INTEGRO-DIFFERENTIAL EQUATION FOR A FINITE CRACK IN A STRIP WITH SURFACE EFFECTS

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Summary

We present a mathematical investigation of a nonstandard boundary value problem for Laplace's equation in an infinite strip containing a finite crack. This unusual problem arises when surface elasticity is incorporated in the description of deformation of the crack faces. We use the Fourier transform to reduce the original boundary value problem to an integral equation that is further shown to reduce to a vector Riemann–Hilbert problem. The latter is solved analytically. The solution consists of two parts. The first is found explicitly, and the second is in a series form. The series converge exponentially, and the series coefficients are obtained by solving an infinite system. It is shown that the rate of convergence of an approximate solution to the exact one is exponential. We illustrate our solution with a particular example which, among other interesting phenomena, reveals the effect of the surface mechanics on the shape of the crack face and shows that the stress is bounded at the crack tip.

1. Introduction

The incorporation of surface mechanics into mathematical models describing deformation of various elastic structures has drawn an increasing amount of attention in the literature (see, for example, (**1** to **11**)). The concept is particularly significant when considering analytical models of deformation at the nanoscale where the high surface area to volume ratio means that the effects of the surface can no longer be neglected.

It is widely accepted that atoms near the surface of a solid material experience a local environment different from those in the bulk material (see, for example, (**12**, **13**)). Consequently, a more accurate and comprehensive analysis of the deformation of an elastic solid with one or more surfaces can be achieved by incorporating a description of the separate surface mechanics near each surface of the solid. In the case of a solid containing a crack, a comprehensive model would include the surface effects corresponding to the two surfaces (faces) of the crack. In the context of continuum-based analytical models, the surface model proposed in (**3**, **4**) has been used extensively in a number of

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studies, including several problems dealing with fracture mechanics (see, for example, (**14**, **15**) and the bibliographies contained therein).

In this paper, the aforementioned surface effects are incorporated using a version of the Gurtin–Murdoch surface elasticity model (see (**16**) for a comprehensive discussion of the various versions of the Gurtin–Murdoch model). More precisely, since it is well known that the thickness, δ, of the 'layer' responsible for surface effects is in the range of a half to a few nanometers (see, for example, (**17**, **18**)), we assume that the crack (including its tips) is 'surrounded' by a correspondingly thin surface layer with elastic properties different from those of the surrounding bulk material. In terms of the Gurtin–Murdoch model, this translates into the assumption of a surface as a negligibly thin elastic membrane adhering to the bulk without slipping. Mathematically, the crack is modelled as an interval of the x-axis onto which we project the properties of the surface layer (much like we project the properties of a thin interphase layer onto the boundary of a fibre in the mathematical modelling of inclusion problems in composite mechanics). Consequently, the bulk stresses are expected to remain finite everywhere, including at the crack tips, as long as $\delta > 0$. When $\delta = 0$ (surface effects are completely neglected), bulk stresses are once again expected to become infinite at the crack tips as we return to the classical model used in linear elastic fracture mechanics. Our analysis demonstrates that stresses at the crack tip do indeed remain finite becoming unbounded only as the surface effects are neglected completely. In addition, it is shown that surface effects decrease the gap between the upper and lower faces of the crack.

In section 2, we formulate the problem for a strip containing a finite mode-III crack with surface effects to a discontinuous boundary value problem for the Laplace operator in a strip. It is reduced to an integro-differential equation of the form

$$\psi'(\xi) + \mu_0 \int_0^\lambda \frac{\psi(\tau)d\tau}{\sinh\frac{\pi}{2}(\tau - \xi)} = g(\xi), \quad 0 < \xi < \lambda, \quad \mu_0 > 0, \tag{1.1}$$

in section 3. Its exact solution is not available in the literature. In section 4, by developing further the method proposed in (**19**, **20**), we show that it can equivalently be written as a Riemann–Hilbert problem for two pairs of functions. This problem is solved analytically. Its solution may be split into two parts. One of them is given by quadratures, and the other is presented by some series that converge exponentially. The series coefficients are the solution of an infinite linear algebraic system. The rate of convergence of an approximate solution to the exact one is exponential as well. We simplify the solution in the case when the loading is constant in section 5.

