\right)d\tau. \tag{4.4}$$

If $\sigma = \sigma_0$, then by the Cauchy theorem

$$I_j(\sigma(z)) = \begin{cases} g_j^-(z_*) - g_j^-(z), & z \in \mathcal{D},\ j = 0, 1, \ldots, n, \\ g_j^-(z_*) - g_j^-(z), & z \in T_0(\mathcal{D}),\ j = 1, 2, \ldots, n, \\ g_0^+(z) + g_0^-(z_*) + ib_0, & z \in T_0(\mathcal{D}),\ j = 0. \end{cases} \tag{4.5}$$

Let now $\sigma \neq \sigma_0$ and $z \in \mathcal{D} \cup T_0(\mathcal{D})$. Then, $\sigma = T_{k_1}T_{k_2}\cdots T_{k_{2\mu}}$, $k_{m+1} \neq k_m$ $(m = 1, 2, \ldots, 2\mu - 1)$, $k_{2\mu} \neq k_1$, and $\sigma(z)$ is an interior point of the disc D_{k_1}. If $k_1 = j$, which means $\sigma = T_jT_{k_2}\cdots T_{k_{2\mu}} \in G_j$, then

$$I_j(\sigma(z)) = g_j^+(\sigma(z)) - g_j^+(\sigma(z_*)). \tag{4.6}$$

When $k_1 \neq j$ and $\sigma \neq \sigma_0$, then $\sigma = T_{k_1}T_{k_2}\cdots T_{k_{2\mu}} \in G\backslash G_j\backslash\sigma_0$, $k_1 \neq k_{2\mu}$ and $k_{m+1} \neq k_m$ $(m = 1, 2, \ldots, 2\mu - 1)$. In this case, the integral I_j is

$$I_j(\sigma(z)) = g_j^-(\sigma(z_*)) - g_j^-(\sigma(z)). \tag{4.7}$$

Thus, the function $\Phi_0(z)$ becomes

$$\begin{aligned}\Phi_0(z) = \frac{1}{2}\sum_{j=0}^{n}\Bigg\{&\sum_{\sigma\in G_j}[g_j^+(\sigma(z)) + \overline{g_j^+(\sigma T_0(z))} - g_j^+(\sigma(z_*)) - \overline{g_j^+(\sigma T_0(z_*))}] \\ &- \sum_{\sigma\in G\backslash G_j}[g_j^-(\sigma(z)) + \overline{g_j^-(\sigma T_0(z))} - g_j^-(\sigma(z_*)) - \overline{g_j^-(\sigma T_0(z_*))}]\Bigg\} \\ &+ \frac{1}{2}[\overline{g_0^+(T_0(z))} + \overline{g_0^-(T_0(z))} - ib_0] + \text{constant}.\end{aligned} \tag{4.8}$$

By substituting the expressions (3.18) for the functions g_j^+ and g_j^- into the last formula,

$$\Phi_0(z) = \frac{1}{2}\sum_{j=0}^{n}\left\{\sum_{\sigma\in G_j}\left[\bar{V}_{k\infty}(\sigma(z)-\sigma(z_*)) + V_{k\infty}(\overline{\sigma T_0(z)}-\overline{\sigma(z_*)})\right.\right.$$
$$+\sum_{\nu=0,\nu\neq j}^{n}\left(\gamma_{k\nu}^+\log\frac{\sigma(z)-z_\nu}{\sigma(z_*)-z_\nu}-\gamma_{k\nu}^-\log\frac{\overline{\sigma T_0(z)}-\bar{z}_\nu}{\overline{\sigma T_0(z_*)}-\bar{z}_\nu}\right)\Bigg]$$
$$-\sum_{\sigma\in G\backslash G_j}\left[V_{k\infty}r_j^2\left(\frac{1}{\sigma(z_*)-z_j}-\frac{1}{\sigma(z)-z_j}\right)+\bar{V}_{k\infty}r_j^2\left(\frac{1}{\overline{\sigma(z_*)}-\bar{z}_j}-\frac{1}{\overline{\sigma T_0(z)}-\bar{z}_j}\right)\right.$$
$$\left.\left.+\sum_{\nu=0,\nu\neq j}^{n}\left(\gamma_{k\nu}^-\log\frac{\overline{T_j\sigma(z)}-\bar{z}_\nu}{\overline{T_j\sigma(z_*)}-\bar{z}_\nu}-\gamma_{k\nu}^+\log\frac{T_j\sigma T_0(z)-z_\nu}{T_j\sigma T_0(z_*)-z_\nu}\right)\right]\right\}$$
$$+\frac{1}{2}\left[V_{k\infty}(\overline{T_0(z)}-\bar{z}_0)-\frac{\bar{V}_{k\infty}r_0^2}{\overline{T_0(z)}-\bar{z}_0}-ib_0\right]$$
$$-\frac{1}{2}\sum_{\nu=1}^{n}[\gamma_{k\nu}^-\log(\overline{T_0(z)}-\bar{z}_\nu)+\gamma_{k\nu}^+\log(z-z_\nu)]+\text{constant.} \tag{4.9}$$

It is possible to simplify the expression for the function $\Phi_0(z)$ further. Prove the following identity:

$$\sum_{j=0}^{n}\sum_{\sigma\in G_j}V_{k\infty}(\overline{\sigma T_0(z)}-\overline{\sigma(z_*)})$$
$$=\sum_{j=0}^{n}\sum_{\sigma\in G\backslash G_j}V_{k\infty}r_j^2\left(\frac{1}{\sigma(z)-z_j}-\frac{1}{\sigma(z_*)-z_j}\right)-\frac{V_{k\infty}r_0^2}{z-z_0}+\text{constant.} \tag{4.10}$$

To do this, make the substitution $\sigma = T_j\omega T_0$. Since $\sigma\in G_j$, $\sigma = T_jT_{k_2}\cdots T_{k_{2\mu}}$ $(k_2\neq j)$, it follows that for $j=0$, $\omega = T_{k_2\cdots T_{k_{2\mu}}}T_0\in G\backslash G_0\backslash\sigma_0$, and $\omega\in G\backslash G_j$ for $j\neq 0$. Therefore,

$$\sum_{j=0}^{n}\sum_{\sigma\in G_j}V_{k\infty}(\overline{\sigma T_0(z)}-\overline{\sigma(z_*)}) = \sum_{j=0}^{n}\sum_{\omega\in G\backslash G_j}V_{k\infty}[\overline{T_j\omega T_0T_0(z)}-\overline{T_j\omega T_0(z_*)}]$$
$$-V_{k\infty}\left(\frac{r_0^2}{z-z_0}-\frac{r_0^2}{z_*-z_0}\right)+\text{constant.} \tag{4.11}$$

