A SYMMETRIC RIEMANN–HILBERT PROBLEM FOR ORDER-4 VECTORS IN DIFFRACTION THEORY

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Summary

A class of $n \times n$ matrices of the form $G = R_1 + aR_2$ (a is a scalar Hölder function defined on a contour L, R_1 and R_2 are rational matrices), which admit a closed-form Wiener–Hopf factorization is analyzed. It is shown that the diffraction problem (E-polarization) for a penetrable right-angled wedge with an electrically resistive and a perfectly magnetically conductive sides can be formulated as a Riemann–Hilbert problem for two order-4 vectors with a 4×4 matrix coefficient $G = R_1 + aR_2$. This problem reduces to two scalar Riemann–Hilbert problems and one vector problem for two order-2 vectors with a certain constraint. The order-2 vector problem is solved by employing the theory of the scalar Riemann–Hilbert problem on a genus-1 Riemann surface. The component E_z of the electric field and the diffraction coefficient are determined.

1. Introduction

Many canonical problems of mathematical diffraction theory require the solution of the associated Riemann–Hilbert problems for n pairs of analytic functions. The cases $n = 1$ and $n = 2$ are well studied. In the former case, the solution of the diffraction problem can always been constructed in closed form. The vector case is more complicated. The main encumbrance is the Wiener–Hopf factorization of the matrix coefficient of the Riemann–Hilbert problem. Chebotarev (**1**) showed that if the logarithm of an $n \times n$ matrix $G(\eta)$ commutes with its Cauchy integral

$$\chi(\eta) = \frac{1}{2\pi i}\int_L \frac{\ln G(t)dt}{t - \eta}, \tag{1.1}$$

that is, if

$$[\ln G(\eta), \chi(\eta)] \equiv \ln G(\eta)\chi(\eta) - \chi(\eta)\ln G(\eta) = 0, \tag{1.2}$$

then the matrix $G(\eta)$ admits a closed-form factorization $G(\eta) = X^+(\eta)[X^-(\eta)]^{-1}$, $\eta \in L$, L is a piecewise smooth contour, by means of the Cauchy integral $X(\eta) = e^{\chi(\eta)}$, $\eta \in \mathbb{C}$, and the theory of functions of matrices. He also derived the general representation of a 2×2 matrix, which satisfies the condition (1.2). It has the form

$$G(\eta) = \varphi_1(\eta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi_2(\eta)\begin{pmatrix} 0 & \lambda(\eta) \\ \mu(\eta) & \nu \end{pmatrix}, \tag{1.3}$$

where $\varphi_1(\eta), \varphi_2(\eta) \in H(L)$ are arbitrary Hölder functions defined in the contour of factorization, ν = const., and $\lambda(\eta)$ and $\mu(\eta)$ are the values in the contour L of arbitrary functions analytic

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everywhere in the complex plane apart from a finite number of poles of multiplicities satisfying certain restrictions, which guarantee that the factors do not have essential singularities.

Khrapkov (**2**) rewrote the Chebotarev matrix (1.3) in the form

$$G(\eta) = \varphi_1(\eta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi_2(\eta) \begin{pmatrix} l(\eta) & m(\eta) \\ n(\eta) & -l(\eta) \end{pmatrix}, \tag{1.4}$$

where $l(\eta)$, $m(\eta)$ and $n(\eta)$ are polynomials. For the case $\deg(l^2 + mn) \leqslant 2$, he derived explicit formulas for the Wiener–Hopf factors. Daniele (**3**) considered the matrix (1.4) in the case $l(\eta) \equiv 0$ when $\deg(mn) = 4$. In this case, the Khrapkov factorization gives rise to an essential singularity at infinity. This singularity was removed in (**3**) by solving a certain nonlinear system in terms of elliptic functions.

There are some other classes of matrices (for example, rational matrices, permutation type matrices and piecewise constant matrices), which admit a closed-form factorization. However, to the author's knowledge, these classes of matrices do not rise in the diffraction theory.

The case $n \geqslant 3$ is less studied. Jones (**4**) described a class of $n \times n$ matrices that admit a closed-form factorization by using functions of matrices. In general, Jones' factors have an inadmissible essential singularity at infinity. By employing the theory of Riemann surfaces of algebraic functions, it is possible to remove such a singularity (**5**). The case of a 3×3 matrix different from the Jones matrices, $G(\eta) = b_1(\eta)Q_1(\eta) + b_2(\eta)Q_2(\eta)$, where $Q_1(\eta)$ and $Q_2(\eta)$ are 3×3 polynomial matrices, $b_1(\eta), b_2(\eta) \in H(L)$, associated with the three-dimensional elastic problem for a semi-infinite interfacial crack, was analyzed in (**6**). A closed-form solution was found by reducing the problem to a scalar 'jump' problem and a vector Riemann–Hilbert problem with a 2×2 Chebotarev–Khrapkov matrix coefficient (1.4) (in this case, the Khrapkov technique does not lead to an essential singularity of the factors at infinity). Recently, the Jones method was modified (**7**) to show that the matrix G from (**6**) could also be factorized in a different way (the actual factorization of the 3×3 matrix coefficient G was not derived).

Canonical problems for wedges, including right-angled wedges, are normally analyzed by using the Maliuzhinets technique (see, for example (**8**)). That tendency of treating every wedge diffraction problem by means of the Sommerfeld integral was changed by Daniele (**9**). He showed how the method of integral transformations and the Wiener–Hopf procedure can be modified and applied to electromagnetic problems for impenetrable wedges. By using this method, he found a closed-form solution of the Maliuzhinets problem for an impedance wedge. For a certain class of diffraction problems, this technique requires (**10**) factorization of $n \times n$ matrices of the form $G(\eta) = b_1(\eta)Q_1(\eta) + b_2(\eta)Q_2(\eta)$, where $b_j(\eta) \in H(L)$, Q_1 and Q_2 are $n \times n$ polynomial matrices and $\det Q_1 \neq 0$. To factorize the matrix, it was proposed (**10**) to use the theory of functions of matrices which, in general, gives factors with an essential singularity at infinity. A system of transcendental equations needs to be analyzed and solved to make them bounded.

Our goal in this paper is to study a class of $n \times n$ matrices arising in diffraction theory for penetrable right-angled wedges reducible to the factorization of m $(0 \leqslant m \leqslant n)$ scalar functions and $\frac{1}{2}(n - m)$ 2×2 matrices. Put more succinctly, we aim to determine a class of matrices $G = b_1(\eta)Q_1 + b_2(\eta)Q_2(\eta)$, which admit the representation $G = T_1 \Gamma T_2$, where T_1 and T_2 are $n \times n$ rational matrices and $\Gamma = \text{diag}\{a_1, \dots, a_m, \mathcal{G}_1, \dots, \mathcal{G}_{(n-m)/2}\}$ is a block diagonal $n \times n$ matrix. Here, a_j are scalar functions, and $\mathcal{G}_s$ are 2×2 matrices that may be factorized in closed form by solving associated Riemann–Hilbert problems on genus h_s-Riemann surfaces (the genera h_s depend on the properties of the matrices $\mathcal{G}_s$). Specifically, we shall be concentrating on the solution of the

Riemann–Hilbert problem $\mathbf{\Phi}^+(\eta) = G(\eta)\mathbf{\Phi}^-(\eta) + \mathbf{g}(\eta)$, $\eta \in L$, L is the real line, subject to the symmetry condition $\mathbf{\Phi}^\pm(\eta) = \mathbf{\Phi}^\mp(-\eta)$, $\eta \in \mathcal{D}^\pm$ ($\mathcal{D}^+$ and $\mathcal{D}^-$ are the upper and lower half-planes, respectively), when

$$G(\eta) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & ar_{14} \\ r_{21} & r_{22} & r_{23} & ar_{24} \\ r_{31} & r_{32} & r_{33} & ar_{34} \\ ar_{41} & ar_{42} & ar_{43} & r_{44} \end{pmatrix}, \tag{1.5}$$

r_{js} are rational functions, and $a \in H(L)$ is a Hölder continuous function. Our main motivation is to solve the electromagnetic diffraction problem for two orthogonal screens. The first one is an electrically resistive (ER) half-plane, and the second one is a perfectly magnetically conductive (PMC) half-plane. The component E_z of the electric screen is continuous, and the component H_x of the magnetic field is discontinuous through the former screen, and its jump is proportional to the component E_z. The component H_y of the magnetic field vanishes on both sides of the second half-plane. In the particular case, $\gamma_* = 2/\sqrt{3}$ $\left(\gamma_* = \frac{1}{2}Z/R,\ Z\right.$ is the intrinsic impedance of the medium, R is the surface resistivity of the ER screen$\left.\right)$, the problem was solved in (**11**) by using the Sommerfeld integral representation and the theory of the second-order difference equations with periodic coefficients (**12**, **13**). In the case, $\gamma_* = 2/\sqrt{3}$, the difference equation reduces to two separate scalar Riemann–Hilbert problems for a segment in a plane. In the general case, when $\gamma_* \neq 2/\sqrt{3}$, the method (**13**) designed for the coupled case cannot directly be applied and should somehow be modified. The main difficulty is caused by the presence of zeros and poles on the branch cuts of the eigenvalues of the matrix coefficient of the Riemann–Hilbert problem. We were unable to overcome that difficulty and find a closed-form solution by using the Sommerfeld integral representation and the method (**13**). Note that an approximate numerical solution based on the Sommerfeld integral can be derived by the method of integral equations (**14** to **16**). In this paper, we propose a different approach that is simpler and does not employ the Sommerfeld integral representation. Instead, it uses the Laplace transformation and the Daniele method (**9**) for formulation of the diffraction problem by means of a Riemann–Hilbert problem for two order-4 vectors. We show how this problem can be reduced to two scalar 'jump' problems and one vector Riemann–Hilbert problem for two pairs of analytic functions subject to a certain constraint. The solution of this vector problem is found by solving a Riemann–Hilbert problem for two analytic scalar functions on a genus-1 Riemann surface. The key step of the procedure is to derive the factor matrices $X^+(\eta)$ and $X^-(\eta)$ and an arbitrary rational vectors in the general representation of the solution, $\boldsymbol{\psi}^+(\eta)$ and $\boldsymbol{\psi}^-(\eta)$, of the vector Riemann–Hilbert problem in the form, which meets the symmetry condition $\boldsymbol{\psi}^+(\eta) = 2\gamma^{-1}(\gamma + \eta)\mathrm{diag}\{-1, 1\}\boldsymbol{\psi}^-(-\eta)$, $\eta \in \mathcal{D}^+$, $\gamma = k\gamma_*$, and k is the wave number. The exact solution to the Riemann–Hilbert problem enables us to derive an integral representation of the E_z-component of the electric field and determine the diffraction coefficient.

2. A Riemann–Hilbert problem for two order-4 vectors with a constraint

Let L be the real axis in a complex η-plane, $a(\eta)$, $\mathbf{g}(\eta) \in H(L)$ be a prescribed Hölder continuous function and an order-4 vector, respectively, and $G(\eta)$ be a 4×4 matrix of the form

$$G(\eta) = R_1(\eta) + a(\eta)R_2(\eta), \quad \eta \in L, \tag{2.1}$$

where

$$R_1(\eta) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & r_{44} \end{pmatrix}, \quad R_2(\eta) = \begin{pmatrix} 0 & 0 & 0 & r_{14} \\ 0 & 0 & 0 & r_{24} \\ 0 & 0 & 0 & r_{34} \\ r_{41} & r_{42} & r_{43} & 0 \end{pmatrix}. \tag{2.2}$$

The elements of the matrices $R_1(\eta)$ and $R_2(\eta)$ are given rational functions, and $\det R_1(\eta) \neq 0$, $\eta \in L$. In addition, we assume that

$$G(\eta)G(-\eta) = I_4, \quad G(\eta)\mathbf{g}(-\eta) + \mathbf{g}(\eta) = 0, \quad \eta \in L, \tag{2.3}$$

where $I_4 = \operatorname{diag}\{1, 1, 1, 1\}$.