2. Formulation

The problem to be considered is a two-dimensional one for an infinite strip $S = \{-\infty < x < \infty,\ -b < y < b\}$, the upper and lower boundaries of which are free of loading, $\tau_{yz} = 0$, $-\infty < x < \infty,\ y = \pm b$. Along the symmetry line of the strip, there is a finite crack $\{0 \leqslant x \leqslant a,\ y = 0^\pm\}$, the sides of which are subject to a loading $\tau_{yz} = p(x)$, where $p(x)$ is a Hölder-continuous function. Everywhere in the strip S, apart from two thin layers $l_\delta^+ = \{0 \leqslant x \leqslant a,\ 0 < y < \delta\}$ and $l_\delta^- = \{0 \leqslant x \leqslant a,\ -\delta < y < 0\}$, the z-component, w, of the displacement vector is a harmonic function. Inside those layers we have

$$\delta\mu_s \frac{d^2}{dx^2}\langle w\rangle^+ + \tau_{yz}|_{y=\delta} - \tau_{yz}|_{y=0^+} = 0, \tag{2.1}$$

$$\delta\mu_s \frac{d^2}{dx^2}\langle w\rangle^- + \tau_{yz}|_{y=0^-} - \tau_{yz}|_{y=-\delta} = 0, \quad 0 < x < a, \tag{2.2}$$

where μ_s is a positive surface parameter (the 'stiffness' of the layers $l_\delta^\pm$) that has to be measured experimentally, $\tau_{yz}(x, 0^\pm) = p(x)$, and $\langle w\rangle^\pm(x)$ are the average values of the displacement $w(x, y)$ taken over the intervals $[0, \delta]$ and $[-\delta, 0]$, respectively. These relations can be retrieved by implementing the following procedure. First, we solve the Laplace equation in the infinite strip $\{-\infty < x < \infty,\ 0 < y < \delta\}$ with the Neumann boundary conditions

$$\mu_s \frac{\partial w}{\partial y}(x, 0) = \tau_0(x), \quad \mu_s \frac{\partial w}{\partial y}(x, \delta) = \tau_1(x), \quad -\infty < x < \infty, \tag{2.3}$$

where $\tau_j(x) = 0$ if $x < 0$ or $x > a$. Then we find the average of the Fourier transform of the function w, $\langle w_\alpha\rangle$, over the interval $(0, \delta)$ as

$$\frac{1}{\delta}\int_0^\delta w_\alpha(y)dy = \frac{\tau_{1\alpha} - \tau_{0\alpha}}{\delta\mu_s\alpha^2}, \tag{2.4}$$

where $\tau_{0\alpha}$ and $\tau_{1\alpha}$ are the Fourier transforms of the functions $\tau_0(x)$ and $\tau_1(x)$. Finally, on applying the inverse Fourier transform we find

$$\delta\mu_s \frac{d^2}{dx^2}\langle w\rangle = \tau_0(x) - \tau_1(x), \quad -\infty < x < \infty. \tag{2.5}$$

This solution can be used to recover (2.1). The same derivations are valid for the lower layer. On the interfaces between the layers $l_\delta^\pm$ and the strip S, the displacements have to be continuous. Because the layers $l_\delta^\pm$ are very thin, we can write

$$\langle w\rangle^\pm = w(x, \pm\delta \pm 0), \quad 0 < x < a. \tag{2.6}$$

This can be rewritten as

$$\delta\mu_s \frac{\partial^2 w}{\partial x^2}(x, 0^+) + \mu \frac{\partial w}{\partial y}(x, 0^+) - p(x) = 0,$$

$$\delta\mu_s \frac{\partial^2 w}{\partial x^2}(x, 0^-) + p(x) - \mu \frac{\partial w}{\partial y}(x, 0^-) = 0, \quad 0 < x < a, \tag{2.7}$$

where μ is the shear modulus of the strip S.