But $T_0T_0(z)\equiv z$, and the relation (4.10) follows from (4.11). Notice next that if the transformation σ runs over the set $G\backslash G_j$, $j\neq 0$, then the map $\omega = T_j\sigma T_0$ runs over the set G_j. If $\sigma\in G\backslash G_0$, then $\omega = T_j\sigma T_0 \in G_0 \cup \sigma_0$. By using this substitution, we get

$$\sum_{j=0}^{n}\sum_{\sigma\in G\setminus G_j}\frac{\bar V_{k\infty} r_j^2}{\overline{\sigma T_0(z)} - \bar z_j} = \sum_{j=0}^{n}\sum_{\sigma\in G_j}\bar V_{k\infty}(\sigma(z) - z_j) + \bar V_{k\infty}(z - z_0) + \text{constant}. \tag{4.12}$$

Analyse now the logarithmic terms in (4.9). Let

$$B = -\sum_{j=0}^{n}\sum_{\sigma\in G_j}\sum_{\nu=0,\nu\neq j}^{n}\gamma_{k\nu}^{-}\log\frac{\overline{\sigma T_0(z)} - \bar z_\nu}{\overline{\sigma T_0(z_*)} - \bar z_\nu}. \tag{4.13}$$

The substitution $\sigma = T_j\omega T_0$ yields

$$B = -\sum_{\nu=0}^{n}\gamma_{k\nu}^{-}\sum_{j=0,j\neq\nu}^{n}\sum_{\omega\in G\setminus G_j}\log\frac{\overline{T_j\omega(z)} - \bar z_\nu}{\overline{T_j\omega(z_*)} - \bar z_\nu} + \sum_{\nu=1}^{n}\gamma_{k\nu}^{-}\log\frac{\overline{T_0(z)} - \bar z_\nu}{\overline{T_0(z_*)} - \bar z_\nu} + \text{constant}. \tag{4.14}$$

On the other hand, by making the substitution $\sigma = T_\nu\omega T_0$,

$$B = \sum_{\nu=0}^{n}\gamma_{k\nu}^{-}\sum_{\omega\in G_\nu}\log\frac{\omega(z) - z_\nu}{\omega(z_*) - z_\nu} + \sum_{\nu=1}^{n}\gamma_{k\nu}^{-}\log\frac{\overline{T_0(z)} - \bar z_\nu}{\overline{T_0(z_*)} - \bar z_\nu} + \text{constant}. \tag{4.15}$$

Collecting all the logarithmic terms with $\gamma_{k\nu}^{-}$ in (4.9) and using formulae (4.14) and (4.15) give the following term:

$$\sum_{\nu=0}^{n}\gamma_{k\nu}^{-}\sum_{\omega\in G_\nu}\log\frac{\omega(z) - z_\nu}{\omega(z_*) - z_\nu} + \text{constant}. \tag{4.16}$$

Similarly,

$$\begin{aligned}&\frac{\gamma_{k0}^{+}}{2}\log(z - z_0) + \frac{1}{2}\sum_{j=0}^{n}\left[\gamma_{kj}^{+}\log(z - z_j) + \sum_{\sigma\in G_j}\sum_{\nu=0,\nu\neq j}^{n}\gamma_{k\nu}^{+}\log\frac{\sigma(z) - z_\nu}{\sigma(z_*) - z_\nu}\right.\\ &\qquad\left. + \sum_{\sigma\in G\setminus G_j}\sum_{\nu=0,\nu\neq j}^{n}\gamma_{k\nu}^{+}\log\frac{T_j\sigma T_0(z) - z_\nu}{T_j\sigma T_0(z_*) - z_\nu}\right]\\ &= \sum_{\nu=0}^{n}\gamma_{k\nu}^{+}\sum_{\sigma\in G\setminus G_\nu}\log\frac{\sigma(z) - z_\nu}{\sigma(z_*) - z_\nu} + \text{constant}.\end{aligned} \tag{4.17}$$

By utilizing the relations (4.3), (4.9), (4.10), (4.12), (4.16) and (4.17), we simplify the expression for the function $\Phi_0(z)$. Finally, we get the following formula for the complex potentials $w_k(z)$:

$$w_k(z) = \bar V_{k\infty} z + \sum_{j=0}^{n}\left(\sum_{\sigma\in G_j}A_{kj}(\sigma(z)) + \sum_{\sigma\in G\setminus G_j}B_{kj}(\sigma(z))\right) + \text{constant}, \tag{4.18}$$

where

$$A_{kj}(\sigma(z)) = \bar{V}_{k\infty}(\sigma(z) - \sigma(z_*)) + \gamma_{kj}^{-} \log \frac{\sigma(z) - z_j}{\sigma(z_*) - z_j} + \Lambda_{kj}(\sigma(z)),$$

$$B_{kj}(\sigma(z)) = V_{k\infty} r_j^2 \left(\frac{1}{\sigma(z) - z_j} - \frac{1}{\sigma(z_*) - z_j} \right) + \gamma_{kj}^{+} \log \frac{\sigma(z) - z_j}{\sigma(z_*) - z_j} + \Lambda_{kj}(\sigma(z)),$$

$$\Lambda_{kj}(\sigma(z)) = \frac{1}{\pi} \int_{L_j} h_{kj}(\tau) \left(\frac{1}{\tau - \sigma(z)} - \frac{1}{\tau - \sigma(z_*)} \right) d\tau. \tag{4.19}$$

If $\sigma = \sigma_0$ and $z \to t \in L$, $z \in \mathcal{D}$, then by the Sokhotski–Plemelj formula the boundary value $\Lambda_{kj}^{-}(t)$ of the above integral can be transformed to the form ($z_* \notin L$)

$$\Lambda_{kj}^{-}(t) = \frac{1}{\pi} \int_{L_j} [h_{kj}(\tau) - h_{kj}(t)] \frac{d\tau}{\tau - t} - \frac{1}{\pi} \int_{L_j} \frac{h_{kj}(\tau) d\tau}{\tau - z_*}.$$

5. Analysis of the solution

5.1 *The case of a single cylinder*

In this particular case, $n = 0$, and the complex potentials $w_k(z)$ can be found without the theory of automorphic functions. The associated Riemann–Hilbert problem for the function $\Phi(z)$ becomes

$$\begin{aligned} &\Phi^{+}(\xi) - \Phi^{-}(\xi) = g(\xi), \quad \xi \in L_0, \\ &\Phi(z) = \overline{\Phi(T_0(z))}, \quad z \in T_0(\mathcal{D}) = D_0, \end{aligned} \tag{5.1}$$

where

$$g(\xi) = 2i h_{k0}(\xi) + \bar{V}_{k\infty}(\xi - z_0) - \frac{V_{k\infty} r_0^2}{\xi - z_0} + i b_0. \tag{5.2}$$

Its solution $\Phi(z)$ is defined by

$$\begin{aligned} &\Phi(z) = \Psi(z) + \overline{\Psi(T_0(z))}, \quad z \in \mathbb{C}, \\ &\Psi(z) = \frac{1}{4\pi i} \int_{L_0} \frac{g(\xi) d\xi}{\xi - z}. \end{aligned} \tag{5.3}$$