It is required to find two order-4 vectors, $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$, analytic in the upper and lower half-planes $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, continuous up to the contour L, satisfying the boundary condition

$$\mathbf{\Phi}^+(\eta) = G(\eta)\mathbf{\Phi}^-(\eta) + \mathbf{g}(\eta), \quad \eta \in L, \tag{2.4}$$

subject to the constraint

$$\mathbf{\Phi}^\pm(\eta) = \mathbf{\Phi}^\mp(-\eta), \quad \eta \in \mathcal{D}^\pm. \tag{2.5}$$

The vectors $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$ may have prescribed poles in $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, and the components $\Phi_j^\pm(\eta)$ $(j = 1, 2, 3, 4)$ of the vectors $\mathbf{\Phi}^\pm(\eta)$ may grow at infinity as polynomials, $\Phi_j^\pm(\eta) \sim c_j \eta^{\kappa_j}$, $\eta \to \infty$, and the numbers κ_j are prescribed.

Note that because of the assumptions (2.3), the necessary conditions for the existence of the solution that meets the symmetry condition (2.5) are satisfied. In what follows, we show that the vector problem (2.4), (2.5) with the 4×4 matrix coefficient (2.1), (2.2) is equivalent to three Riemann–Hilbert problems. The first two are scalar problems, and the third one is an order-2 vector Riemann–Hilbert problem. The 2×2 matrix coefficient of the latter problem admits a closed-form factorization.

It is an easy matter to reduce the factorization of the matrix $G(\eta)$ to factorization of a matrix $I_4 + a(\eta)B(\eta)$,

$$R_1^{-1}(\eta)\mathbf{\Phi}^+(\eta) = [I_4 + a(\eta)B(\eta)]\mathbf{\Phi}^-(\eta) + R_1^{-1}(\eta)\mathbf{g}(\eta), \quad \eta \in L, \tag{2.6}$$

where

$$B(\eta) = \begin{pmatrix} 0 & 0 & 0 & \beta_{14} \\ 0 & 0 & 0 & \beta_{24} \\ 0 & 0 & 0 & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & 0 \end{pmatrix},$$

$$\beta_{14} = \frac{1}{\delta}[r_{34}(r_{13}r_{22} - r_{12}r_{23}) - r_{24}(r_{13}r_{32} - r_{12}r_{33}) + r_{14}(r_{23}r_{32} - r_{22}r_{33})],$$

$$\beta_{24} = \frac{1}{\delta}[r_{34}(r_{13}r_{21} - r_{11}r_{23}) - r_{24}(r_{13}r_{31} - r_{11}r_{33}) + r_{14}(r_{23}r_{31} - r_{21}r_{33})],$$

$$\beta_{34} = \frac{1}{\delta}[r_{34}(r_{12}r_{21} - r_{11}r_{22}) - r_{24}(r_{12}r_{31} - r_{11}r_{32}) + r_{14}(r_{22}r_{31} - r_{21}r_{32})],$$

$$\beta_{41} = \frac{r_{41}}{r_{44}}, \quad \beta_{42} = \frac{r_{42}}{r_{44}}, \quad \beta_{43} = \frac{r_{43}}{r_{44}},$$

$$\delta = \det \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}. \tag{2.7}$$

To factorize the new matrix $I_4 + a(\eta)B(\eta)$, we find the characteristic polynomial of the matrix $B(\eta)$

$$\det[\mu I_4 - B(\eta)] = \mu^2 \phi(\mu), \tag{2.8}$$

where

$$\phi(\mu) = \mu^2 - \beta, \quad \beta = \beta_{14}\beta_{41} + \beta_{24}\beta_{42} + \beta_{34}\beta_{43}. \tag{2.9}$$

Because of the structure of the characteristic polynomial, it is possible to determine a matrix of transformation, $T(\eta)$, such that

$$I_4 + a(\eta)B(\eta) = T(\eta)\Gamma(\eta)T^{-1}(\eta), \tag{2.10}$$

where the matrix $\Gamma(\eta)$ has the form

$$\Gamma(\eta) = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & \mathcal{G}(\eta) \end{pmatrix}, \tag{2.11}$$

where I_2, 0_2 and $\mathcal{G}(\eta)$ are 2×2 matrices, I_2 is the unit matrix and 0_2 is the zero matrix. The first two columns of the matrix $T(\eta)$ are the eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$, of the matrix $B(\eta)$ corresponding to the multiplicity-2 eigenvalue $\mu = 0$,

$$\mathbf{x}_1 = \begin{pmatrix} \beta_{42} \\ -\beta_{41} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \beta_{43} \\ 0 \\ -\beta_{41} \\ 0 \end{pmatrix}. \tag{2.12}$$

The other two columns can be reconstructed by determining the kernel of the matrix operator

$$\phi(B) = B^2 - \beta I_4 = \begin{pmatrix} -\beta_{24}\beta_{42} - \beta_{34}\beta_{43} & \beta_{14}\beta_{42} & \beta_{14}\beta_{43} & 0 \\ \beta_{24}\beta_{41} & -\beta_{14}\beta_{41} - \beta_{34}\beta_{43} & \beta_{24}\beta_{43} & 0 \\ \beta_{34}\beta_{41} & \beta_{34}\beta_{42} & -\beta_{14}\beta_{41} - \beta_{24}\beta_{42} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

This is a rank-2 4×4 matrix, and a basis of the kernel of the operator $\phi(B)$ can be chosen as follows:

$$\mathbf{x}_3 = \begin{pmatrix} \beta_{14} \\ \beta_{24} \\ \beta_{34} \\ 0 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.14}$$

We then write the transformation matrix T and its inverse

$$T(\eta) = \begin{pmatrix} \beta_{42} & \beta_{43} & \beta_{14} & 0 \\ -\beta_{41} & 0 & \beta_{24} & 0 \\ 0 & -\beta_{41} & \beta_{34} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}(\eta) = \frac{1}{\beta\beta_{41}} \begin{pmatrix} \beta_{41}\beta_{24} & -\beta_{14}\beta_{41} - \beta_{34}\beta_{43} & \beta_{24}\beta_{43} & 0 \\ \beta_{41}\beta_{34} & \beta_{34}\beta_{42} & -\beta_{14}\beta_{41} - \beta_{24}\beta_{42} & 0 \\ \beta_{41}^2 & \beta_{41}\beta_{42} & \beta_{41}\beta_{43} & 0 \\ 0 & 0 & 0 & \beta\beta_{41} \end{pmatrix}. \quad (2.15)$$

By using formulas (2.7), (2.10) and (2.15), it is a matter of simple algebra to determine the matrix $\Gamma(\eta)$

$$\Gamma(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a(\eta) \\ 0 & 0 & a(\eta)\beta(\eta) & 1 \end{pmatrix}. \quad (2.16)$$

The function $\beta(\eta)$ defined by (2.9) and (2.7) admits the representation $\beta(\eta) = p_1(\eta)/p_2(\eta)$, where $p_1(\eta)$ and $p_2(\eta)$ are polynomials, and $p_1(\eta)/p_2(\eta)$ is a irreducible fraction. The factorization of the matrix

$$\mathcal{G}(\eta) = \begin{pmatrix} 1 & a(\eta) \\ a(\eta)\beta(\eta) & 1 \end{pmatrix} \quad (2.17)$$

can be found by solving a Riemann–Hilbert problem on a hyperelliptic surface of the algebraic function $w^2 = p_1(\eta)p_2(\eta)$. The genus of the surface, h, is determined by the formula $h = \mathrm{int}\left(\frac{\kappa-1}{2}\right)$, $\kappa = \deg p_1(\eta) + \deg p_2(\eta)$, and $\mathrm{int}(b)$ is the integer part of a number b. We emphasize that the method (**5**, **17**, **18**) for the solution of the Riemann–Hilbert problem on a Riemann surface needs to be modified because of the symmetry condition (2.5). To analyze this constraint, we replace the matrix $I_4 + aB$ by the matrix $T\Gamma T^{-1}$ and obtain from (2.6)

$$\mathbf{\Omega}^+(\eta) = \Gamma(\eta)\mathbf{\Omega}^-(\eta) + \mathbf{h}(\eta), \quad \eta \in L, \quad (2.18)$$

where

$$\mathbf{\Omega}^+(\eta) = T^{-1}(\eta)R_1^{-1}(\eta)\mathbf{\Phi}^+(\eta), \quad \mathbf{\Omega}^-(\eta) = T^{-1}(\eta)\mathbf{\Phi}^-(\eta),$$
$$\mathbf{h}(\eta) = T^{-1}(\eta)R_1^{-1}(\eta)\mathbf{g}(\eta). \quad (2.19)$$

In comparison to (2.5), the new unknown vectors, $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$, satisfy a more complicated constraint, namely,

$$\mathbf{\Omega}^\pm(\eta) = [T(\eta)]^{-1}[R_1(\pm\eta)]^{\mp 1}T(-\eta)\mathbf{\Omega}^\mp(-\eta), \quad \eta \in \mathcal{D}^\pm. \quad (2.20)$$

In section 3, we shall use the method outlined for the solution of a model problem of diffraction theory.

3. Diffraction of a plane wave by an ER–PMC right-angled screen: the formulation

Consider a planar structure consisting of two orthogonal half-planes, an ER screen $W_1 = \{(x, y)\colon 0 < x < +\infty, y = \pm 0\}$ and a PMC screen $W_2 = \{(x, y)\colon x = \pm 0, 0 < y < +\infty\}$. The structure is illuminated by a line electric current of uniform excitation, taken as unity (the factor $e^{-i\omega t}$, ω is the angular frequency, is suppressed)

$$s(x, y) = \delta(x - a)\delta(y - b), \tag{3.1}$$

where (a, b) is a point lying in one of the following three domains: $S_1 = \{(x, y)\colon 0 < x < +\infty, 0 < y < +\infty\}$, $S_2 = \{(x, y)\colon 0 < x < +\infty, -\infty < y < 0\}$ or $S_3 = \{(x, y)\colon -\infty < x < 0, -\infty < y < +\infty\}$. Let $k = \omega\sqrt{\varepsilon\mu}$ and $Z = \sqrt{\mu/\varepsilon}$ be the wave number and the intrinsic impedance of the medium, respectively, where μ, the magnetic permeability, and ε, the electric permittivity, are complex. The parameters and the branches $\sqrt{\varepsilon\mu}$ and $\sqrt{\mu/\varepsilon}$ are chosen such that $k = k_1 + ik_2$, $k_2 > 0$, $kZ = \omega\mu$ and $k/Z = \omega\varepsilon$. The Maxwell equations read

$$E_x = -\frac{Z}{ik}\frac{\partial H_z}{\partial y}, \quad E_y = \frac{Z}{ik}\frac{\partial H_z}{\partial x}, \quad E_z = -\frac{Z}{ik}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right),$$
$$H_x = \frac{1}{ikZ}\frac{\partial E_z}{\partial y}, \quad H_y = -\frac{1}{ikZ}\frac{\partial E_z}{\partial x}, \quad H_z = \frac{1}{ikZ}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right), \tag{3.2}$$

and the component E_z of the electric field satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) E_z = \delta(x - a)\delta(y - b), \quad (x, y) \in S_1 \cup S_2 \cup S_3. \tag{3.3}$$

Here, $\delta(z)$ is the Dirac's delta function. In the ER screen W_1, the E_z-component of the electric field is continuous and proportional to the jump of the H_x-component of the magnetic field,

$$E_z|_{y=0^\pm} = -R[H_x|_{y=0^+} - H_x|_{y=0^-}], \quad 0 < x < +\infty, \tag{3.4}$$

where R is the surface resistivity. In the PMC screen W_2, the H_y-component is continuous and vanishes,

$$H_y|_{x=0^\pm} = 0, \quad 0 < y < +\infty. \tag{3.5}$$

In the rest of this section, we aim to convert (3.2) and (3.3) and the boundary conditions (3.4) and (3.5) into a Riemann–Hilbert problem for two order-4 vectors. Assume first that $x > 0$ and $y > 0$ ($(x, y) \in S_1$). By following (**9**), we introduce the Laplace transforms with respect to y

$$\begin{pmatrix} \tilde{\mathbf{E}}_+ \\ \tilde{\mathbf{H}}_+ \end{pmatrix}(x, \eta) = \int_0^\infty e^{i\eta y}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y)dy \tag{3.6}$$

and apply the Laplace transform to the Helmholtz equation (3.3). This enables us to write

$$\tilde{E}_{z+}(x, \eta) = A(\eta)e^{-\xi x} - \frac{1}{2\xi}\int_0^\infty q(x_1)e^{-\xi|x-x_1|}dx_1 - \frac{\kappa_1}{2\xi}e^{-\xi|x-a_1|+i\eta b_1}, \quad \eta \in L, \tag{3.7}$$

where L is the real axis, $(a_1, b_1) = (a, b)$, $\kappa_1 = 1$ if $(a, b) \in S_1$, and $\kappa_1 = 0$ otherwise. The function $\xi(\eta) = (\eta^2 - k^2)^{1/2}$ is a branch of the two-valued function $\xi^2 = \eta^2 - k^2$ defined in the