Now, the condition (2.6) enable us to reduce the problem to the solution of the following discontinuous boundary value problem for the Laplace operator Δ in the strip S:

$$\Delta w(x, y) = 0, \quad (x, y) \in S \setminus \{0 \leqslant x \leqslant a,\ y = 0^\pm\}, \tag{2.8}$$

$$\left.\frac{\partial w}{\partial y}\right|_{y=\pm b} = 0, \quad -\infty < x < \infty, \tag{2.9}$$

$$\delta\mu_s \frac{\partial^2 w}{\partial x^2}(x, 0^\pm) \pm \left[\mu \frac{\partial w}{\partial y}(x, 0^\pm) - p(x)\right] = 0, \quad 0 < x < a, \tag{2.10}$$

$$w|_{y=0^+} - w|_{y=0^-} = \chi(x), \quad -\infty < x < \infty, \tag{2.11}$$

$$\frac{\partial w}{\partial y}(x, 0^+) - \frac{\partial w}{\partial y}(x, 0^-) = 0, \quad -\infty < x < \infty. \tag{2.12}$$

The displacement jump $\chi(x)$ is unknown for $0 < x < a$ and $\chi(x) = 0$ if $x \leqslant 0$ or $x \geqslant a$. Notice that another interpretation of the same interface conditions based on the results in (**11**) was recently given in (**14**). We emphasize that the conditions in (2.10) are different from the 'weak' interface conditions (**21**, **22**)

$$\mu_+^\circ \frac{\partial w}{\partial y}(x, 0^+) = \mu_-^\circ \frac{\partial w}{\partial y}(x, 0^-) = w(x, 0^+) - w(x, 0^-), \quad 0 < x < a, \tag{2.13}$$

and the 'stiff' interface conditions (**23**)

$$w(x, 0^+) = w(x, 0^-), \quad \mu_0^+ \frac{\partial w}{\partial y}(x, 0^+) - \mu_0^- \frac{\partial w}{\partial y}(x, 0^-) + \frac{\partial^2 w}{\partial x^2}(x, 0^+) = 0, \quad 0 < x < a.$$

The interface boundary conditions (2.10) require the continuity of the traction component τ_{yz}. The displacement w is discontinuous, and the second tangential derivatives of the limiting values of the displacement w, $\partial^2 w/\partial x^2(x, 0^+)$ and $\partial^2 w/\partial x^2(x, 0^-)$, are connected to the traction $\mu \partial w/\partial y(x, 0^\pm)$ by (2.10).

It will be convenient to introduce dimensionless coordinates, functions and parameters

$$\xi = \frac{x}{b}, \ \eta = \frac{y}{b}, \ \hat{w}(\xi, \eta) = \frac{w(b\xi, b\eta)}{b}, \ \hat{p}(\xi) = \frac{p(b\xi)}{\mu}, \ \hat{\chi}(\xi) = \frac{\chi(b\xi)}{b}, \ \hat{\mu}_s = \frac{\delta \mu_s}{b\mu}, \ \lambda = \frac{a}{b}.$$

Then we integrate (2.10), differentiate (2.11) and rewrite (2.8)–(2.12) in the form

$$\Delta \hat{w}(\xi, \eta) = 0, \quad \{-\infty < \xi < \infty, \ -1 < \eta < 1\} \setminus \{0 \leqslant \xi \leqslant 1, \ \eta = 0^\pm\}, \tag{2.14}$$

$$\left.\frac{\partial \hat{w}}{\partial \eta}\right|_{\eta = \pm 1} = 0, \quad -\infty < \xi < \infty, \tag{2.15}$$

$$\int_0^\xi \frac{\partial \hat{w}}{\partial \eta}(\xi_1, 0^\pm) \, d\xi_1 \pm \hat{\mu}_s \frac{\partial \hat{w}}{\partial \xi}(\xi, 0^\pm) = -p^\circ(\xi) + C, \quad 0 < \xi < \lambda, \tag{2.16}$$

$$\frac{\partial \hat{w}}{\partial \xi}(\xi, 0^+) - \frac{\partial \hat{w}}{\partial \xi}(\xi, 0^-) = \psi(\xi), \quad \frac{\partial \hat{w}}{\partial \eta}(\xi, 0^+) - \frac{\partial \hat{w}}{\partial \eta}(\xi, 0^-) = 0, \quad -\infty < \xi < \infty. \tag{2.17}$$

Here,

$$\psi(\xi) = \hat{\chi}'(\xi), \quad p^\circ(\xi) = \int_0^\xi \hat{p}(\xi_1) \, d\xi_1, \tag{2.18}$$

and C is a constant of integration to be determined from the condition

$$\int_0^\lambda \psi(\xi) \, d\xi = 0 \tag{2.19}$$

that follows from the fact that $\chi(0) = \chi(a) = 0$.