By using the Cauchy theorem, we find for $z \in \mathcal{D}$

$$\begin{aligned} \Psi(z) &= \frac{V_{k\infty} r_0^2}{2(z - z_0)} + \frac{1}{2\pi} \int_{L_0} \frac{h_{k0}(\xi) d\xi}{\xi - z}, \\ \Psi(T_0(z)) &= \frac{i b_0}{2} + \frac{V_{k\infty}}{2} \left(z_0 + \frac{r_0^2}{\bar{z} - \bar{z}_0} \right) + \frac{1}{2\pi} \int_{L_0} \frac{h_{k0}(\xi) d\xi}{\xi - T_0(z)}. \end{aligned} \tag{5.4}$$

Since the Cauchy kernel $K(z, \xi) = 1/(\xi - z)$ satisfies the identity

$$\overline{K(T_0(z), \xi) d\xi} = K(z, \xi) d\xi - K(z_0, \xi) d\xi, \tag{5.5}$$

the complex potentials $w_k(z)$ can be written in the form

$$w_k(z) = \bar{V}_{k\infty} z + \frac{V_{k\infty} r_0^2}{z - z_0} + \frac{\gamma_{k0} + i\eta_{k0}}{2\pi i} \log(z - z_0)$$
$$+ \frac{1}{\pi} \int_{L_0} \frac{h_{k0}(\xi) d\xi}{\xi - z} + \text{constant}, \quad z \in \mathcal{D}, \quad k = 0, 1, \ldots. \tag{5.6}$$

The total complex potential is

$$w(z) = \phi(x, y) + i\psi(x, y) \sim V_\infty [w_0(z) + \kappa w_1(z) + \kappa^2 w_2(z) + \cdots], \tag{5.7}$$

which defines the streamlines $\text{Im} w(z) = \text{constant}$.

Let X_0, Y_0 be the drag and lift forces acting on the cylinder and let M_0 be the moment about the origin of the pressure thrusts on the cylinder. By the Blasius theorem,

$$X_0 - iY_0 = \frac{i\rho}{2} \int_{L_0} \left(\frac{dw^-(t)}{dt} \right)^2 dt, \quad M_0 = -\frac{\rho}{2} \text{Re} \int_{L_0} \left(\frac{dw^-(t)}{dt} \right)^2 t dt, \tag{5.8}$$

where $dw^-(t)/dt$ is the boundary value of the function $w'(z) = V_\infty [w_0'(z) + \kappa w_1'(z) + \cdots]$,

$$w_k'(z) = \bar{V}_{k\infty} - \frac{V_{k\infty} r_0^2}{(z - z_0)^2} + \frac{\gamma_{k0} + i\eta_{k0}}{2\pi i (z - z_0)} - \frac{1}{\pi} \int_{L_0} \frac{[f_{k0}(\xi) - (2\pi r_0)^{-1} \eta_{k0}] d\xi}{\xi - z}, \quad z \notin L_0, \tag{5.9}$$

as $z \to t \in L_0$ and $z \in \mathcal{D}$. Here, we have used the relation

$$\frac{1}{\pi} \int_{L_0} \frac{h_{k0}(\xi) d\xi}{(\xi - z)^2} = \frac{1}{\pi} \int_{L_0} \frac{h_{k0}'(\xi) d\xi}{\xi - z}, \quad z \notin L_0, \tag{5.10}$$

and $h_{k0}'(\xi) = (2\pi r_0)^{-1} \eta_{k0} - f_{k0}(\xi)$. The boundary value of the function (5.9) is defined by the Sokhotski–Plemelj formula

$$\frac{dw_k^-(t)}{dt} = \bar{V}_{k\infty} - \frac{V_{k\infty} r_0^2}{(t - z_0)^2} + \frac{\gamma_{k0} + i\eta_{k0}}{2\pi i (t - z_0)} - \frac{1}{\pi} \int_{L_0} [f_{k0}(\xi) - f_{k0}(t)] \frac{d\xi}{\xi - t}, \quad t \in L_0, \tag{5.11}$$

and the integral in (5.11) is not singular.

In the particular case of a single non-penetrable cylinder, $F_0(\Delta p_0) \equiv 0$, $w_k(z) = 0$, $k \geqslant 1$, and $\eta_{00} = 0$, $h_{00}(\tau) \equiv 0$. In this case, we deduce the known formula for the complex potential

$$w(z) = V_\infty \left(z + \frac{r_0^2}{z - z_0} \right) + \frac{\Gamma_0}{2\pi i} \log(z - z_0) + \text{constant}. \tag{5.12}$$

Here, we have used $w(z) = V_\infty w_0(z)$, $V_{0\infty} = 1$, $\gamma_{00} = V_\infty^{-1} \Gamma_0$. In combination, formulae (5.8) and (5.12) give zero drag force (d'Alambert's paradox)

$$X_0 + iY_0 = -i\rho V_\infty \Gamma_0, \tag{5.13}$$

while for a porous cylinder, in general, this is not true. In Fig. 3, the dependence of the drag and lift forces upon the parameter $\kappa = \kappa_0$ ($\alpha_0 = 1$) when $\beta_0 = 0$ and $\beta_0 = 0{\cdot}01$ is given. The

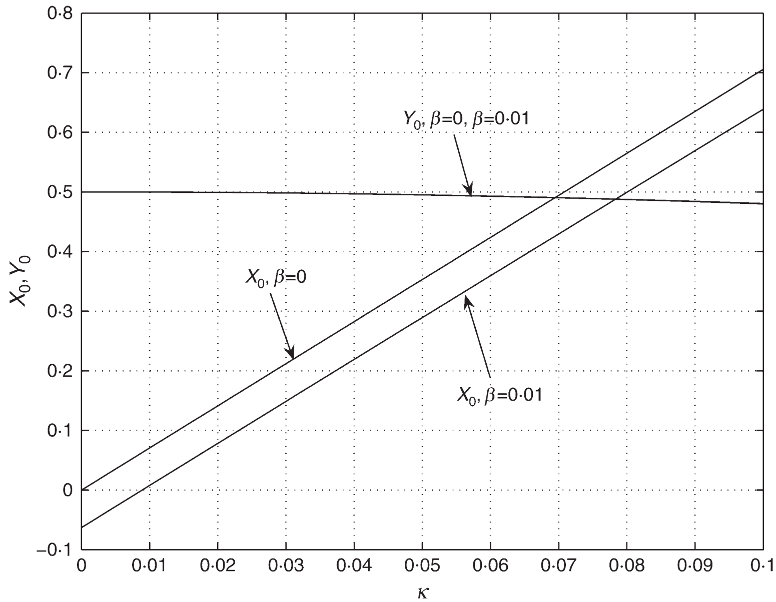

Fig. 3 The lift and drag forces for a single porous cylinder D_0

other parameters of the problem are chosen to be $\mu_0 = 1{\cdot}2$, $r_0 = 1$, $z_0 = 0$, $V_\infty = 1$ and $\Gamma_0 = -0{\cdot}5$. Numerical results implemented for different numbers ($N = 1, 2, \ldots, 100$) of terms in the approximation of the velocity vector

$$V_\infty^{-1}\mathbf{u}(x, y) \sim \mathbf{U}_0(x, y) + \kappa\mathbf{U}_1(x, y) + \kappa^2\mathbf{U}_2(x, y) + \cdots + \kappa^N\mathbf{U}_N(x, y), \tag{5.14}$$

and for $\kappa < 1$ reveal the fast convergence of the asymptotic algorithm proposed. The results of computations for the drag force for the case $\beta_0 = 0$ for some values of the parameter κ (the other parameters are the same as in Fig. 3) are shown in the table below.