η-plane cut along a line joining the branch points $\eta = k$ and $\eta = -k$ and passing through the infinite point. It will be convenient in future to choose the cut as the polygonal line that passes through the points $\pm\gamma = kZ/(2R)$ (without loss of generality, we assume $\mathrm{Im}\,\gamma > 0$). The branch is fixed by the condition $\xi(0) = -ik$. This choice guarantees that $\mathrm{Re}\,\xi > 0$ for all $\eta \in L$ if $k_2 > 0$. In the case $k_2 = 0^+$, $\xi = |\xi|$ for $\eta \in (-\infty, -k) \cup (k, +\infty)$, and $\xi = -i|\xi|$ for $-k < \eta < k$. The function $q(x)$ is given by

$$q(x) = ikZH_x(x, 0^+) - i\eta E_z(x, 0^+), \tag{3.8}$$

and the function $A(\eta)$ is arbitrary. By using the relations (3.2) and putting $x = 0^+$, we obtain from (3.7)

$$\begin{aligned} \tilde{E}_{z+}(0^+, \eta) &= A(\eta) - \frac{1}{2\xi}\int_0^\infty q(x_1)e^{-\xi x_1}dx_1 - \frac{\kappa_1}{2\xi}e^{-\xi a_1 + i\eta b_1}, \\ ikZ\tilde{H}_{y+}(0^+, \eta) &= \xi A(\eta) + \frac{1}{2}\int_0^\infty q(x_1)e^{-\xi x_1}dx_1 + \frac{\kappa_1}{2}e^{-\xi a_1 + i\eta b_1}. \end{aligned} \tag{3.9}$$

The integral term in formulas (3.9) can be expressed through the Laplace transforms with respect to x

$$\begin{pmatrix} \hat{\mathbf{E}}_+ \\ \hat{\mathbf{H}}_+ \end{pmatrix}(i\xi, y) = \int_0^\infty e^{-\xi x}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y)dx \tag{3.10}$$

as follows:

$$\int_0^\infty q(x_1)e^{-\xi x_1}dx_1 = ikZ\hat{H}_{x+}(i\xi, 0^+) - i\eta\hat{E}_{z+}(i\xi, 0^+). \tag{3.11}$$

By using the two relations (3.9), we exclude the function $A(\eta)$ and obtain

$$\begin{aligned} kZ\tilde{H}_{y+}(0^+, \eta) + i\xi\tilde{E}_{z+}(0^+, \eta) &= kZ\hat{H}_{x+}(i\xi, 0^+) - \eta\hat{E}_{z+}(i\xi, 0^+) - i\kappa_1 e^{-\xi a_1 + i\eta b_1}, \\ kZ\tilde{H}_{y+}(0^+, -\eta) + i\xi\tilde{E}_{z+}(0^+, -\eta) &= kZ\hat{H}_{x+}(i\xi, 0^+) + \eta\hat{E}_{z+}(i\xi, 0^+) - i\kappa_1 e^{-\xi a_1 - i\eta b_1}. \end{aligned} \tag{3.12}$$

The second equation is obtained from the first one by replacing η by $-\eta$. Introduce now the following Laplace transforms:

$$\begin{pmatrix} \tilde{\mathbf{E}}_- \\ \tilde{\mathbf{H}}_- \end{pmatrix}(x, \eta) = \int_{-\infty}^0 e^{i\eta x}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y)dy \tag{3.13}$$

and consider the cases when $x > 0$, $y < 0$ ($(x, y) \in S_2$) and $x < 0$, $-\infty < y < +\infty$ ($(x, y) \in S_3$). In a similar manner, we obtain

$$\begin{aligned} kZ\tilde{H}_{y-}(0^+, \eta) + i\xi\tilde{E}_{z-}(0^+, \eta) &= -kZ\hat{H}_{x+}(i\xi, 0^-) + \eta\hat{E}_{z+}(i\xi, 0^-) - i\kappa_2 e^{-\xi a_2 + i\eta b_2}, \\ kZ\tilde{H}_{y-}(0^+, -\eta) + i\xi\tilde{E}_{z-}(0^+, -\eta) &= -kZ\hat{H}_{x+}(i\xi, 0^-) - \eta\hat{E}_{z+}(i\xi, 0^-) - i\kappa_2 e^{-\xi a_2 - i\eta b_2}, \end{aligned} \tag{3.14}$$

and

$$-ikZ[\tilde{H}_{y+}(0^-,\eta)+\tilde{H}_{y-}(0^-,\eta)=\xi\tilde{E}_{z+}(0^-,\eta)+\xi\tilde{E}_{z-}(0^-,\eta)+\kappa_3 e^{\xi a_3+i\eta b_3},$$
$$-ikZ[\tilde{H}_{y+}(0^-,-\eta)+\tilde{H}_{y-}(0^-,-\eta)=\xi\tilde{E}_{z+}(0^-,-\eta)+\xi\tilde{E}_{z-}(0^-,-\eta)+\kappa_3 e^{\xi a_3-i\eta b_3}. \tag{3.15}$$

Here, $(a_j,b_j)=(a,b)$ and $\kappa_j=1$ if $(a,b)\in S_j$, and $\kappa_j=0$ when $(a,b)\in S_m$, $m\neq j$ ($j=2,3$, $m=1,2,3$). We emphasize that (3.12), (3.14) and (3.15) are deduced from the Maxwell equations alone. By employing now the boundary conditions (3.5), we conclude that

$$\tilde{H}_{y+}(0^+,\eta)=\tilde{H}_{y+}(0^-,\eta)=0. \tag{3.16}$$

The use of the boundary conditions (3.4) or, equivalently,

$$\hat{E}_{z+}(i\xi,0^+)=\hat{E}_{z+}(i\xi,0^-)=-R[\hat{H}_{x+}(i\xi,0^+)-\hat{H}_{x+}(i\xi,0^-)], \tag{3.17}$$

and the relations (3.13) makes it possible to exclude the four terms $\hat{E}_{z+}(i\xi,0^\pm)$ and $\hat{H}_{x+}(i\xi,0^\pm)$,

$$\begin{aligned}
\hat{E}_{z+}(i\xi,0^\pm) &= -\frac{i\xi}{2\eta}[\tilde{E}_{z+}(0^+,\eta)-\tilde{E}_{z+}(0^+,-\eta)]+\frac{\kappa_1}{\eta}e^{-\xi a_1}\sin\eta b_1,\\
\hat{H}_{x+}(i\xi,0^+) &= \frac{i\xi}{2kZ}[\tilde{E}_{z+}(0^+,\eta)+\tilde{E}_{z+}(0^+,-\eta)]+\frac{i\kappa_1}{kZ}e^{-\xi a_1}\cos\eta b_1,\\
\hat{H}_{x+}(i\xi,0^-) &= \frac{i\xi}{2kZ\eta R}[(kZ+\eta R)\tilde{E}_{z+}(0^+,-\eta)-(kZ-\eta R)\tilde{E}_{z+}(0^+,\eta)]\\
&\quad+\frac{\kappa_1 e^{-\xi a_1}}{kZ\eta R}(kZ\sin\eta b_1+i\eta R\cos\eta b_1).
\end{aligned} \tag{3.18}$$

By utilizing these relations and the following notations:

$$\Phi_1^+(\eta)=\tilde{E}_{z+}(0^+,\eta),\quad \Phi_2^+(\eta)=\tilde{E}_{z+}(0^-,\eta),$$
$$\Phi_3^+(\eta)=\tilde{E}_{z-}(0^+,-\eta),\quad \Phi_4^+(\eta)=-ikZ\tilde{H}_{y-}(0^+,-\eta) \tag{3.19}$$

and

$$\mathbf{\Phi}^-(\eta)=\mathbf{\Phi}^+(-\eta),\quad \eta\in\mathcal{D}^-, \tag{3.20}$$

we derive from (3.14) and (3.15), the following four boundary conditions for the components of the vectors $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$:

$$\begin{aligned}
-\frac{\gamma}{\eta}\Phi_1^-(\eta)+\frac{\xi}{\eta}(\gamma-\eta)\Phi_1^+(\eta)-\xi\Phi_3^-(\eta)-\Phi_4^-(\eta) &= \kappa_1^\circ(\eta)+\kappa_2^\circ(\eta),\\
\frac{\gamma}{\eta}\Phi_1^+(\eta)-\frac{\xi}{\eta}(\gamma+\eta)\Phi_1^-(\eta)-\xi\Phi_3^+(\eta)-\Phi_4^+(\eta) &= \kappa_1^\circ(-\eta)+\kappa_2^\circ(-\eta),\\
\xi\Phi_2^+(\eta)+\xi\Phi_3^-(\eta)-\Phi_4^-(\eta) &= \kappa_3^\circ(\eta),\\
\xi\Phi_2^-(\eta)+\xi\Phi_3^+(\eta)-\Phi_4^+(\eta) &= \kappa_3^\circ(-\eta),\quad \eta\in L,
\end{aligned} \tag{3.21}$$

where

$$\kappa_1^\circ(\eta) = \kappa_1 e^{-\xi a_1}\left[\frac{\eta-\gamma}{\eta}e^{i\eta b_1}+\frac{\gamma}{\eta}e^{-i\eta b_1}\right],$$
$$\kappa_2^\circ(\eta) = \kappa_2 e^{-\xi a_2+i\eta b_2},$$
$$\kappa_3^\circ(\eta) = -\kappa_3 e^{\xi a_3+i\eta b_3}. \tag{3.22}$$

Simple manipulations reduce these relations to the Riemann–Hilbert problem for two vectors, $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$, analytic in the upper and lower half-planes, $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, continuous up to the boundary L, whose limit values satisfy the boundary condition

$$\mathbf{\Phi}^+(\eta) = G(\eta)\mathbf{\Phi}^-(\eta)+\mathbf{g}(\eta), \quad \eta\in L, \tag{3.23}$$

subject to the symmetry condition (2.5). Here,

$$G(\eta)=\frac{1}{2(\gamma-\eta)}\begin{pmatrix}2\gamma & 0 & 2\eta & 2\eta/\xi\\ 0 & 0 & -2(\gamma-\eta) & 2(\gamma-\eta)/\xi\\ \eta & -(\gamma-\eta) & \gamma & \gamma/\xi\\ \eta\xi & (\gamma-\eta)\xi & \gamma\xi & \gamma\end{pmatrix}, \quad \det G(\eta)=\frac{\eta+\gamma}{\eta-\gamma},$$

$$\mathbf{g}(\eta)=\frac{1}{2(\gamma-\eta)\xi}\begin{pmatrix}2\eta[\kappa_1^\circ(\eta)+\kappa_2^\circ(\eta)]\\ 2(\gamma-\eta)\kappa_3^\circ(\eta)\\ \gamma[\kappa_1^\circ(\eta)+\kappa_2^\circ(\eta)]-(\gamma-\eta)[\kappa_1^\circ(-\eta)+\kappa_2^\circ(-\eta)-\kappa_3^\circ(-\eta)]\\ \gamma\xi[\kappa_1^\circ(\eta)+\kappa_2^\circ(\eta)]-(\gamma-\eta)\xi[\kappa_1^\circ(-\eta)+\kappa_2^\circ(-\eta)+\kappa_3^\circ(-\eta)]\end{pmatrix}. \tag{3.24}$$

Since the electric field component E_z and magnetic field component H_y are bounded as $x=0^\pm$ and $y\to 0$, by the Abelian theorem for the Laplace transform, the vectors $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$ vanish at the infinite point: $|\mathbf{\Phi}^\pm(\eta)|\leqslant c|\eta|^{-1}$ as $\eta\to\infty$ ($c=$ const.).

The matrix coefficient $G(\eta)$ has the structure (2.1), (2.2), the necessary conditions of solvability (2.3) are satisfied, and the method of section 2 can be applied.