3. Integro-differential equation in a segment

The boundary value problem (2.14)–(2.17) may be further transformed by using the Fourier transform

$$\hat{w}_\alpha(\eta) = \int_{-\infty}^\infty \hat{w}(\xi, \eta) e^{i\alpha\xi} \, d\xi. \tag{3.1}$$

To be able to apply the Fourier transform to (2.16), we extend its definition to the whole real axis by introducing a new unknown function $f(\xi)$, such that $f(\xi) = 0$ if $0 < \xi < \lambda$. Then

$$\int_{-\infty}^{\xi} \frac{\partial \hat{w}}{\partial \eta}(\xi_1, 0^{\pm})\, d\xi_1 \pm \hat{\mu}_s \frac{\partial \hat{w}}{\partial \xi}(\xi, 0^{\pm}) = -p_0^{\circ}(\xi) + C_0(\xi) + f(\xi), \quad -\infty < \xi < \infty, \tag{3.2}$$

where

$$p_0^{\circ}(\xi) = \begin{cases} p^{\circ}(\xi), & 0 < \xi < \lambda, \\ 0, & \text{otherwise}, \end{cases} \qquad C_0(\xi) = \begin{cases} C, & 0 < \xi < \lambda, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

Applying the Fourier transform gives the following discontinuous boundary value problem

$$\frac{d^2 \hat{w}_\alpha}{d\eta^2} - \alpha^2 \hat{w}_\alpha = 0, \quad \{-1 < \eta < 1\} \setminus \{0\}, \tag{3.4}$$

$$\left.\frac{d\hat{w}_\alpha}{d\eta}\right|_{\eta=\pm 1} = 0, \tag{3.5}$$

$$-\frac{1}{i\alpha}\frac{d\hat{w}_\alpha}{d\eta}(0^{\pm}) \mp i\alpha\hat{\mu}_s \hat{w}_\alpha(0^{\pm}) = -p_\alpha^{\circ} + \frac{C}{i\alpha}(e^{i\alpha\lambda} - 1) + f_\alpha, \tag{3.6}$$

$$-i\alpha[\hat{w}_\alpha(0^+) - \hat{w}_\alpha(0^-)] = \psi_\alpha, \qquad \frac{d\hat{w}_\alpha}{d\eta}(0^+) - \frac{d\hat{w}_\alpha}{d\eta}(0^-) = 0. \tag{3.7}$$

Here, p_α°, f_α and ψ_α are the Fourier transforms of the functions $p_0^{\circ}(\xi)$, $f(\xi)$ and $\psi(\xi)$, respectively. The general solution of the differential equation can be written in the form:

$$\hat{w}_\alpha(\eta) = \begin{cases} C_1^+ \cosh\alpha(\eta - 1) + C_2^+ \sinh\alpha(\eta - 1), & 0 < \eta \leqslant 1 \\ C_1^- \cosh\alpha(\eta + 1) + C_2^- \sinh\alpha(\eta + 1), & -1 \leqslant \eta < 0, \end{cases} \tag{3.8}$$

where $C_1^{\pm}$ and $C_2^{\pm}$ are functions of α determined from (3.5) and (3.7) as

$$C_2^+ = C_2^- = 0, \quad C_1^+ = -C_1^- = -\frac{\psi_\alpha}{2i\alpha\cosh\alpha}. \tag{3.9}$$

This implies that $\hat{w}_\alpha(0^+) = -\hat{w}_\alpha(0^-)$ and therefore

$$\hat{w}(\xi, 0^+) = -\hat{w}(\xi, 0^-) = \frac{1}{2}\hat{\chi}(\xi). \tag{3.10}$$