κ	$N = 1$	$N = 2$	$N = 5$	$N = 10$	$N = 20$	$N = 50$	$N = 100$
0·1	0·706640	0·706623	0·706625	0·706625	0·7066245	0·706625	0·706625
0·5	3·533279	3·531339	3·532001	3·531921	3·531924	3·531924	3·531924
0·9	6·360042	6·350027	6·357796	6·352153	6·353356	6·354012	6·354042
0·99	6·996080	6·983141	6·995741	6·983480	6·983883	6·984945	6·986256

5.2 *The case of* $n + 1$ ($n \geqslant 1$) *cylinders*

Denote by X_j, Y_j the drag and lift forces acting on the jth cylinder. Then

$$X_j - iY_j = \frac{i\rho}{2}\int_{L_j}\left(\frac{dw^-(t)}{dt}\right)^2 dt. \tag{5.15}$$

Here $dw^-(t)/dt$ ($t \in L_j$) is the boundary value of the function dw/dz as $z \to t \in L_j$ and $z \in \mathcal{D}$. The pressure p in the flow domain $\mathcal{D}$ is recovered by the Bernoulli theorem

$$p = p_\infty + \frac{\rho}{2}(V_\infty^2 - |w'(z)|^2). \tag{5.16}$$

Formulae (5.15) and (5.16) require the derivative of the complex potentials $w_k(z)$

$$w_k'(z) = \bar{V}_{k\infty} + \sum_{j=0}^{n} \left(\sum_{\sigma \in G_j} \hat{A}_{kj}(\sigma(z)) + \sum_{\sigma \in G \setminus G_j} \hat{B}_{kj}(\sigma(z)) \right), \tag{5.17}$$

where

$$\hat{A}_{kj}(\sigma(z)) = \left[\bar{V}_{k\infty} + \frac{\gamma_{kj}^-}{\sigma(z) - z_j} \right] \sigma'(z) + \hat{\Lambda}_{kj}(\sigma(z)),$$

$$\hat{B}_{kj}(\sigma(z)) = \left[-\frac{V_{k\infty} r_j^2}{(\sigma(z) - z_j)^2} + \frac{\gamma_{kj}^+}{\sigma(z) - z_j} \right] \sigma'(z) + \hat{\Lambda}_{kj}(\sigma(z)),$$

$$\sigma'(z) = (\sigma(z) - z_*) \left(\frac{1}{z - \sigma^{-1}(z_*)} - \frac{1}{z - \sigma^{-1}(\infty)} \right),$$

$$\hat{\Lambda}_{kj}(\sigma(z)) = \sigma'(z) \Lambda_{kj}'(\sigma(z)),$$

$$\Lambda_{kj}'(\sigma(z)) = \frac{1}{\pi} \int_{L_j} \frac{h_{kj}(\tau) d\tau}{(\tau - \sigma(z))^2} = \frac{1}{\pi} \int_{L_j} \frac{h_{kj}'(\tau) d\tau}{\tau - \sigma(z)}, \quad \sigma(z) \notin L_j. \tag{5.18}$$

Here, $h_{kj}'(\tau) = (2\pi r_j)^{-1} \eta_{kj} - f_{kj}(\tau)$. If $\sigma = \sigma_0$ and $z = t^- \in L$, then the boundary value of the derivative $\Lambda_{kj}'(z)$ can be computed by the formula

$$\frac{d\Lambda_j^-(t)}{dt} = \frac{1}{\pi} \int_{L_j} [f_{kj}(t) - f_{kj}(\tau)] \frac{d\tau}{\tau - t}. \tag{5.19}$$

For implementation of numerics, we need the inverse transformations. Let $\sigma \in G_j$, where

$$G_j = \{T_j T_{k_2}, T_j T_{k_2} T_{k_3} T_{k_4}, T_j T_{k_2} T_{k_3} T_{k_4} T_{k_5} T_{k_6}, \ldots\}, \quad k_2 \neq j, \; k_3 \neq k_2, \; k_4 \neq k_3, \ldots. \tag{5.20}$$

Then, the set of the inverse maps σ^{-1} is defined by

$$\sigma^{-1} \in \hat{G}_j = \{T_{k_2} T_j, T_{k_4} T_{k_3} T_{k_2} T_j, T_{k_6} T_{k_5} T_{k_4} T_{k_3} T_{k_2} T_j, \ldots\}, \quad j \neq k_2, \; k_2 \neq k_3, \; k_3 \neq k_4, \ldots. \tag{5.21}$$

Let now $\sigma \in G \setminus G_j \setminus \sigma_0$, where

$$G \setminus G_j \setminus \sigma_0 = \{T_{k_1} T_{k_2}, T_{k_1} T_{k_2} T_{k_3} T_{k_4}, \ldots\}, \quad k_1 \neq j, \; k_2 \neq k_1, \; k_3 \neq k_2, \; k_4 \neq k_3, \ldots. \tag{5.22}$$

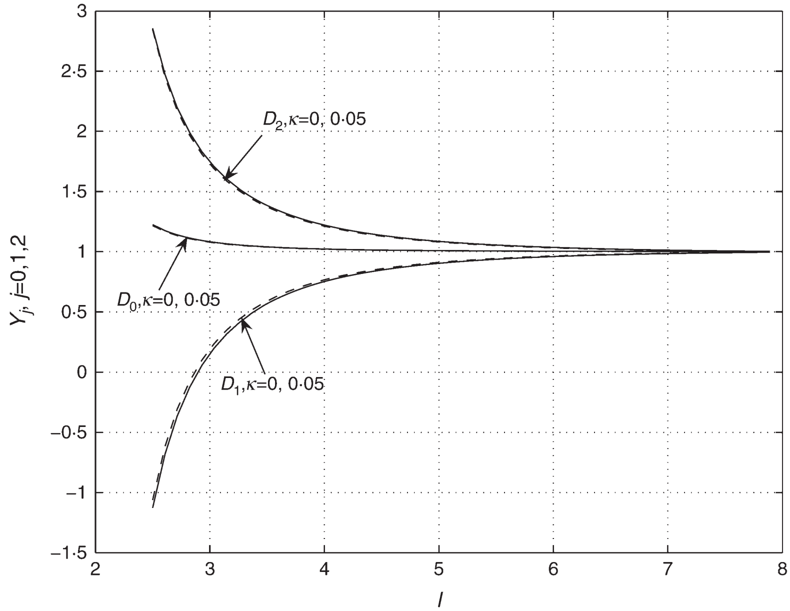

Fig. 4 The lift forces for three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = il$ and $z_2 = -il$