4. Solution of the Riemann–Hilbert problem associated with the diffraction problem

4.1 *Reduction to a Riemann–Hilbert problem for order-2 vectors*

By following the procedure of section 2, we can split the matrix $G(\eta)$ as follows:

$$G(\eta)=R(\eta)\left[I_4+\frac{1}{\xi}B(\eta)\right]=R(\eta)T(\eta)\Gamma(\eta)T^{-1}(\eta), \tag{4.1}$$

where

$$R(\eta)=\frac{1}{2(\gamma-\eta)}\begin{pmatrix}2\gamma & 0 & 2\eta & 0\\ 0 & 0 & 2(\eta-\gamma) & 0\\ \eta & \eta-\gamma & \gamma & 0\\ 0 & 0 & 0 & \gamma\end{pmatrix},$$

$$T(\eta) = \frac{1}{\gamma}\begin{pmatrix} (\gamma-\eta)\xi^2 & \gamma\xi^2 & 2\eta & 0 \\ -\eta\xi^2 & 0 & -2(\gamma+\eta) & 0 \\ 0 & -\eta\xi^2 & -\gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix},$$

$$\Gamma(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/\xi \\ 0 & 0 & 4\zeta^{-2}\xi/\gamma^2 & 1 \end{pmatrix}, \quad \zeta^2 = \eta^2 - \gamma_0^2, \quad \gamma_0 = \frac{\sqrt{3}\gamma}{2}. \tag{4.2}$$

Substitution of the splitting (4.1) into (3.23) gives a new Riemann–Hilbert problem with the matrix coefficient $\Gamma(\eta)$

$$\mathbf{\Omega}^+(\eta) = \Gamma(\eta)\mathbf{\Omega}^-(\eta) + \mathbf{h}(\eta), \quad \eta \in L, \tag{4.3}$$

where the new unknown vectors $\mathbf{\Omega}^\pm(\eta) = \{\Omega_j(\eta)\}$ and the vector $\mathbf{h}(\eta) = \{h_j(\eta)\}$ $(j = 1, 2, 3, 4)$ are

$$\mathbf{\Omega}^+(\eta) = U_0(\eta)\mathbf{\Phi}^+(\eta), \quad \mathbf{\Omega}^-(\eta) = U_1(\eta)\mathbf{\Phi}^-(\eta), \quad \mathbf{h}(\eta) = U_0(\eta)\mathbf{g}(\eta), \tag{4.4}$$

and

$$U_0(\eta) = \begin{pmatrix} \gamma & -\gamma(\gamma+\eta)/\eta & 2(\gamma^2-2\eta^2)/\eta & 0 \\ 2(\gamma-\eta) & (\gamma^2-2\eta^2)/\eta & -2\gamma(\gamma-\eta)/\eta & 0 \\ -2(\gamma-\eta)\eta\xi^2/\gamma & 2(\gamma^2-\eta^2)\xi^2/\gamma & 2(\gamma-\eta)\xi^2 & 0 \\ 0 & 0 & 0 & -8(\gamma-\eta)\zeta^2\xi^2/\gamma^2 \end{pmatrix},$$

$$U_1(\eta) = \begin{pmatrix} 2(\gamma+\eta) & -(\gamma^2-2\eta^2)/\eta & 2\gamma(\gamma+\eta)/\eta & 0 \\ \gamma & \gamma(\gamma-\eta)/\eta & -2(\gamma^2-2\eta^2)/\eta & 0 \\ -\eta\xi^2 & -(\gamma-\eta)\xi^2 & -\gamma\xi^2 & 0 \\ 0 & 0 & 0 & -4\zeta^2\xi^2/\gamma \end{pmatrix}. \tag{4.5}$$

The vectors $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$ are meromorphic in the upper and lower half-planes, respectively, continuous up to the real axis L, and their limit values meet the boundary condition (4.3). Because of formulas (3.20) and (4.4), these vectors must satisfy the following symmetry condition:

$$\mathbf{\Omega}^+(\eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2(1-\eta/\gamma) & 0 \\ 0 & 0 & 0 & 2(1-\eta/\gamma) \end{pmatrix}\mathbf{\Omega}^-(-\eta), \quad \eta \in \mathcal{D}^+. \tag{4.6}$$

It will be convenient to introduce two new order-2 vectors, $\boldsymbol{\psi}^+(\eta)$ and $\boldsymbol{\psi}^-(\eta)$, whose components are

$$\psi_j^\pm(\eta) = \frac{\gamma}{2(\gamma-\eta)\xi^2}\Omega_{j+2}^\pm(\eta), \quad j = 1, 2. \tag{4.7}$$

In view of (4.4) and (4.7), we are able to express the components of the original vectors $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$ through the functions $\Omega_j^\pm(\eta)$ and $\psi_j^\pm(\eta)$ $(j = 1, 2)$

$$\begin{aligned}
\Phi_1^+(\eta) &= -\frac{\gamma}{4\zeta^2}\left[\Omega_1^+(\eta) + \left(1+\frac{\eta}{\gamma}\right)\Omega_2^+(\eta) + \frac{2\eta}{\gamma}\psi_1^+(\eta)\right],\\
\Phi_2^+(\eta) &= -\frac{\gamma}{4\zeta^2}\left[\frac{\eta}{\gamma}\Omega_2^+(\eta) + 2\left(1-\frac{\eta}{\gamma}\right)\psi_1^+(\eta)\right],\\
\Phi_3^+(\eta) &= -\frac{\gamma}{4\zeta^2}\left[\frac{\eta}{\gamma}\Omega_1^+(\eta) + \psi_1^+(\eta)\right],\\
\Phi_4^+(\eta) &= -\frac{\gamma}{4\zeta^2}\psi_2^+(\eta),\\
\Phi_1^-(\eta) &= -\frac{\gamma}{4\zeta^2}\left[\left(1-\frac{\eta}{\gamma}\right)\Omega_1^-(\eta) + \Omega_2^-(\eta) + \frac{4\eta}{\gamma}\left(1-\frac{\eta}{\gamma}\right)\psi_1^-(\eta)\right],\\
\Phi_2^-(\eta) &= -\frac{\gamma}{4\zeta^2}\left[-\frac{\eta}{\gamma}\Omega_1^-(\eta) - 4\left(1-\frac{\eta^2}{\gamma^2}\right)\psi_1^-(\eta)\right],\\
\Phi_3^-(\eta) &= -\frac{\gamma}{4\zeta^2}\left[-\frac{\eta}{\gamma}\Omega_2^-(\eta) - 2\left(1-\frac{\eta}{\gamma}\right)\psi_1^-(\eta)\right],\\
\Phi_4^-(\eta) &= -\frac{\gamma-\eta}{2\zeta^2}\psi_2^-(\eta).
\end{aligned} \tag{4.8}$$

First, we determine the functions $\Omega_1(\eta)$ and $\Omega_2(\eta)$. Its limiting values $\Omega_j^+(\eta)$ and $\Omega_j^-(\eta)$ satisfy the boundary conditions

$$\Omega_j^+(\eta) - \Omega_j^-(\eta) = h_j(\eta), \quad \eta \in L, \quad j = 1, 2. \tag{4.9}$$

The functions should also meet the symmetry conditions

$$\Omega_1^+(\eta) = \Omega_2^-(-\eta), \quad \Omega_2^+(\eta) = \Omega_1^-(-\eta), \quad \eta \in \mathcal{D}^+. \tag{4.10}$$

We shall later show that the functions $\psi_j^\pm(\eta) \sim \text{const.}\eta^{1/2+n}$ as $\eta \to \infty$ $(j = 1, 2$, and n is an integer). On the other hand, the functions $|\Phi_m^\pm(\eta)| \leqslant c|\eta^{-1}|$ as $\eta \to \infty$ $(m = 1, 2, 3, 4)$. It follows from the relations (4.4), (4.5) and (4.8) that the functions $\Omega_j^+(\eta)$ and $\Omega_j^-(\eta)$ are bounded at infinity, and $\Omega_j^\pm(\eta) = O(\eta^{-1})$ as $\eta \to 0$ $(j = 1, 2)$. Therefore,

$$\begin{aligned}
\Omega_1(\eta) &= \frac{1}{2\pi i}\int_L \frac{h_1(\tau)d\tau}{\tau-\eta} + C_0 + \frac{C_1}{\eta},\\
\Omega_2(\eta) &= \frac{1}{2\pi i}\int_L \frac{h_2(\tau)d\tau}{\tau-\eta} + C_0 - \frac{C_1}{\eta},
\end{aligned} \tag{4.11}$$

where C_0 and C_1 are arbitrary constants. Note that $h_1(\eta) = -h_2(-\eta)$, and that is why the symmetry conditions (4.10) are satisfied.

From (4.8), the functions $\Phi_j^{\pm}(\eta)$ have removable singularities at the point $\eta = 0$ and inadmissible simple poles at the points $\eta = \pm\gamma_0$. To eliminate these poles, we require

$$\begin{aligned}
\Omega_1^+(\gamma_0) &= -\frac{\gamma}{\gamma_0}\psi_1^+(\gamma_0),\\
\Omega_2^+(\gamma_0) &= -2\left(\frac{\gamma}{\gamma_0} - 1\right)\psi_1^+(\gamma_0),\\
\psi_2^+(\gamma_0) &= 0.
\end{aligned} \tag{4.12}$$

Notice that if the conditions (4.12) are satisfied, then

$$\Omega_1^+(\gamma_0) + \frac{\gamma + \gamma_0}{\gamma}\Omega_2^+(\gamma_0) + \frac{2\gamma_0}{\gamma}\psi_1^+(\gamma_0) = 0, \tag{4.13}$$

and the points $\eta = \gamma_0$ and $\eta = -\gamma_0$ are removable singular points of the functions $\Phi_j^+(\eta)$ and $\Phi_j^-(\eta)$ ($j = 1, 2, 3, 4$).

The functions $\boldsymbol{\psi}_1^{\pm}(\eta)$ and $\boldsymbol{\psi}_2^{\pm}(\eta)$ cannot be derived in such a simple way as the functions $\Omega_1^{\pm}(\eta)$ and $\Omega_2^{\pm}(\eta)$. The vectors $\boldsymbol{\psi}^+(\eta)$ and $\boldsymbol{\psi}^-(\eta)$ are analytic everywhere in the domain $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively. They are continuous up to the contour L, and their limit values satisfy the boundary condition

$$\boldsymbol{\psi}^+(\eta) = \mathcal{G}(\eta)\boldsymbol{\psi}^-(\eta) + \mathbf{v}(\eta), \quad \eta \in L, \tag{4.14}$$

the symmetry condition

$$\boldsymbol{\psi}^+(\eta) = \frac{2(\gamma + \eta)}{\gamma}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{\psi}^-(-\eta), \quad \eta \in \mathcal{D}^+, \tag{4.15}$$

and as $\eta \to \infty$,

$$\begin{aligned}
|\psi_1^+(\eta)| \leqslant c_1^+, &\quad |\psi_1^-(\eta)| \leqslant c_1^-|\eta|^{-1},\\
|\psi_2^+(\eta)| \leqslant c_2^+|\eta|, &\quad |\psi_2^-(\eta)| \leqslant c_2^-,
\end{aligned} \tag{4.16}$$

where $c_1^{\pm}$ and $c_2^{\pm}$ are constants. Here,

$$\begin{aligned}
\mathcal{G}(\eta) &= \begin{pmatrix} 1 & 1/\xi \\ 4\zeta^2\xi/\gamma^2 & 1 \end{pmatrix},\\
v_1(\eta) &= -\eta g_1(\eta) + (\gamma + \eta)g_2(\eta) + \gamma g_3(\eta),\\
v_2(\eta) &= -\frac{4\zeta^2}{\gamma}g_4(\eta).
\end{aligned} \tag{4.17}$$

Because of the symmetry condition (4.15) that can be written in the form $\boldsymbol{\psi}^+(\eta) = N(\eta)\boldsymbol{\psi}^-(-\eta)$, where $N(\eta) = 2(1 + \eta/\gamma)\mathrm{diag}\{-1, 1\}$, the matrix coefficient $\mathcal{G}(\eta)$ and the vector $\mathbf{v}(\eta)$ must satisfy the following necessary solvability conditions of the Riemann–Hilbert problem (4.14), (4.15):

$$N(\eta)[\mathcal{G}(-\eta)]^{-1}N(-\eta) = \mathcal{G}(\eta), \quad -N(\eta)[\mathcal{G}(-\eta)]^{-1}\mathbf{v}(-\eta) = \mathbf{v}(\eta). \tag{4.18}$$

It is a matter of simple algebra to verify that these conditions are met identically.