To employ (3.6), we introduce the following notations:

$$\Psi^+(\alpha) = \psi_\alpha = \int_0^{\lambda} \psi(\xi) e^{i\alpha\xi}\, d\xi, \quad P^+(\alpha) = p_\alpha^{\circ} = \int_0^{\lambda} p^{\circ}(\xi) e^{i\alpha\xi}\, d\xi, \tag{3.11}$$

$$F^-(\alpha) = \int_{-\infty}^{0} f(\xi) e^{i\alpha\xi}\, d\xi, \quad F^+(\alpha) = \int_0^{\infty} f(\xi + \lambda) e^{i\alpha\xi}\, d\xi, \tag{3.12}$$

and obtain

$$\frac{\hat{\mu}_s}{2} G(\alpha)\Psi^+(\alpha) = F^-(\alpha) + e^{i\alpha\lambda}F^+(\alpha) - P^+(\alpha) + \frac{C}{i\alpha}(e^{i\alpha\lambda} - 1), \quad -\infty < \alpha < \infty, \tag{3.13}$$

where

$$G(\alpha) = 1 + \frac{\tanh\alpha}{\hat{\mu}_s \alpha}. \tag{3.14}$$

Notice that, because of the definitions (3.11) and (3.12), the functions $\Psi^+(\alpha)$, $F^+(\alpha)$ and $P^+(\alpha)$ are analytic in the upper half-plane $\mathbb{C}^+ = \{\operatorname{Im}\alpha > 0\}$, while the function $F^-(\alpha)$ is analytic in the lower half-plane $\mathbb{C}^- = \{\operatorname{Im}\alpha < 0\}$. Apply now the inverse Fourier transform to (3.13) in the segment $0 < \xi < \lambda$. This gives

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\tanh\alpha}{\alpha}\Psi^+(\alpha)e^{-i\alpha\xi}\,d\alpha + \frac{\hat{\mu}_s}{2}\psi(\xi) = -p^\circ(\xi) + C, \quad 0 < \xi < \lambda. \tag{3.15}$$

If we use the table integral

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh\alpha}{\alpha}e^{-i\alpha\xi}\,d\alpha = \ln\left|\coth\frac{\pi\xi}{4}\right| \tag{3.16}$$

and the convolution theorem, we obtain the integral equation

$$\frac{\hat{\mu}_s}{2}\psi(\xi) + \frac{1}{2\pi}\int_0^\lambda \ln\coth\frac{\pi|\tau-\xi|}{4}\psi(\tau)\,d\tau = -p^\circ(\xi) + C, \quad 0 < \xi < \lambda, \tag{3.17}$$

subject to the condition (2.19). By differentiating the left- and right-hand sides of (3.17) with respect to ξ we arrive at the singular integro-differential equation

$$\frac{\hat{\mu}_s}{2}\psi'(\xi) + \frac{1}{4}\int_0^\lambda \frac{\psi(\tau)\,d\tau}{\sinh\frac{\pi}{2}(\tau-\xi)} = -p(\xi), \quad 0 < \xi < \lambda. \tag{3.18}$$

This equation is solvable up to an arbitrary constant, and since the constant C in (3.17) is arbitrary, (3.17) and (3.18) are equivalent. In what follows we use the form (3.17) since it is more convenient for the method to be proposed.

On satisfying the condition (2.19) it is possible to express the constant C through the solutions of the two integral equations

$$\frac{\hat{\mu}_s}{2}\psi_m(\xi) + \frac{1}{2\pi}\int_0^\lambda \ln\coth\frac{\pi|\tau-\xi|}{4}\psi_m(\tau)\,d\tau = p_m(x), \quad 0 < \xi < \lambda, \quad m = 0, 1, \tag{3.19}$$

as

$$C = -\int_0^\lambda \psi_0(\xi)\,d\xi\left(\int_0^\lambda \psi_1(\xi)\,d\xi\right)^{-1}. \tag{3.20}$$

Here,

$$\psi(\xi) = \psi_0(\xi) + C\psi_1(\xi), \quad p_0(\xi) = -p^\circ(\xi), \quad p_1(\xi) = 1. \tag{3.21}$$

An alternative derivation of the integral equation (3.17) is given in Appendix A.

4. Vector Riemann–Hilbert problem

In this section we shall reduce the boundary value problem (2.14)–(2.17) or, equivalently, the integral equation (3.17), to a vector Riemann–Hilbert problem and solve it.