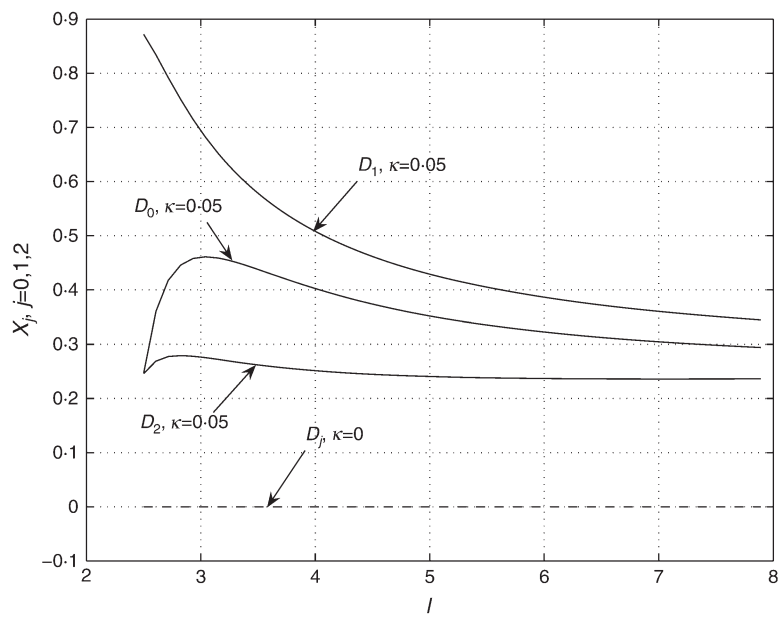

Fig. 5 The drag forces for three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = il$ and $z_2 = -il$

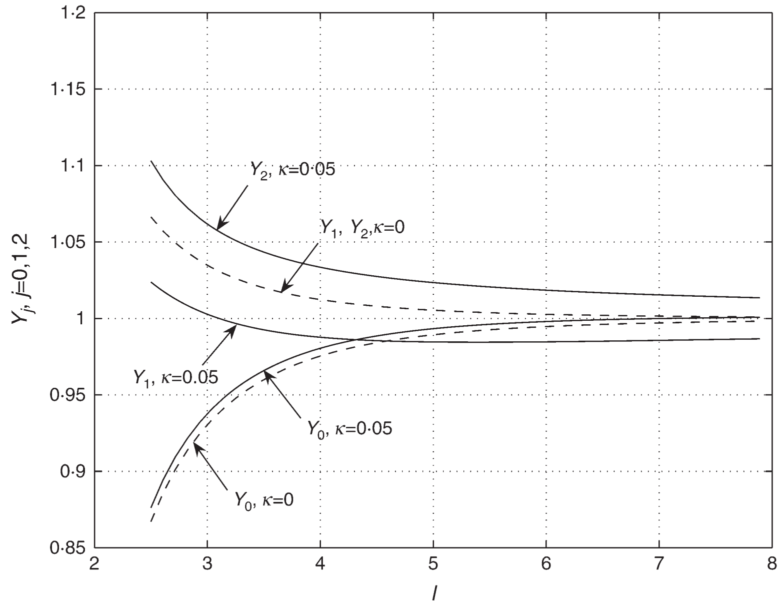

Fig. 6 The lift forces for three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = l$ and $z_2 = -l$

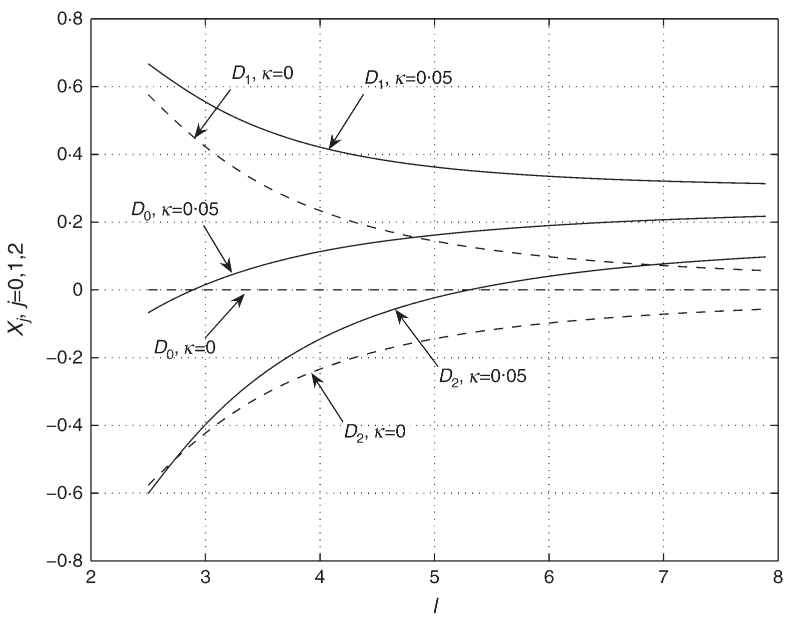

Fig. 7 The drag forces for three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = l$ and $z_2 = -l$

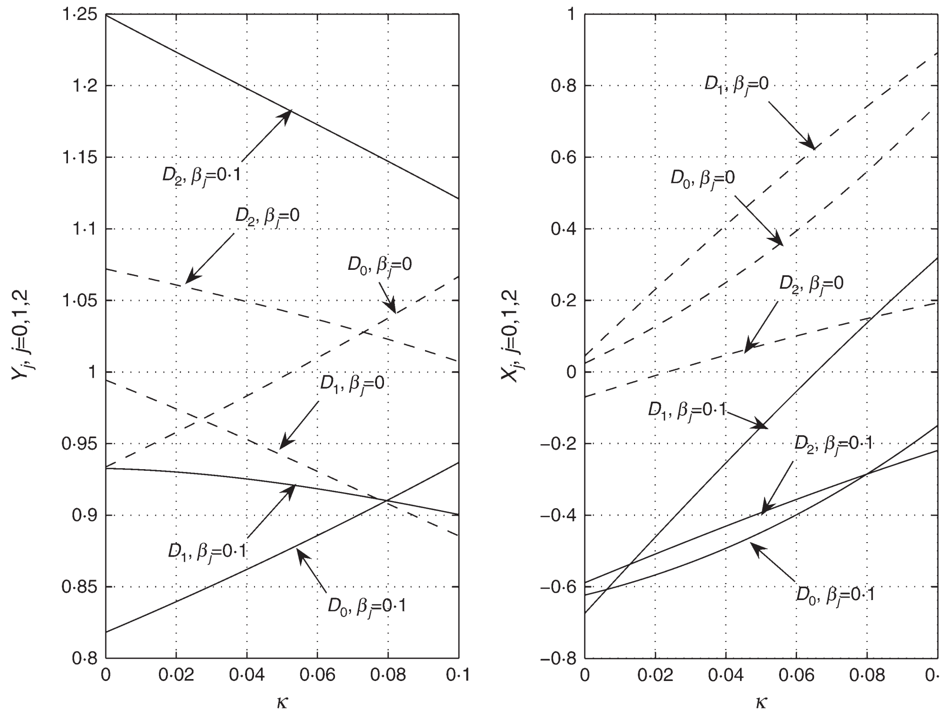

Fig. 8 Dependence of the lift and drag forces on the parameter κ for three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = 3e^{i\pi/6}$ and $z_2 = \bar{z}_1$

In this case, the inverse transformations are

$$\sigma^{-1} \in \tilde{G}_j = \{T_{k_2}T_{k_1}, T_{k_4}T_{k_3}T_{k_2}T_{k_1}, \ldots\}, \quad k_1 \neq j, \; k_1 \neq k_2, \; k_2 \neq k_3, \; k_3 \neq k_4, \ldots. \qquad (5.23)$$

If $\sigma_0(z) \equiv z$, then the inverse map is the identity: $\sigma_0^{-1}(z) = z$.