4.2 *Factorization of the matrix $\mathcal{G}(\eta)$*

It is evident that the matrix coefficient $\mathcal{G}(\eta)$ in (4.17) of the Riemann–Hilbert problem (4.14) has the Chebotarev–Khrapkov structure (**1**, **2**). It will be convenient to split the matrix as follows:

$$\mathcal{G}(\eta) = \frac{2\zeta}{\gamma i}\mathcal{G}_0(\eta), \tag{4.19}$$

and factorize the matrix $\mathcal{G}_0(\eta)$ defined by

$$\mathcal{G}_0(\eta) = b(\eta)I_2 + c(\eta)Q(\eta), \quad b(\eta) = \frac{\gamma i}{2\zeta}, \quad c(\eta) = \frac{i}{\zeta\xi},$$

$$Q(\eta) = \begin{pmatrix} l(\eta) & m_+(\eta) \\ m_-(\eta) & -l(\eta) \end{pmatrix}, \quad l(\eta) = 0, \quad m_+(\eta) = \frac{\gamma}{2}, \quad m_-(\eta) = \frac{2}{\gamma}\zeta^2\xi^2. \tag{4.20}$$

Here, $\zeta = (\eta^2 - \gamma_0^2)^{1/2}$ is the single branch of the two-valued function $\zeta^2 = \eta^2 - \gamma_0^2$ fixed by the condition $\zeta(0) = i\gamma_0$ in the η-plane cut along the straight line joining the branch points γ_0 and $-\gamma_0$ and passing through the infinite point. Let $f(\eta) = l^2 + m_+m_- = (\eta^2 - k^2)(\eta^2 - \gamma_0^2)$. Since $\deg f(\eta) = 4$, the factorization problem

$$\mathcal{G}_0(\eta) = X_0^+(\eta)[X_0^-(\eta)]^{-1}, \quad \eta \in L, \tag{4.21}$$

reduces to a scalar Riemann–Hilbert problem on a genus-1 Riemann surface (**17**). The solution to this problem needs to be constructed in a form that makes it possible to satisfy the symmetry condition (4.15).

First we fix a branch of the algebraic function $w^2 = f(\eta)$ in the extended η-plane cut along the two segments $l_+ = [k, \gamma_0] \subset \mathcal{D}^+$ and $l_- = [-k, -\gamma_0] \subset \mathcal{D}^-$ by enforcing the condition $f^{1/2}(\eta) \sim -\eta^2$, $\eta \to \infty$. This branch satisfies the relation $f^{1/2}(\eta) = \xi\zeta$. Let $\mathcal{R}$ be the elliptic surface generated by the algebraic function $w^2 = f(\eta)$ and formed by gluing two copies, $\mathcal{D}_1$ and $\mathcal{D}_2$, of the extended η-plane $\mathbb{C} \cup \{\infty\}$ cut along the segments l_+ and l_- according to the rule

$$w = \begin{cases} \sqrt{f(\eta)}, & \eta \in \mathcal{D}_1, \\ -\sqrt{f(\eta)}, & \eta \in \mathcal{D}_2. \end{cases} \tag{4.22}$$

Determine next the eigenvalues of the matrix $\mathcal{G}_0(\eta)$

$$\lambda_1(\eta) = \frac{\gamma i}{2\zeta} + i, \quad \lambda_2(\eta) = \frac{\gamma i}{2\zeta} - i, \tag{4.23}$$

and, on the surface $\mathcal{R}$, define the function

$$\lambda(\eta, w) = \begin{cases} \lambda_1(\eta), & \eta \in \mathcal{D}_1, \\ \lambda_2(\eta), & \eta \in \mathcal{D}_2. \end{cases} \tag{4.24}$$

The factors $X_0^+(\eta)$ and $X_0^-(\eta)$ are given by (**5**, **17**, **18**)

$$\begin{aligned} X_0^\pm(\eta) &= X_0(\eta), \quad \eta \in \mathcal{D}^\pm, \\ X_0(\eta) &= F(\eta, w)Y(\eta, w) + F(\eta, -w)Y(\eta, -w), \\ [X_0(\eta)]^{-1} &= \frac{Y(\eta, w)}{F(\eta, w)} + \frac{Y(\eta, -w)}{F(\eta, -w)}, \\ F(\eta, w) &= e^{\chi(\eta, w)}, \quad Y(\eta, w) = \frac{1}{2}\left[I_2 + \frac{1}{w}Q(\eta)\right], \end{aligned} \tag{4.25}$$

The function $\chi(\eta, w)$ is a solution to the Riemann–Hilbert problem on the surface $\mathcal{R}$

$$\chi^+(\eta, w) = \lambda(\eta, w)\chi^-(\eta, w), \quad (\eta, w) \in \mathcal{L}, \tag{4.26}$$

where $\mathcal{L} = (L \subset \mathcal{D}_1) \cup (L \subset \mathcal{D}_2)$. Referring to the symmetry condition (4.15), we choose the solution to the problem (4.26) in the form

$$\chi(\eta, w) = \frac{1}{2\pi i}\int_{\mathcal{L}} \log \lambda(t, u) dW + \int_{q_0}^{q_1} dW(\eta) - \int_{q_0}^{q_1} dW(-\eta) + m\oint_{\mathbf{a}} dW(\eta) - m\oint_{\mathbf{a}} dW(-\eta) + n\oint_{\mathbf{b}} dW(\eta) - n\oint_{\mathbf{b}} dW(-\eta), \tag{4.27}$$

where

$$dW(\eta) = \frac{w+u}{2u}\frac{dt}{t-\eta} \tag{4.28}$$

is the Weierstrass kernel, $u = w(t)$, $(t, u) \in \mathcal{L}$, m and n are integers to be fixed, $q_0 = (\sigma_0, w_0) \in \mathcal{D}_1$, $q_1 = (\sigma_1, w_1) \in \mathcal{R}$, $w_j = w(\sigma_j)$, $j = 0, 1$. The point q_0 is an arbitrary fixed point in the first sheet, while the point q_1 may lie on either sheet of the surface and has to be determined. The contours **a** and **b** form a system of canonical cross-sections of the elliptic surface $\mathcal{R}$ (Fig. 1). The contour **a** is a loop formed by the branch cut $[k, \gamma_0]$, and its positive direction is chosen such that the sheet $\mathcal{D}_1$ is on the left. The contour **b** is the loop that is formed by the segment $[\gamma_0, -\gamma_0] \subset \mathcal{D}_1$ passing through the infinite point $(\infty, \infty)_1 \in \mathcal{D}_1$ and by the segment $[-\gamma_0, \gamma_0] \subset \mathcal{D}_2$ passing through the infinite point $(\infty, \infty)_2 \in \mathcal{D}_2$. The loop **b** crosses the loop **a** from the right to the left.

The expression (4.27) for the function $\chi(\eta, w)$ may be put into the form

$$\chi(\eta, w) = \chi_1(\eta) + w\chi_2(\eta), \tag{4.29}$$

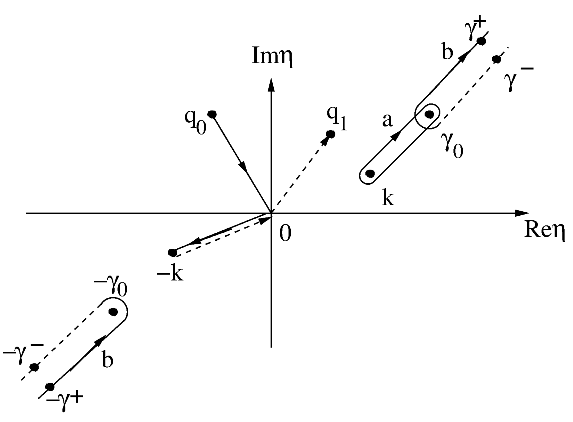

Fig. 1 The canonical cross-sections **a** and **b** and the contour q_0q_1 in the case $q_1 \in \mathcal{D}_2$

where $\chi_1(\eta)$ and $\chi_2(\eta)$ are odd functions given by

$$\chi_1(\eta) = \frac{\eta}{\pi i}\int_0^\infty \frac{\varepsilon_+(t)dt}{t^2-\eta^2} + \eta\int_{\sigma_0}^{\sigma_1}\frac{dt}{t^2-\eta^2},$$

$$\chi_2(\eta) = \frac{\eta}{\pi i}\int_0^\infty \frac{\varepsilon_-(t)dt}{\sqrt{f(t)}(t^2-\eta^2)} + \eta\int_{q_0}^{q_1}\frac{dt}{u(t)(t^2-\eta^2)} + m\eta\oint_{\mathbf{a}}\frac{dt}{u(t)(t^2-\eta^2)} + n\eta\oint_{\mathbf{b}}\frac{dt}{u(t)(t^2-\eta^2)}. \tag{4.30}$$

Here, the following notations are introduced:

$$\varepsilon_\pm(t) = \frac{1}{2}[\log\lambda_1(t) \pm \log\lambda_2(t)]. \tag{4.31}$$

Notice that the winding numbers of the eigenvalues vanish

$$\frac{1}{2\pi}\,[\arg\lambda_j(t)]\big|_L = 0, \quad j = 1, 2, \tag{4.32}$$

and the branches of the logarithmic functions $\log\lambda_j(\eta) = \log|\lambda_j(\eta)| + i\alpha_j$ are fixed by the conditions

$$-\pi \leqslant \alpha_j \leqslant \pi, \quad j = 1, 2. \tag{4.33}$$

Analysis of the asymptotics of the function $\chi_2(\eta)$ at infinity shows that

$$\chi_2(\eta) \sim -\frac{1}{\eta}\left\{\frac{1}{\pi i}\int_0^\infty \frac{\varepsilon_-(t)dt}{\sqrt{f(t)}} + \int_{q_0}^{q_1}\frac{dt}{u(t)} + m\oint_{\mathbf{a}}\frac{dt}{u(t)} + n\oint_{\mathbf{b}}\frac{dt}{u(t)}\right\} + O\left(\frac{1}{\eta^3}\right), \quad \eta\to\infty. \tag{4.34}$$

Since $dW(\eta) = O(\eta^2)$, $(\eta, w) \to (\infty,\infty)_j$, $j = 1, 2$, in general, the function $\chi(\eta, w)$ has a simple pole at the two infinite points of the surface, and the solution $X_0(\eta)$, given by (4.25), has an inadmissible essential singularity at the infinite point in the η-plane. The function $F(\eta, w)$ is bounded as $(\eta, w) \to (\infty,\infty)_j$, $j = 1, 2$, if and only if

$$\frac{1}{\pi i}\int_0^\infty \frac{\varepsilon_-(t)dt}{\sqrt{f(t)}} + \int_{q_0}^{q_1}\frac{dt}{u(t)} + m\oint_{\mathbf{a}}\frac{dt}{u(t)} + n\oint_{\mathbf{b}}\frac{dt}{u(t)} = 0. \tag{4.35}$$

This is a genus-1 Jacobi inversion problem that can be solved by inversion of an elliptic integral. It will be convenient to make the substitution $\tau = t/k$, $k_0 = k/\gamma_0$ and introduce the function

$$\sqrt{f_0(\tau)} = \sqrt{(1-\tau^2)(1-k_0^2\tau^2)}, \tag{4.36}$$

the branch of the two-valued function $f_0(\tau) = (1-\tau^2)(1-k_0^2\tau^2)$ fixed by the condition $\sqrt{f_0(0)} = 1$ in the τ-plane cut along two segments $[1, k_0]$ and $[-1, -k_0]$. By using the connection $\sqrt{f(t)} = k\gamma_0\sqrt{f_0(\tau)}$, we can express the A- and B periods in (4.35) through the elliptic integrals

$$\oint_{\mathbf{a}}\frac{dt}{u(t)} = \frac{2i\mathbf{K}'}{\gamma_0}, \quad \oint_{\mathbf{b}}\frac{dt}{u(t)} = -\frac{4\mathbf{K}}{\gamma_0}, \tag{4.37}$$

where $\mathbf{K} = \mathbf{K}(k_0)$ is the complete elliptic integral of the first kind, and $\mathbf{K}' = \mathbf{K}\big(\sqrt{1-k_0^2}\big)$. Equation (4.35) now reads

$$\int_{(0,1)}^{(\tilde{\sigma}_1,\tilde{w}_1)} \frac{d\tau}{\tilde{w}(\tau)} = d + 4\mathbf{K}n - 2i\mathbf{K}'m, \tag{4.38}$$

where

$$d = -\frac{\gamma_0}{\pi i}\int_0^\infty \frac{\varepsilon_-(t)dt}{\sqrt{f(t)}} + \gamma_0\int_0^{\sigma_0} \frac{dt}{\sqrt{f(t)}}, \tag{4.39}$$

$\tilde{w} = (-1)^{j-1}\sqrt{f_0(t/k)}$, $t \in \mathcal{D}_j$, $j = 1, 2$, $\tilde{\sigma}_1 = \sigma_1/k$, $\tilde{w}_1 = \tilde{w}(\tilde{\sigma}_1)$. By using the solution obtained in (**12**), we invert the elliptic integral in (4.38) and find $\sigma_1 = k\operatorname{sn} d$. The integers m and n are defined either by

$$m = -\frac{\operatorname{Im}[(I-d)\overline{\mathbf{K}}]}{2\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}, \quad n = \frac{\operatorname{Re}[(I-d)\overline{\mathbf{K}'}]}{4\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]} \tag{4.40}$$

or by

$$m = \frac{\operatorname{Im}[(I+d)\overline{\mathbf{K}}]}{2\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}, \quad n = -\frac{\operatorname{Re}[(I+d)\overline{\mathbf{K}'}]}{4\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]} - \frac{1}{2}. \tag{4.41}$$

One and only one pair (m, n) is integer. If the first set of numbers is integer, then $w_1 = \sqrt{f(\sigma_1)}$ $(q_1 \in \mathcal{D}_1)$, otherwise both numbers, m and n, in (4.41) are integer, and $w_1 = -\sqrt{f(\sigma_1)}$ $(q_1 \in \mathcal{D}_2)$.