Equation (3.13) can be considered as the first boundary condition of a vector Riemann–Hilbert problem. After some rearrangement there is obtained another boundary condition

$$\frac{\hat{\mu}_s}{2}G(\alpha)\Psi^-(\alpha) = e^{-i\alpha\lambda}F^-(\alpha) + F^+(\alpha) - P^-(\alpha) + \frac{C}{i\alpha}(1 - e^{-i\alpha\lambda}), \quad -\infty < \alpha < \infty. \quad (4.1)$$

Here,

$$\Psi^-(\alpha) = \int_{-\lambda}^{0} \psi(\xi+\lambda)e^{i\alpha\xi}\,d\xi, \quad P^-(\alpha) = \int_{-\lambda}^{0} p_0^{\circ}(\xi+\lambda)e^{i\alpha\xi}\,d\xi. \quad (4.2)$$

The functions $\Psi^-(\alpha)$ and $P^-(\alpha)$ are analytic in the half-plane $\mathbb{C}^-$. To solve the Riemann–Hilbert problem, we need to determine two vectors $\{\Psi^+(\alpha), F^+(\alpha)\}^\top$ and $\{\Psi^-(\alpha), F^-(\alpha)\}^\top$ analytic in the upper and lower half planes, respectively, continuous up to the boundary $L = \{-\infty < \operatorname{Re}\alpha < \infty,\ \operatorname{Im}\alpha = 0\}$ and whose limit values on the contour L satisfy the boundary relations (3.13) and (4.1).

First, we factorize the function $G(\alpha)$ as

$$G(\alpha) = \frac{X^+(\alpha)}{X^-(\alpha)}, \quad \alpha \in L, \quad (4.3)$$

where

$$X(\alpha) = \exp\left\{\frac{\alpha}{\pi i}\int_0^\infty \frac{\ln G(\beta)\,d\beta}{\beta^2-\alpha^2}\right\}, \quad \alpha \in \mathbb{C}\setminus L, \quad (4.4)$$

and

$$X^\pm(\alpha) = \exp\left\{\pm\frac{\ln G(\alpha)}{2} + \frac{1}{\pi i}\mathrm{PV}\int_0^\infty \frac{\alpha\ln G(\beta)\,d\beta}{\beta^2-\alpha^2}\right\}, \quad \alpha \in L, \quad (4.5)$$

where $\mathbb{C}$ is the complex plane. By substituting (4.3) into (3.13) and (4.1) and eliminating the unknown constant C, a more convenient form of the boundary conditions may be obtained

$$\frac{\hat{\mu}_s}{2}X^+(\alpha)\Psi_m^+(\alpha) - \frac{e^{i\alpha\lambda}X^+(\alpha)F_m^+(\alpha)}{G(\alpha)} = X^-(\alpha)F_m^-(\alpha) + X^-(\alpha)P_m^+(\alpha),$$

$$\frac{\hat{\mu}_s\Psi_m^-(\alpha)}{2X^-(\alpha)} - \frac{e^{-i\alpha\lambda}F_m^-(\alpha)}{X^-(\alpha)G(\alpha)} = \frac{F_m^+(\alpha)}{X^+(\alpha)} + \frac{P_m^-(\alpha)}{X^+(\alpha)}, \quad \alpha \in L, \quad (4.6)$$

where

$$\Psi^\pm(\alpha) = \Psi_0^\pm(\alpha) + C\Psi_1^\pm(\alpha), \quad F^\pm(\alpha) = F_0^\pm(\alpha) + CF_1^\pm(\alpha),$$

$$P_0^\pm(\alpha) = -P^\pm(\alpha), \quad P_1^\pm(\alpha) = \pm\frac{e^{\pm i\alpha\lambda}-1}{i\alpha}, \quad m = 0, 1. \quad (4.7)$$