Figures 4 and 5 show the dependence of the lift and drag forces on the distance l between the centres of the discs in the case of flow past three unit discs aligned vertically centred at $z_0 = 0$, $z_1 = li$ and $z_2 = -li$. The circulations are taken to be $\Gamma_j = -1$, $j = 0, 1, 2$. The internal pressure in each cylinder is the same and it is less than the pressure at infinity: the dimensionless parameters μ_j are assumed to be equal to 1·2. The parameters β_j vanish and $\alpha_0 = 0{\cdot}98$, $\alpha_1 = 1$ and $\alpha_2 = 1{\cdot}02$. The density of the fluid is $\rho = 1$ and the speed at infinity $V_\infty = 1$. As in the case of a single cylinder, the asymptotic algorithm converges for $\kappa < 1$. To make computations in the case $\kappa < 1$, we took four terms ($N = 3$) in formula (2.8). For the particular case κ_0 (solid walls), the graphs for the lift forces coincide with those presented in (**18**). It turns out that for this arrangement of the discs, the lift force changes only slightly when the porosity parameters $\kappa_j = \alpha_j\kappa$ are non-zero. Again, the drag forces X_j vanish for $\kappa = 0$, while they change substantially even for small values of the parameter κ (Fig. 5).

The dependence of the forces on the distance l between the centres of the discs was studied for another arrangement of the discs. In Figs 6 and 7, the unit discs are aligned horizontally centred at

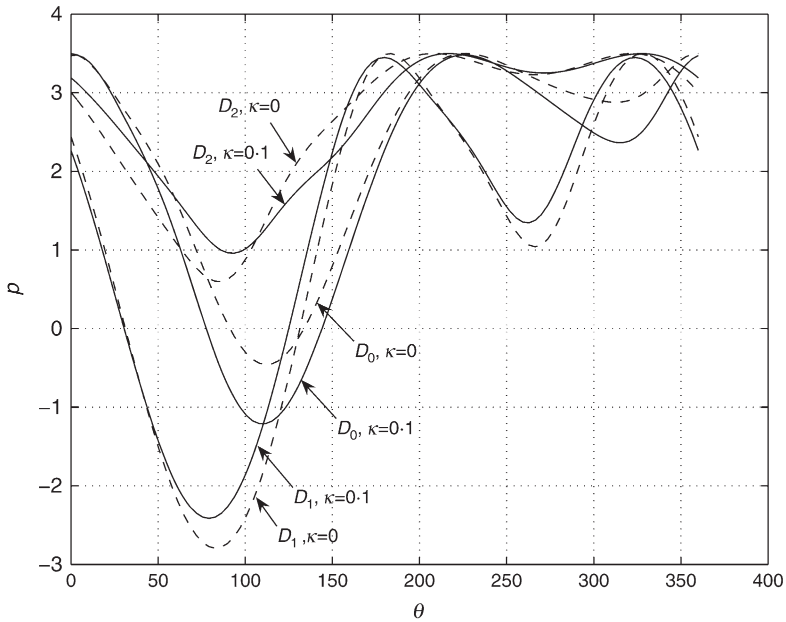

Fig. 9 The pressure p on the external surface: three porous cylinders D_j of radius $r_j = 1$ for $z_0 = 0$, $z_1 = 3e^{i\pi/6}$ and $z_2 = \bar{z}_1$

$z_0 = 0$, $z_1 = l$ and $z_2 = -l$. The parameters of the problem V_∞, ρ, Γ_j, μ_j, β_j and α_j are taken to be the same as in the previous case. It is seen that in this case, the lift forces (apart from Y_2) vary with the change of the parameter κ.

Figure 8 presents the dependence of the forces on the parameter κ for three unit discs centred at $z_0 = 0$, $z_1 = 3e^{i\pi/6}$ and $z_2 = \bar{z}_1$. The parameters V_∞, ρ, Γ_j, μ_j and α_j are still the same as in Figs 4 to 7, whilst β_j are taken to be 0 and 0·1. Both components of the force acting on the cylinders depend on the parameter κ.

For the same arrangement of the unit discs as in Fig. 8, we studied the variation of the pressure $p = p_j^-$ on the external surfaces of the cylinders when $p_\infty = 3$ (Fig. 9). It is assumed that $\Gamma_j = -5$, and the parameters V_∞, ρ, α_j, β_j and μ_j are the same as in Figs 4 to 7. The minimum of the pressure is attained at the angle which in general is different from $\pi/2$ and depends on the parameters of the problem.

Figure 10 presents the dependence of the forces on the distances between the discs for four cylinders of different radii arranged as in Fig. 1. The radii of the discs are $r_0 = 1$, $r_1 = 2$, $r_2 = 0{\cdot}5$ and $r_3 = 1{\cdot}5$. The centres of the discs D_0, D_1 and D_3 are fixed: $z_0 = 0$, $z_1 = 3 + 2i$, $z_3 = 9 + i$. The disc D_2 traverses the circle centred at $Z = 2 + 1{\cdot}2i$, radius $R = 4{\cdot}5$, so that $z_2 = Z + Re^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$. The parameters R and Z are chosen such that the disc D_2 does not collide with the others. In this case, $\mathcal{D}$ is a four-connected domain, and it can be split into triple-connected domains as shown in Fig. 1. The associated symmetric Schottky group is of the convergent type (a first class group), and the convergence of the series solution is guaranteed.