The choice of the point $q_1 \in \mathcal{R}$ and the integers m and n we have made guarantees that the function $F(\eta, w)$ and the elements of the matrix $X_0(\eta)$ are bounded at infinity. Formula (4.25) for the matrix $X_0(\eta)$ may be written in the form

$$\begin{aligned} X_0(\eta) &= e^{\chi_1(\eta)}\left[\cosh\left(f^{1/2}(\eta)\chi_2(\eta)\right)I_2 + \frac{1}{f^{1/2}(\eta)}\sinh\left(f^{1/2}(\eta)\chi_2(\eta)\right)Q(\eta)\right] \\ &= e^{\chi_1(\eta)}\begin{pmatrix} \cosh(\xi\zeta\chi_2(\eta)) & \frac{\gamma}{2\xi\zeta}\sinh(\xi\zeta\chi_2(\eta)) \\ \frac{2\xi\zeta}{\gamma}\sinh(\xi\zeta\chi_2(\eta)) & \cosh(\xi\zeta\chi_2(\eta)) \end{pmatrix}. \end{aligned} \tag{4.42}$$

This completes the factorization of the matrix $\mathcal{G}_0(\eta)$. The original matrix $\mathcal{G}(\eta)$ can be immediately split as follows:

$$\mathcal{G}(\eta) = X^+(\eta)[X^-(\eta)]^{-1}, \quad \eta \in L, \tag{4.43}$$

where

$$\begin{aligned} X^+(\eta) &= \zeta^+(\eta)X_0^+(\eta), \quad \eta \in \mathcal{D}^+, \\ X^-(\eta) &= \frac{\gamma i}{2\zeta^-(\eta)}X_0^-(\eta), \quad \eta \in \mathcal{D}^-, \\ \zeta^\pm(\eta) &= \sqrt{\eta \pm \gamma_0}, \end{aligned}$$

$$\arg(\eta+\gamma_0) \in [-\pi+\alpha_0, \alpha_0], \quad \arg(\eta-\gamma_0) \in [\alpha_0, 2\pi+\alpha_0], \quad \alpha_0 = \arg\gamma_0 \in (0, \pi). \tag{4.44}$$

4.3 *The symmetric homogeneous Riemann–Hilbert problem*

Consider now the homogeneous analogue of the Riemann–Hilbert problem (4.14) subject to the symmetry condition (4.15). By using the factorization (4.43) of the matrix coefficient $\mathcal{G}(\eta)$, the continuity principle and the Liouville theorem we represent the general solution, $\boldsymbol{\psi}_h^{\pm}(\eta)$, of the homogeneous problem in the form

$$\boldsymbol{\psi}_h^{+}(\eta) = \zeta^{+}(\eta)X_0^{+}(\eta)\mathbf{M}(\eta), \quad \eta \in \mathcal{D}^{+},$$
$$\boldsymbol{\psi}_h^{-}(\eta) = \frac{\gamma i}{2\zeta^{-}(\eta)}X_0^{-}(\eta)\mathbf{M}(\eta), \quad \eta \in \mathcal{D}^{-}, \tag{4.45}$$

where $\mathbf{M}(\eta)$ is an order-2 vector whose components, $M_1(\eta)$ and $M_2(\eta)$, are rational functions to be determined. The solution of the homogeneous Riemann–Hilbert problem must satisfy the symmetry condition (4.15) which can be written in the form

$$\zeta^{+}(\eta)X_0^{+}(\eta)\mathbf{M}(\eta) = \frac{(\gamma+\eta)i}{\zeta^{-}(-\eta)}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}X_0^{-}(-\eta)\mathbf{M}(-\eta), \quad \eta \in \mathcal{D}^{+}. \tag{4.46}$$

Fortunately, the integrals in the representation (4.30) of the function $\chi_1(\eta)$ can be explicitly computed, and the condition (4.46) can be simplified. To carry out the integration, we analyze the function

$$\varepsilon_{+}(\eta) = \frac{1}{2}\log\left|\frac{\eta^2-\gamma^2}{\eta^2-\gamma_0^2}\right| + i\theta_{+}, \quad -\pi \leqslant \theta_{+} \leqslant \pi. \tag{4.47}$$

Denote (Fig. 1) by γ^{+} and γ^{-} the two points on the upper and lower sides of the cut $[\gamma_0,\infty)$, respectively, such that $|\gamma^{+}| = |\gamma^{-}| = |\gamma|$. It is directly verified that the points γ^{-} and $-\gamma^{-}$ are simple zeros of the function $\lambda_1(\eta)$, and the points γ^{+} and $-\gamma^{+}$ are simple zeros of the function $\lambda_2(\eta)$. Referring to (4.23), we compute

$$\lambda_1(\eta) = i\left(1 \pm \left|\frac{\gamma}{2\zeta}\right|\right), \quad \lambda_2(\eta) = i\left(-1 \pm \left|\frac{\gamma}{2\zeta}\right|\right), \quad \eta \in [\gamma_0,\gamma]^{\pm}. \tag{4.48}$$

For the ranges of the arguments α_1 and α_2 chosen in (4.33) and θ_{+} chosen in (4.47), we find

$$\alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = \frac{\pi}{2}, \quad \theta_{+} = \pi, \quad \eta \in [\gamma_0,\gamma]^{+},$$
$$\alpha_1 = -\frac{\pi}{2}, \quad \alpha_2 = -\frac{\pi}{2}, \quad \theta_{+} = -\pi, \quad \eta \in [\gamma_0,\gamma]^{-},$$
$$\alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = -\frac{\pi}{2}, \quad \theta_{+} = 0, \quad \eta \in [\gamma,\infty]^{\pm}. \tag{4.49}$$

Application of the Cauchy theorem gives

$$\chi_1^{\pm}(\eta) = \frac{1}{2}\log\frac{(\sigma_1-\eta)(\sigma_0+\eta)}{(\sigma_1+\eta)(\sigma_0-\eta)} \pm \frac{1}{2}\log\frac{\eta\pm\gamma}{\eta\pm\gamma_0}, \quad \eta \in \mathcal{D}^{\pm}. \tag{4.50}$$

The branches of the logarithmic functions are fixed such that the function $\chi_1(\eta) = \chi_1^{\pm}(\eta)$, $\eta \in \mathcal{D}^{\pm}$, remains odd. Formulas (4.2) and (4.50) imply that

$$X_0^{-}(-\eta) = e^{-\chi_1^{+}(\eta)}\left[\cosh\left(f^{1/2}(\eta)\chi_2^{+}(\eta)\right)I_2 - \frac{1}{f^{1/2}(\eta)}\sinh\left(f^{1/2}(\eta)\chi_2^{+}(\eta)\right)Q(\eta)\right], \quad \eta \in \mathcal{D}^{+}. \tag{4.51}$$

Replacing $X^-(-\eta)$ and $\zeta^-(-\eta)$ in (4.46) by its expression (4.51) and $i\zeta^+(\eta)$, respectively, we have finally

$$(\eta-\sigma_1)(\eta+\sigma_0)M_1(\eta) = -(\eta+\sigma_1)(\eta-\sigma_0)M_1(-\eta),$$
$$(\eta-\sigma_1)(\eta+\sigma_0)M_2(\eta) = (\eta+\sigma_1)(\eta-\sigma_0)M_2(-\eta), \quad \eta\in\mathcal{D}^\pm. \tag{4.52}$$

Since the matrix $X_0(\eta)$ has simple zeros at the points $\eta=\sigma_1$ and $\eta=-\sigma_0$, the rational functions $M_1(\eta)$ and $M_2(\eta)$ may have simple poles at these points. From the symmetry conditions (4.52), we can deduce the most general form of these functions

$$M_1(\eta) = \frac{p_1(\eta)}{(\eta-\sigma_1)(\eta+\sigma_0)},$$
$$M_2(\eta) = \frac{p_2(\eta)}{(\eta-\sigma_1)(\eta+\sigma_0)}, \tag{4.53}$$

where $p_1(\eta)$ and $p_2(\eta)$ are arbitrary polynomials, odd and even, respectively.

4.4 *The vectors $\psi^+(\eta)$ and $\psi^-(\eta)$*

Since the matrix $\mathcal{G}(\eta)$ has been factorized, we can find a particular solution of the inhomogeneous symmetric Riemann–Hilbert problem (4.14), (4.15). Introduce the Cauchy integral

$$\mathbf{\Psi}(\eta) = \frac{1}{2\pi i}\int_L \frac{[X_0^+(t)]^{-1}\mathbf{v}(t)dt}{\zeta^+(t)(t-\eta)} \tag{4.54}$$

and represent the vector $[\zeta^+(t)X_0^+(t)]^{-1}\mathbf{v}(t)$ through the limit values $\mathbf{\Psi}^\pm(t)=\mathbf{\Psi}(t\pm i0)$ of the integral (4.54)

$$\frac{[X_0^+(t)]^{-1}\mathbf{v}(t)}{\zeta^+(t)} = \mathbf{\Psi}^+(t) - \mathbf{\Psi}^-(t), \quad t\in L. \tag{4.55}$$

This gives us a particular solution of the problem (4.14), (4.15)

$$\boldsymbol{\psi}_p^+(\eta) = \zeta^+(\eta)X_0^+(\eta)\mathbf{\Psi}^+(\eta), \quad \eta\in\mathcal{D}^+,$$
$$\boldsymbol{\psi}_p^-(\eta) = \frac{\gamma i}{2\zeta^-(\eta)}X_0^-(\eta)\mathbf{\Psi}^-(\eta), \quad \eta\in\mathcal{D}^-. \tag{4.56}$$

It can be directly verified that this solution satisfies the symmetry condition (4.15). The sum $\boldsymbol{\psi}^\pm(\eta) = \boldsymbol{\psi}_h^\pm(\eta)+\boldsymbol{\psi}_p^\pm(\eta)$, the general solution to the problem (4.14), (4.15), must have a certain finite order as $\eta\to\infty$. To satisfy the specific conditions (4.16) at infinity, we need to fix the polynomials $p_1(\eta)$ and $p_2(\eta)$ in the representation of the general solution of the problem (4.14), (4.15)

$$\boldsymbol{\psi}^+(\eta) = \zeta^+(\eta)X_0^+(\eta)\left[\mathbf{\Psi}^+(\eta)+\frac{1}{(\eta-\sigma_1)(\eta+\sigma_0)}\begin{pmatrix}p_1(\eta)\\p_2(\eta)\end{pmatrix}\right], \quad \eta\in\mathcal{D}^+,$$
$$\boldsymbol{\psi}^-(\eta) = \frac{\gamma i}{2\zeta^-(\eta)}X_0^-(\eta)\left[\mathbf{\Psi}^-(\eta)+\frac{1}{(\eta-\sigma_1)(\eta+\sigma_0)}\begin{pmatrix}p_1(\eta)\\p_2(\eta)\end{pmatrix}\right], \quad \eta\in\mathcal{D}^-. \tag{4.57}$$