Next, we use the Sokhotski–Plemelj formulas in order to split the functions $X^-(\alpha)P_m^+(\alpha)$ and $P_m^-(\alpha)/X^+(\alpha)$ into the '+' and '−' parts as

$$X^-(\alpha)P_m^+(\alpha) = \Lambda_{m-}^+(\alpha) - \Lambda_{m-}^-(\alpha),$$

$$\frac{P_m^-(\alpha)}{X^+(\alpha)} = \Lambda_{m+}^+(\alpha) - \Lambda_{m+}^-(\alpha), \quad \alpha \in L, \quad (4.8)$$

where

$$\Lambda_{m\pm}(\alpha) = \frac{1}{2\pi i}\int_L \frac{[X^\pm(\beta)]^{\mp 1} P_m^\mp(\beta)\, d\beta}{\beta - \alpha}, \quad \alpha \in \mathbb{C} \setminus L, \tag{4.9}$$

and remove the simple poles, $\pm i\beta_n$ $(n = 1, 2, \dots)$, of the function $1/G(\alpha)$ in (4.6) by subtracting from the left- and the right-hand sides of (4.6) the functions

$$\Omega_m^-(\alpha) = \sum_{n=1}^\infty \frac{i A_{mn}^-}{\alpha - i\beta_n}, \quad \Omega_m^+(\alpha) = \sum_{n=1}^\infty \frac{i A_{mn}^+}{\alpha + i\beta_n}. \tag{4.10}$$

Here, $\alpha = i\beta_n$ $(n = 1, 2, \dots)$ are the roots of the transcendental equation

$$1 + \frac{\tanh\alpha}{\hat{\mu}_s\alpha} = 0 \tag{4.11}$$

lying in the upper half plane. These roots can be computed by employing the following procedure of successive approximations

$$\beta_n^{(0)} = \pi n, \quad \beta_n^{(s)} = \pi n + \frac{1}{2i}\ln\frac{1 - i\hat{\mu}_s\beta_n^{(s-1)}}{1 + i\hat{\mu}_s\beta_n^{(s-1)}}, \quad s = 1, 2, \dots; \quad n = 1, 2, \dots, \tag{4.12}$$

where $i\beta_n^{(s)}$ is the sth approximation of the nth root $i\beta_n$ of (4.11). The single branch of the logarithmic function in (4.12) is fixed by the condition $\ln 1 = 0$. The coefficients A_{mn}^- and A_{mn}^+ will be fixed *a posteriori* such that

$$\operatorname*{res}_{s=i\beta_k}\left[\frac{e^{i\alpha\lambda}X^+(\alpha)F_m^+(\alpha)}{G(\alpha)} + \Omega_m^-(\alpha)\right] = 0, \tag{4.13}$$

$$\operatorname*{res}_{s=-i\beta_k}\left[\frac{e^{-i\alpha\lambda}F_m^-(\alpha)}{X^-(\alpha)G(\alpha)} + \Omega_m^+(\alpha)\right] = 0, \quad k = 1, 2, \dots; \quad m = 0, 1. \tag{4.14}$$

Now, the functions $\psi(\xi)$ and $f(\xi)$ must be integrable in the vicinity of the points $\xi = 0$ and $\xi = 1$. Therefore, by the Abelian theorem the integrals $\Psi^\pm(\alpha)$ and $F^\pm(\alpha)$ must vanish as $\alpha \to \infty$, $\alpha \in \mathbb{C}^\pm$. Consequent application of the continuity principle and the Liouville's theorem enables us to conclude that

$$\frac{\hat{\mu}_s}{2}X^+(\alpha)\Psi_m^+(\alpha) - \frac{e^{i\alpha\lambda}X^+(\alpha)F_m^+(\alpha)}{G(\alpha)} - \Omega_m^-(\alpha) - \Lambda_{m-}^+(\alpha)$$
$$= X^-(\alpha)F_m^-(\alpha) - \Omega_m^-(\alpha) - \Lambda_{m-}^-(\alpha) = 0, \quad \alpha \in \mathbb{C}, \tag{4.15}$$

$$\frac{\hat{\mu}_s\Psi_m^-(\alpha)}{2X^-(\alpha)} - \frac{e^{-i\alpha\lambda}F_m^-(\alpha)}{X^-(\alpha)G(\alpha)} - \Omega_m^+(\alpha) + \Lambda_{m+}^-(\alpha)$$
$$= \frac{F_m^+(\alpha)}{X^+(\alpha)} - \Omega_m^+(\alpha) + \Lambda_{m+}^+(\alpha) = 0, \quad \alpha \in \mathbb{C}, \quad m = 0, 1. \tag{4.16}$$