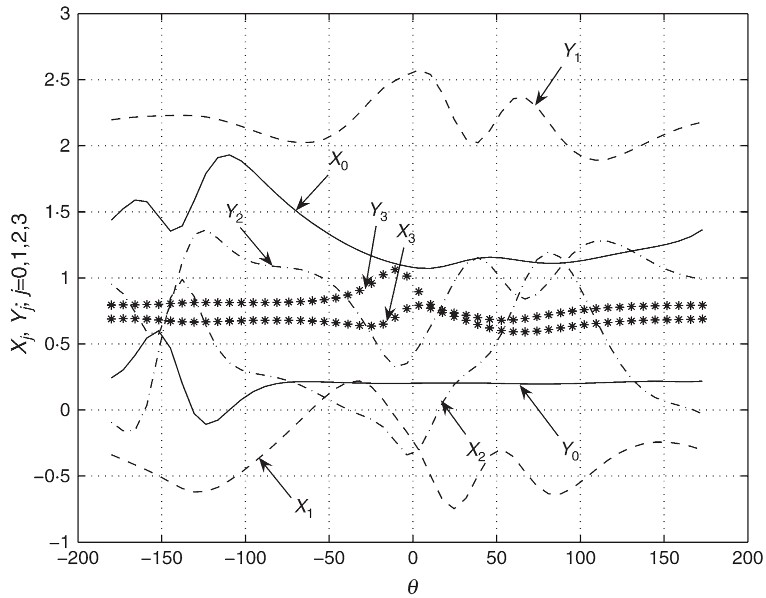

Fig. 10 The drag and lift forces for four porous cylinders for $\kappa = 0{\cdot}1$, $r_0 = 1$, $r_1 = 2$, $r_2 = 0{\cdot}5$, $r_3 = 1{\cdot}5$, $z_0 = 0$, $z_1 = 3 + 2i$, $z_3 = 9 + i$ and $z_2 = Z + Re^{i\theta}$, $Z = 2 + 1{\cdot}2i$, $R = 4{\cdot}5$, $-\pi < \theta < \pi$

6. Conclusions

In this paper, we have examined the steady flow of an ideal fluid around a group of parallel cylinders with porous walls. It has been found that for small values of the porosity parameters κ_j, the boundary condition $u\cos\theta + v\sin\theta + \kappa_j(u^2 + v^2) = \beta_j + \kappa_j\mu_j$, initially nonlinear, can be linearized, and the physical problem can be reduced to a sequence of the Hilbert problems for the multiply connected flow domain. By using symmetric Möbius transformations which generate a Schottky symmetry group, the problem is converted into a Riemann–Hilbert problem for symmetric automorphic functions. Its solution has been derived analytically in a series form. The expressions for the coefficients of the series consist of two parts. The first one is given explicitly. The second part is expressed through quasiautomorphic analogues of the Cauchy integrals. These integrals vanish if the walls are solid, and the solution reduces to the form known in the literature. By applying this method, we have derived asymptotic formulae for the drag and lift forces for a group of porous cylinders. It has been shown that the drag force depends on the porosity parameters and does not vanish when it is zero for the solid walls.

It is worth noticing that the convergence of the solution is guaranteed when the associated Schottky group is a first class group. This occurs, for example, for two and three cylinders, for $n + 1$ ($n \geqslant 3$) cylinders if their centres lie on the same straight line, when the flow domain can be split into a union of triply connected domains, and in some other cases. We have constructed an example of a symmetric Schottky group which is not a first class group. It does not mean, however, that if the flow domain does not meet the sufficient conditions described in the paper, then the series is necessarily divergent. Derivation of a necessary and sufficient condition for a Schottky symmetric group to be a first class group is still an open problem.

Finally, we point out that the method proposed can also be generalized for a stack of cylinders with porous walls which move in an accelerating flow and whose radii oscillate.

Acknowledgments

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APPENDIX

An example of the symmetric Schottky group of the divergent type

Consider a group G of linear transformations $\sigma(z)$ whose elements apart from the identity transformation $\sigma_0(z) \equiv z$ are written as

$$\sigma(z) = \frac{a_\sigma z + b_\sigma}{c_\sigma z + d_\sigma}, \quad a_\sigma d_\sigma - b_\sigma c_\sigma = 1, \quad c_\sigma \neq 0. \tag{A.1}$$

The convergence of the series

$$\sum_{\sigma \in G \setminus \sigma_0} |c_\sigma|^{-m} \tag{A.2}$$

(m is an integer and positive number) is equivalent to the uniform convergence of the series

$$\sum_{\sigma \in G} |\sigma'(z)|^{-m/2} \tag{A.3}$$

in closed domains excluding the limit points of the group and the points where the term $\sigma'(z) = (c_\sigma z + d_\sigma)^{-2}$ has poles. Thus, the group G is a first class group if, for $m = 2$, the series (A.2) or, equivalently, the series (A.3) is convergent. Following (**33**), introduce the m-dimensional computing functions of the group G

$$f_\sigma^{(m)}(z) = \sum_{\omega \in G_* \setminus \sigma} \frac{R_\omega^m}{|g_\omega - z|^m}, \quad z \in E_\sigma, \ \ \sigma \in G_*, \tag{A.4}$$

where G_* is the set of all generators and their inverses, E_σ is the set of the limit points of the group G lying inside the isometric circle $I_\sigma = \{z \in \mathbb{C}\colon |c_\sigma z + d_\sigma| = 1\}$ of the transformation $\sigma \in G_*$ and R_ω and g_ω are the radius and the centre of the isometric circle I_ω, respectively. Combining the results by Akaza (**33**, pp. 138, 139, 131), we have the following.

THEOREM A.1 *If for each $\sigma \in G_*$ there exists a constant $\alpha_\sigma > 1$, such that*

$$f_\sigma^{(m)}(z) \geqslant \alpha_\sigma \tag{A.5}$$

for all $z \in E_\sigma$, then the series (A.2) *and* (A.3) *diverge.*

In particular, if $m = 2$ and the condition (A.5) is satisfied, then the group G is a divergence group (not a first class group).

Myrberg (**31**) and Akaza (**32**) gave examples of the Schottky groups which meet the condition (A.5) for $m = 2$. However, their examples were not symmetric Schottky groups. Notice that the fundamental domain of the group G in (**32**) is bounded by 36 circles. In the later examples (**33**, **34**) of Kleinian groups that are not first class, the fundamental domains are bounded by five and four circles, respectively. In both cases, the groups are neither symmetric groups nor Schottky groups.

In what follows, we present an example of a symmetric Schottky group of a line $L = L_0 \cup L_1 \cup \cdots \cup L_n$ which is not a first class group. Let L_0 be the real axis, L_j ($j = 1, 2, \ldots, 8$) be the circles radius $r_j = 1-\varepsilon$ (ε is a small positive number), centred at $z_j = 2j-5+3i$ ($j = 1, \ldots, 4$) and $z_j = 2j-13+i$ ($j = 5, \ldots, 8$). The set G_* consists of 16 transformations $\sigma_j = T_0 T_j$, $\sigma_{j+8} = T_j T_0$ ($j = 1, 2, \ldots, 8$), where $T_0(z) = \bar{z}$ and $T_j(z) = z_j + r_j^2/(\bar{z} - \bar{z}_j)$, $j = 1, 2, \ldots, 8$. The isometric circles I_σ of the transformations σ_j coincide with the circles L_j ($j = 1, 2, \ldots, 8$). The isometric circles L_{j+8} of the transformations σ_{j+8} are the mirror images of the L_j with respect to the real axis L_0 (Fig. 11), and $z_{j+8} = \bar{z}_j$ ($j = 1, 2, \ldots, 8$). The group G has 16 two-dimensional computing functions

$$f_j^{(2)}(z) = f_{\sigma_j}^{(2)}(z) = \sum_{\nu=1, \nu\neq j}^{16} \frac{(1-\varepsilon)^2}{|z_\nu - z|^2}, \quad j = 1, 2, \ldots, 16. \tag{A.6}$$