Since the functions $\chi_1(\eta)$ are $\chi_2(\eta)$ are odd, for the chosen point (σ_1, w_1) and the integers m and n, we obtain

$$X_0(\eta) \sim \begin{pmatrix} 1 & c_{12}\eta^{-3} \\ c_{21}\eta & 1 \end{pmatrix}, \quad \eta \to \infty, \tag{4.58}$$

where c_{12} and c_{21} are some nonzero constants. From the analysis of the behavior at infinity of the vectors (4.56), we may conclude on the ground of the inequalities (4.16) that

$$p_1(\eta) = C_2\eta, \quad p_2(\eta) = C_3 + C_4\eta^2, \tag{4.59}$$

where C_2, C_3 and C_4 are arbitrary constants. Then, as $\eta \to \infty$,

$$\boldsymbol{\psi}_1^+(\eta) \sim (C_2 - \Psi_1^\circ)\eta^{-1/2}, \quad \boldsymbol{\psi}_2^+(\eta) \sim [(C_2 - \Psi_1^\circ)c_{21} + C_4]\eta^{1/2},$$

$$\boldsymbol{\psi}_1^-(\eta) \sim \frac{\gamma i}{2}(C_2 - \Psi_1^\circ)\eta^{-3/2}, \quad \boldsymbol{\psi}_2^+(\eta) \sim \frac{\gamma i}{2}[(C_2 - \Psi_1^\circ)c_{21} + C_4]\eta^{-1/2}, \tag{4.60}$$

where

$$\begin{pmatrix} \Psi_1^\circ \\ \Psi_2^\circ \end{pmatrix} = \frac{1}{2\pi i}\int_L \frac{[X_0^+(t)]^{-1}\mathbf{v}(t)dt}{\zeta^+(t)}. \tag{4.61}$$

The above result enables us to deduce from formulas (4.8) the asymptotic relations for the functions $\Phi_j^\pm(\eta)$ $(j = 1, 2, 3, 4)$ as $\eta \to \infty$

$$\Phi_1^\pm(\eta) \sim \frac{K_{11}^\pm}{\eta} + \frac{K_{12}^\pm}{\eta^{3/2}}, \quad \eta \in \mathcal{D}^\pm,$$

$$\Phi_2^\pm(\eta) \sim \frac{K_{21}^\pm}{\eta} + \frac{K_{22}^\pm}{\eta^{3/2}}, \quad \eta \in \mathcal{D}^\pm,$$

$$\Phi_3^\pm(\eta) \sim \frac{K_{31}^\pm}{\eta} + \frac{K_{32}^\pm}{\eta^2} + \frac{K_{33}^\pm}{\eta^{5/2}}, \quad \eta \in \mathcal{D}^\pm,$$

$$\Phi_4^\pm(\eta) \sim \frac{K_{41}^\pm}{\eta^{3/2}}, \quad \eta \in \mathcal{D}^\pm, \quad \eta \to \infty, \tag{4.62}$$

where $K_{jm}^\pm$ are some nonzero constants.

Now, it is easily seen that the elements of the matrix $X_0(\eta)$ have simple poles at the points $\eta = \sigma_0$ and $\eta = -\sigma_1$. This gives rise to simple poles of the vectors $\boldsymbol{\psi}^+(\eta)$ or $\boldsymbol{\psi}^-(\eta)$ depending on if σ_0 and $-\sigma_1$ are points of the upper or lower half-planes. To remove these inadmissible poles, we require

$$\operatorname*{res}_{\eta=\sigma_0} \psi_j(\eta) = 0, \quad \operatorname*{res}_{\eta=-\sigma_1} \psi_j(\eta) = 0, \quad j = 1, 2. \tag{4.63}$$

At the points $\eta = \sigma_0$ and $\eta = -\sigma_1$, the matrix $X_0(\eta)$ has the following asymptotics:

$$X_0(\eta) \sim \frac{X_*(\sigma)}{\eta - \sigma}\begin{pmatrix} 1 & \gamma[2\sqrt{f(\sigma)}]^{-1} \\ 2\gamma^{-1}\sqrt{f(\sigma)} & 1 \end{pmatrix}, \quad \eta \to \sigma, \tag{4.64}$$

where $\sigma = \sigma_0$ or $\sigma = -\sigma_1$, and $X_*(\sigma)$ is a function bounded at the points σ_0 and $-\sigma_1$ and does not vanish at these points. The matrix in the right-hand side is a rank-1 matrix. Therefore, there are only two linearly independent conditions in (4.63). These conditions reduce to

$$\begin{aligned} C_2 + \frac{\gamma}{2\sigma_0\sqrt{f(\sigma_0)}}(C_3+\sigma_0^2 C_4) &= -2(\sigma_0-\sigma_1)\left[\Psi_1(\sigma_0)+\frac{\gamma\,\Psi_2(\sigma_0)}{2\sqrt{f(\sigma_0)}}\right],\\ C_2 - \frac{\gamma}{2\sigma_1\sqrt{f(-\sigma_1)}}(C_3+\sigma_1^2 C_4) &= -2(\sigma_0-\sigma_1)\left[\Psi_1(-\sigma_1)+\frac{\gamma\,\Psi_2(-\sigma_1)}{2\sqrt{f(-\sigma_1)}}\right]. \end{aligned} \tag{4.65}$$

The solution (4.8) to the Riemann–Hilbert problem (3.23), (2.5) possesses five unknown constants, $C_0, C_1, \ldots, C_4$. For their definition, we have the same number of linear conditions, the two equations (4.65) and the following three equations:

$$\begin{aligned} &\left(\frac{2\gamma_0}{\gamma}-1\right)C_0+\left(\frac{2}{\gamma}-\frac{3}{\gamma_0}\right)C_1=\frac{1}{2\pi i}\int_L\left[2\left(1-\frac{\gamma_0}{\gamma}\right)h_1(\tau)-h_2(\tau)\right]\frac{d\tau}{\tau-\gamma_0},\\ &\gamma_0C_0+C_1+\frac{\gamma\,\zeta^+(\gamma_0)}{(\gamma_0-\sigma_1)(\gamma_0+\sigma_0)}\left[\gamma_0\chi_{11}^+(\gamma_0)C_2+\chi_{12}^+(\gamma_0)(C_3+\gamma_0^2C_4)\right]\\ &=-\frac{\gamma_0}{2\pi i}\int_L\frac{h_1(\tau)d\tau}{\tau-\gamma_0}-\gamma\,\zeta^+(\gamma_0)[\chi_{11}^+(\gamma_0)\Psi_1^+(\gamma_0)+\chi_{12}^+(\gamma_0)\Psi_2^+(\gamma_0)],\\ &\gamma_0\chi_{21}^+(\gamma_0)C_2+\chi_{22}^+(\gamma_0)(C_3+\gamma_0^2C_4)=-(\gamma_0-\sigma_1)(\gamma_0+\sigma_0)[\chi_{21}^+(\gamma_0)\Psi_1^+(\gamma_0)+\chi_{22}^+(\gamma_0)\Psi_2^+(\gamma_0)], \end{aligned} \tag{4.66}$$

which come from conditions (4.12). On fixing the constants $C_0, C_1, \ldots, C_4$, we complete the solution of the Riemann–Hilbert problem (3.23).

5. The electric field component E_z

In this section, we aim to express the electric field component E_z through the solution to the Riemann–Hilbert problem (3.23), the vectors $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$. We start with the quadrant S_1: $x > 0$, $y > 0$. From formulas (3.6) and (3.7), we have

$$E_z(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(\eta)e^{-\xi x}-\frac{1}{2\xi}\int_0^{\infty}q(x_1)e^{-\xi|x-x_1|}dx_1-\frac{\kappa_1}{2\xi}e^{-\xi|x-a_1|+i\eta b_1}\right]e^{-i\eta y}d\eta. \tag{5.1}$$

Referring to (3.8) and (3.9), we obtain first

$$A(\eta)=\tilde{E}_{z+}(0^+,\eta)+\frac{ikZ}{2\xi}\hat{H}_{x+}(i\xi,0^+)-\frac{i\eta}{2\xi}\hat{E}_{z+}(i\xi,0^+)+\frac{\kappa_1}{2\xi}e^{-\xi a_1+i\eta b_1} \tag{5.2}$$

and then determine

$$A(\eta)=\frac{1}{2}\Phi_1^+(\eta). \tag{5.3}$$

Because of the definition

$$\begin{aligned} H_x(x_1,0^+)&=\frac{1}{2\pi i}\int_{\mathcal{T}}\hat{H}_{x+}(i\xi_1,0^+)e^{\xi_1x_1}d\xi_1,\\ E_z(x_1,0^+)&=\frac{1}{2\pi i}\int_{\mathcal{T}}\hat{E}_{z+}(i\xi_1,0^+)e^{\xi_1x_1}d\xi_1 \end{aligned} \tag{5.4}$$

and formula (3.8), the electric component E_z in (5.1) possesses triple integrals. Here, $\mathcal{T} = \{\xi_1 \in \mathbb{C}\colon \operatorname{Re}\xi_1 = \sigma\}$ is the Bromwich contour associated with the inverse Laplace transform. The number σ is chosen so that the contour lies in the ξ_1-plane to the right of all singularities of the functions $\hat{H}_{x+}(i\xi_1, 0^+)$ and $\hat{E}_{z+}(i\xi_1, 0^+)$. By evaluating the integral with respect to x_1 in (5.1), we obtain

$$E_z(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\Phi_1^+(\eta)e^{-\xi x} - \frac{\kappa_1}{\xi}e^{-\xi|x-a_1|+i\eta b_1} + V_1(x,\eta)\right]e^{-i\eta y}d\eta, \quad (x,y)\in S_1, \tag{5.5}$$

where $\operatorname{Re}\xi_1 < \operatorname{Re}\xi$ and

$$V_1(x,\eta) = \frac{1}{4\pi i\xi}\int_{\mathcal{T}}\left(\frac{e^{\xi_1 x}}{\xi-\xi_1} + \frac{e^{\xi_1 x} - e^{-\xi x}}{\xi+\xi_1}\right)\left\{\left(1+\frac{\eta}{\eta_1}\right)[\xi_1\Phi_1^+(\eta_1) + \kappa_1 e^{-\xi_1 a_1 + i\eta_1 b_1}]\right.$$
$$\left. + \left(1-\frac{\eta}{\eta_1}\right)[\xi_1\Phi_1^-(\eta_1) + \kappa_1 e^{-\xi_1 a_1 - i\eta_1 b_1}]\right\}d\xi_1. \tag{5.6}$$

For the quadrant S_2: $x > 0$, $y < 0$, we have

$$E_z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(\eta)e^{-\xi x} - \frac{1}{2\xi}\int_0^{\infty} q(x_1)e^{-\xi|x-x_1|}dx_1 - \frac{\kappa_2}{2\xi}e^{-\xi|x-a_2|+i\eta b_2}\right]e^{-i\eta y}d\eta, \tag{5.7}$$

where

$$A(\eta) = \tilde{E}_{z-}(0^+,\eta) + \frac{1}{2\xi}\left[-ikZ\hat{H}_{x+}(i\xi,0^-) + i\eta\hat{E}_{z+}(i\xi,0^-)\right] + \frac{\kappa_2}{2\xi}e^{-\xi a_2 + i\eta b_2},$$
$$q(x_1) = -ikZH_x(x_1,0^-) + i\eta E_z(x_1,0^-). \tag{5.8}$$

As in the previous case, we find

$$A(\eta) = -\frac{1}{2}\Phi_2^+(\eta), \tag{5.9}$$

and then, finally,

$$E_z(x,y) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\Phi_2^+(\eta)e^{-\xi x} + \frac{\kappa_2}{\xi}e^{-\xi|x-a_2|+i\eta b_2} + V_2(x,\eta)\right]e^{-i\eta y}d\eta, \quad (x,y)\in S_2, \tag{5.10}$$

where $\operatorname{Re}\xi_1 < \operatorname{Re}\xi$ and

$$V_2(x,\eta) = \frac{1}{4\pi i\xi}\int_{\mathcal{T}}\left(\frac{e^{\xi_1 x}}{\xi-\xi_1} + \frac{e^{\xi_1 x} - e^{-\xi x}}{\xi+\xi_1}\right)\left\{\left(\frac{1}{2} - \frac{\gamma}{\eta_1} + \frac{\eta}{2\eta_1}\right)[\xi_1\Phi_1^+(\eta_1) + \kappa_1 e^{-\xi_1 a_1 + i\eta_1 b_1}]\right.$$
$$\left. + \left(\frac{1}{2} + \frac{\gamma}{\eta_1} - \frac{\eta}{2\eta_1}\right)[\xi_1\Phi_1^-(\eta_1) + \kappa_1 e^{-\xi_1 a_1 - i\eta b_1}]\right\}d\xi_1. \tag{5.11}$$

The convergence of the integrals $V_1(x,\eta)$ and $V_2(x,\eta)$ at infinity is guaranteed since

$$\Phi_1^+(\eta) + \Phi_1^-(\eta) \sim \Phi_1^-(\eta) - \Phi_3^-(\eta) \sim \frac{\eta}{4\zeta^2}\left[\Omega_1^-(\eta) - \Omega_2^-(\eta) + \frac{4\eta}{\gamma}\psi_1^-(\eta)\right]$$
$$= O(\eta^{-3/2}), \quad \eta\to\infty, \quad \eta\in\mathcal{D}^-. \tag{5.12}$$

The above relations follow from (4.8), (4.11) and (4.60). A similar result can be derived in $\mathcal{D}^+$, and we have $\Phi_1^+(\eta) + \Phi_1^-(\eta) = O(\eta^{-3/2})$, $\eta \to \infty$, $\eta \in \mathcal{D}^\pm$.