We aim to find the lower bound of the function $f_j(z)$ on the set E_j of the limit points of the group G inside the circle L_j. The limit points of the group G coincide with the limit points of the set $\sigma(\infty)$, σ runs over the whole group G. Since any transformation $\sigma \in G$ is a composition of an even number of symmetry transformations of the line L, then all points $\sigma(\infty)$ lie inside the square with vertices z_1, z_4, z_9 and z_{12}. Therefore, the limit points of the group lie in the interior of this square and its boundary. Because of the symmetry, the functions $f_j^{(2)}(z)$ ($j = 1, 4, 9, 12$) are bounded from below on the corresponding sets E_j by the same constant. To find this constant, notice that since $E_1 \subset Q_1$, where Q_1 is the closed quarter-disc (Fig. 11), then it follows that

$$\inf_{z\in E_1} f_1^{(2)}(z) \geqslant \min_{z\in Q_1} f_1^{(2)}(z), \tag{A.7}$$

where

$$f_1^{(2)}(z) = (1-\varepsilon)^2 \sum_{\nu=2}^{16} \frac{1}{|z_\nu - z|^2}. \tag{A.8}$$

Next, group the terms of the sum (A.8) with indices 2 and 5, 3 and 13, 4 and 9, 7 and 14, 8 and 10, 11 and 16. Either two-term sum attains its minimum on the set Q_1 at the point $z_1 = -3 + 3i$. The same property is valid for the other terms of the sum (A.8) with indices 6, 12 and 15 considered separately. Thus,

$$\min_{z\in Q_1} f_1^{(2)}(z) = (1-\varepsilon)^2 \sum_{\nu=2}^{16} \frac{1}{|z_\nu - z_1|^2} = \frac{19453}{18720}(1-\varepsilon)^2 = \alpha_j. \tag{A.9}$$

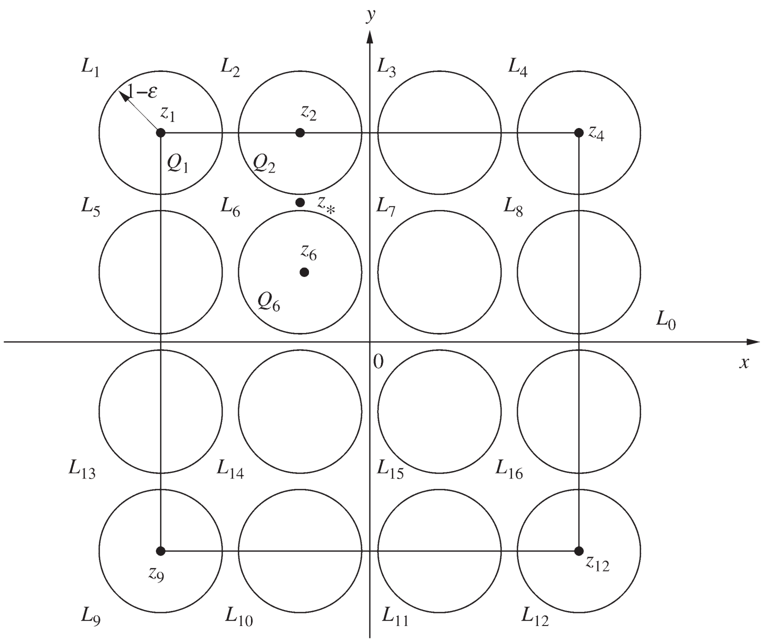

Fig. 11 Isometric circles of the transformations $\sigma_j\,(\,j = 1, 2, \ldots, 16)$

Take

$$\varepsilon < 1 - \sqrt{\frac{18720}{19453}} = 0{\cdot}019021\cdots. \tag{A.10}$$

Then if $j = 1, 4, 9, 12$, $f_j^{(2)}(z) \geqslant \alpha_j$ for any $z \in E_j$, where $\alpha_j > 1$ for the choice (A.10) of ε.

For $j = 2, 3, 5, 8, 10, 11, 13, 16$, because of the symmetry, the functions $f_j^{(2)}(z)$ have the same low bound on the sets E_j. Therefore, it is sufficient to study the function

$$f_2^{(2)}(z) = (1-\varepsilon)^2 \sum_{\nu=1,\nu\neq 2}^{16} \frac{1}{|z_\nu - z|^2} \tag{A.11}$$

on the set E_2. Notice that $E_2 \subset Q_2 \subset Q_2'$, where Q_2 and Q_2' are the lower semi-discs, centred at z_2, radius $1-\varepsilon$ and 1, respectively. Then

$$\inf_{z\in E_2} f_2^{(2)}(z) \geqslant \min_{z\in Q_2'} f_2^{(2)}(z). \tag{A.12}$$

The sum of the terms in (A.11) with indices 1 and 3 attains its minimum $\frac{2}{5}$ at the point $z_* = -1 + 2i$. Each two-term sum of the terms with indices 5 and 7, 9 and 11, 13 and 15 attains its minimum in the semi-disc Q_2' at the point $z_2 = -1 + 3i$. The minimum of the sum of all the mentioned terms is again $\frac{2}{5}$. Each term in (A.11) with indices 6, 10 and 14 considered separately attains it minimum in Q_2' at the points $-2 + 3i$ or $3i$. The minimum for the sum of these three terms is equal to $\frac{1}{5} + \frac{1}{17} + \frac{1}{37}$. Thus, the minimum of the sum (A.12) without positive terms with indices 4, 8, 12 and 16 in Q_2' is equal to $\frac{683}{629} > \frac{19453}{18720}$, and $f_j^{(2)}(z) \geqslant \alpha_j$ on E_j, where $\alpha_j = \frac{683}{629}(1-\varepsilon)^2 > 1$, ε satisfies the inequality (A.10) and $j = 2, 3, 5, 8, 10, 11, 13, 16$.

Finally, study the functions $f_j^{(2)}(z)$ on the sets E_j for $j = 6, 7, 14$ and 15. Again, because of symmetry, we confine our task to analysing the minimum of the function

$$f_6^{(2)}(z) = (1-\varepsilon)^2 \sum_{\nu=1,\nu\neq 6}^{16} \frac{1}{|z_\nu - z|^2} \tag{A.13}$$

in the disc Q_6. Since $E_6 \subset Q_6$, it follows that

$$\inf_{z\in E_6} f_6^{(2)}(z) \geqslant \min_{z\in Q_6} f_6^{(2)}(z). \tag{A.14}$$

Similarly to the previous cases, $f_j^{(2)}(z) \geqslant \alpha_j = \frac{3}{2}(1-\varepsilon)^2 > 1$ for any $z \in E_j$, ε chosen in (A.10) and $j = 6, 7, 14, 15$.

Therefore, all the two-dimensional computing functions of the group G satisfy the inequality $f_j^{(2)}(z) \geqslant \alpha_j > 1$ for any $z \in E_j$, $j = 1, 2, \ldots, 16$. The series (A.2) and (A.3) are divergent, the symmetric Schottky group considered is not a group of the first class and the associated Poincaré theta series of dimension 2 is not absolutely convergent.