In the last case, $(x, y) \in S_3$: $x < 0, -\infty < y < +\infty$, the electric component E_z is defined through just one quadrature

$$E_z(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left[\Phi_2^+(\eta) + \Phi_3^-(\eta)\right] e^{\xi x} - \frac{\kappa_3}{2\xi} \left[e^{-\xi|x-a_3|} - e^{\xi(x+a_3)}\right] \right\}, \quad (x, y) \in S_3. \tag{5.13}$$

Derive next the diffraction coefficient for the domain S_3. For $x < a_3$ $(|x| > |a_3|)$, formula (5.13) can be written as follows:

$$E_z(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_0(\eta) e^{\xi x - i\eta y} d\eta, \tag{5.14}$$

where

$$\Phi_0(\eta) = \Phi_2^+(\eta) + \Phi_3^-(\eta) + \frac{\kappa_3}{\xi} \sinh a_3 \xi e^{i\eta b_3}. \tag{5.15}$$

Let $x = -r\cos\varphi$, $y = r\sin\varphi$, $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$. By applying the method of steepest descent (**19**) to the integral (5.14), we find

$$E_z(x, y) \sim \frac{e^{i(kr-\pi/4)}}{\sqrt{r}} D(\varphi), \quad r \to \infty. \tag{5.16}$$

where the diffraction coefficient, $D(\varphi)$, is given by

$$D(\varphi) = \sqrt{\frac{k}{2\pi}} \cos\varphi \, \Phi_0(-k\sin\varphi). \tag{5.17}$$

If $y > 0$ $\left(0 < \varphi < \frac{1}{2}\pi\right)$, then we replace the function $\Phi_2^+(\eta)$ in (5.15) by its analytic continuation into the half-plane $\operatorname{Im}\eta < 0$

$$\Phi_2^+(\eta) = -\Phi_3^-(\eta) + \frac{1}{\xi}\Phi_4^-(\eta) + \frac{\kappa_3^\circ(\eta)}{\xi}, \quad \operatorname{Im}\eta < 0. \tag{5.18}$$

For negative y $\left(-\frac{1}{2}\pi < \varphi < 0\right)$, we need to replace the function $\Phi_3^-(\eta)$. From (3.23), (3.24) its analytic continuation in the upper half-plane is

$$\Phi_3^-(\eta) = \frac{1}{2(\gamma + \eta)} \left[-\eta\Phi_1^+(\eta) - (\gamma + \eta)\Phi_2^+(\eta) + \gamma\,\Phi_3^+(\eta) + \frac{\gamma}{\xi}\Phi_4^+(\eta)\right]$$
$$-\frac{1}{2\xi}[\kappa_1^\circ(\eta) + \kappa_2^\circ(\eta) - \kappa_3^\circ(\eta)] + \frac{\gamma}{2(\gamma + \eta)\xi}[\kappa_1^\circ(-\eta) + \kappa_2^\circ(-\eta)], \quad \operatorname{Im}\eta > 0. \tag{5.19}$$

Notice that by changing the system of coordinates such that the ER screen W_1 is the half-plane $\{(x, y)\colon x = \pm 0, 0 < y < +\infty\}$, and the PMC screen W_2 is the half-plane $\{(x, y)\colon 0 < x < +\infty, y = \pm 0\}$ and solve the new problem by the method described, we can determine the electric field E_z for $x < 0$ through one quadrature. Therefore, by combining the results, it is possible to find the function E_z everywhere in the exterior of the wedge $\{(x, y)\colon x > 0, y > 0\}$ in terms of a single integral and determine the diffraction coefficient.

6. Conclusions

We have found a closed-form solution of a diffraction problem for a planar structure consisting of two orthogonal half-planes. The first one is an ER screen, and the second one is a PMC screen. We have shown how the problem can be formulated as a Riemann–Hilbert problem for two order-4 analytic vector functions. We have split the matrix coefficient of the problem, $G(\eta)$, into three matrix factors, $G(\eta) = T_1(\eta)\Gamma(\eta)T_2(\eta)$, where $T_1(\eta)$ and $T_2(\eta)$ are 4×4 rational matrices, $\Gamma(\eta) = \operatorname{diag}\{I_2, \mathcal{G}(\eta)\}$, $\mathcal{G}(\eta) = I_2 + a(\eta)Q(\eta)$, $a(\eta) \in H(L)$, $I_2 = \operatorname{diag}\{1, 1\}$, $Q(\eta)$ is a polynomial matrix, $\operatorname{tr} Q(\eta) = 0$ and $\deg \det Q(\eta) = 4$. One of the differences between the Riemann–Hilbert problems associated with a half-plane and a quarter-plane is the presence of the symmetry condition $\mathbf{\Phi}^+(\eta) = \mathbf{\Phi}^-(-\eta)$ for the unknown vectors. This simple constraint changes its structure for new auxiliary vectors, $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$, introduced after the splitting of the matrix $G(\eta) = T_1(\eta)\Gamma(\eta)T_2(\eta)$. The new symmetry condition has the form: $\Omega_j^+(\eta) = \Omega_{3-j}^-(-\eta)$ $(j = 1, 2)$ and $\Omega_j^+(\eta) = 2(-1)^j(1 - \eta/\gamma)\Omega_j^-(-\eta)$ $(j = 3, 4)$, $\eta \in \mathcal{D}^+$. Initially, the factors $X^+(\eta)$ and $X^-(\eta)$ we have derived had an essential singularity at infinity. It has been removed by solving a certain genus-1 Jacobi inversion problem. The matrices $X^+(\eta)$ and $X^-(\eta)$ and a rational vector with prescribed poles due to the generalized Liouville theorem have been found in the form, which satisfies the constraints for the vectors $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$. Having solved the governing order-4 Riemann–Hilbert problem, we have shown that the electric field component E_z is bounded at the junction of the two half-planes, derived the asymptotics of the function E_z at infinity and determined the diffraction coefficient in the exterior of the right-angled wedge.

We have generalized the method developed for 4×4 matrices to the case of $n \times n$ matrices by describing in Appendix a class of matrices $G(\eta) = R_1(\eta) + a(\eta)R_2(\eta)$ $(a(\eta) \in H(L)$, $R_1(\eta)$ and $R_2(\eta)$ are $n \times n$ rational matrices), which admit the representation $G(\eta) = T_1(\eta)\Gamma(\eta)T_2(\eta)$. Here, $T_1(\eta)$ and $T_2(\eta)$ are rational $n \times n$ matrices, and $\Gamma(\eta)$ is a block diagonal matrix. Each block is either a scalar or a 2×2 matrix whose factorization reduces to the Wiener–Hopf splitting of a Chebotarev–Khrapkov 2×2 matrix of the structure $\mathcal{G}(\eta) = I_2 + a_0(\eta)Q(\eta)$ (a_0 is a scalar Hölder function, $Q(\eta)$ is a polynomial matrix and $\operatorname{tr} Q(\eta) = 0$). Its factorization can be implemented by solving an associated Riemann–Hilbert problem on a hyperelliptic surface. The coefficient of this problem coincides with the first eigenvalue of the matrix $\mathcal{G}(\eta)$ in the upper sheet and with the second one in the lower sheet. The genus of the surface is equal to the degree of the polynomial $\det Q(\eta)$.

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APPENDIX

An $n \times n$ Matrix Factorization

The factorization method for the matrix (2.1), (2.2) can be generalized for $n \times n$ matrices. Let L be a piecewise smooth closed curve that splits an η-plane into the interior domain $\mathcal{D}^+$ and the exterior one $\mathcal{D}^-$. Given an $n \times n$ matrix $G(\eta) = R_1(\eta) + a(\eta)R_2(\eta)$, where $R_1(\eta)$ and $R_2(\eta)$ are arbitrary rational $n \times n$ matrices, $a(\eta)$ is a Hölder function in L, and the matrices $G(\eta)$ and $R_1(\eta)$ are nonsingular in L, we want to find two matrices $X^+(\eta)$ and $X^-(\eta)$ analytic everywhere in the η-plane continuous up to the contour L except at most a finite number of poles in $\mathcal{D}^+$ and $\mathcal{D}^-$. The matrices $X^+(\eta)$ and $X^-(\eta)$ are nonsingular in $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, except at most a finite number of points. Their limit values satisfy the boundary condition

$$X^+(\eta) = G(\eta)X^-(\eta), \quad \eta \in L. \tag{A.1}$$

THEOREM A.1. *Let the characteristic polynomial of the matrix $B(\eta) = R_1^{-1}(\eta)R_2(\eta)$ admit the representation*

$$\det[\mu I_n - B(\eta)] = \prod_{j=1}^{m} \omega_j(\mu) \prod_{s=1}^{(n-m)/2} \phi_s(\mu),$$

where I_n is the $n \times n$ unit matrix, $n - m \in [0, n]$ is an even number,

$$\omega_j(\mu) = \mu - \rho_j, \quad j = 1, \ldots, m,$$
$$\phi_s(\mu) = \mu^2 + \nu_s\mu + \sigma_s, \quad s = 1, \ldots, (n-m)/2, \tag{A.3}$$

ρ_j, ν_s and σ_s are rational functions of η (the scalars ρ_j, $j = 1, \ldots, m$, and the pairs (ν_s, σ_s), $s = 1, \ldots, (n-m)/2$, are not necessarily distinct), and $\phi_s(\mu)$ are irreducible polynomials over the field of rational functions of η. Then, the matrices $X^+(\eta)$ and and $X^-(\eta)$ are given by

$$X^+(\eta) = R_1(\eta)T(\eta)X_\Gamma^+(\eta), \quad X^-(\eta) = T(\eta)X_\Gamma^-(\eta), \tag{A.4}$$

where

$$T(\eta) = \left(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_1', \mathbf{x}_1'', \ldots, \mathbf{x}_{(n-m)/2}', \mathbf{x}_{(n-m)/2}''\right), \tag{A.5}$$

$\mathbf{x}_j$ $(j = 1, \ldots, m)$ are the eigenvectors associated with the linear polynomials $\omega_j(\mu)$, and the vectors $\mathbf{x}_s'$, $\mathbf{x}_s''$ form the basis of the kernel of the matrix operator

$$\phi_s(B) = B^2 + \nu_s B + \sigma_s I_n, \quad s = 1, \ldots, (n-m)/2, \tag{A.6}$$

$X_\Gamma^+(\eta)$ and $X_\Gamma^-(\eta)$ factorize the following matrix with a block diagonal structure:

$$\Gamma(\eta) = \mathrm{diag}\{a_1, \ldots, a_m, \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{(n-m)/2}\}, \tag{A.7}$$

where $a_j(\eta)$ $(j = 1, \ldots, m)$ are scalar functions, $\mathcal{G}_s(\eta)$ $(s = 1, \ldots, (n-m)/2)$ are 2×2 matrices of the form $\mathcal{G}_s(\eta) = R_s'(\eta) + a(\eta)R_s''(\eta)$, $R_s'(\eta)$ and $R_s''(\eta)$ are 2×2 rational matrices.

To complete the factorization, one needs to factorize the scalar functions $a_j(\eta)$ $(j = 1, 2, \ldots, m)$ and the matrices $\mathcal{G}_s(\eta)$ $(s = 1, 2, \ldots, (n-m)/2)$. The last step is reducible to the factorization of matrices that have the form (1.4).