The expressions for $\Psi^{\pm}(\alpha)$ and $F^{\pm}(\alpha)$ can be easily retrieved from (4.15) and (4.16) as

$$F_m^-(\alpha) = \frac{\Omega_m^-(\alpha) + \Lambda_{m-}^-(\alpha)}{X^-(\alpha)}, \quad \alpha \in \mathbb{C}^-, \tag{4.17}$$

$$F_m^+(\alpha) = X^+(\alpha)[\Omega_m^+(\alpha) - \Lambda_{m+}^+(\alpha)], \quad \alpha \in \mathbb{C}^+, \tag{4.18}$$

$$\Psi_m^-(\alpha) = \frac{2}{\hat{\mu}_s} X^-(\alpha)[\Omega_m^+(\alpha) - \Lambda_{m+}^-(\alpha)] + \frac{2e^{-i\alpha\lambda} F_m^-(\alpha)}{\hat{\mu}_s G(\alpha)}, \quad \alpha \in \mathbb{C}^-, \tag{4.19}$$

$$\Psi_m^+(\alpha) = \frac{2}{\hat{\mu}_s X^+(\alpha)}[\Omega_m^-(\alpha) + \Lambda_{m-}^+(\alpha)] + \frac{2e^{i\alpha\lambda} F_m^+(\alpha)}{\hat{\mu}_s G(\alpha)}, \quad \alpha \in \mathbb{C}^+, \quad m = 0, 1. \tag{4.20}$$

We would have written that these formulas completely define the solution to the Riemann–Hilbert problem if they had not possessed the undetermined coefficients $A_{mk}^{\pm}$. These coefficients must be recovered from (4.13) and (4.14) that guarantee that the points $\pm i\beta_k$ $(k = 1, 2, \ldots)$ are removable points for the functions $\Psi_m^{\pm}(\alpha)$. Substituting the expressions for $F_m^{\pm}(\alpha)$ into (4.13) and (4.14) gives the following system for the coefficients $A_{mk}^{\pm}$:

$$A_{mk}^- + e^{-\beta_k\lambda} d_k \left(\sum_{n=1}^{\infty} \frac{A_{mn}^+}{\beta_n + \beta_k} - g_{mk}^+ \right) = 0, \tag{4.21}$$

$$A_{mk}^+ + e^{-\beta_k\lambda} d_k \left(\sum_{n=1}^{\infty} \frac{A_{mn}^-}{\beta_n + \beta_k} - g_{mk}^- \right) = 0, \tag{4.22}$$

where $k = 1, 2, \ldots, m = 0, 1,$

$$d_k = \frac{X_k^2}{v_k}, \quad X_k = X^+(i\beta_k) = \exp\left\{ \frac{\beta_k}{\pi} \int_0^{\infty} \frac{\ln G(\beta)\, d\beta}{\beta^2 + \beta_k^2} \right\}, \tag{4.23}$$

$$g_{mk}^{\pm} = \Lambda_{m\pm}^{\pm}(\pm i\beta_k), \quad v_k = \frac{1}{\hat{\mu}_s \beta_k}\left(1 + \tan^2\beta_k - \frac{\tan\beta_k}{\beta_k}\right). \tag{4.24}$$

It is seen that since $\lambda = a/b > 0$, $\beta_k > 0$, and $\beta_k \sim \pi k$ as $k \to \infty$, the rate of convergence of an approximate solution of the infinite system to the exact one is exponential. That is why, regardless of the method for the numerical solution chosen, whether it is the method of reduction, the method of successive approximations, or the method of asymptotic expansions, it will be very sufficient. Formula (3.20) for the constant C can be expressed through the solution of the vector Riemann–Hilbert problem as

$$C = -\frac{\Psi_0^+(0)}{\Psi_1^+(0)}. \tag{4.25}$$

5. Particular case $p(x) = \text{constant}$

In the particular case, $p(x) = p = \text{constant}$, $0 < x < a$, the solution may be simplified by evaluating the integrals $p^{\circ}(\xi)$, $P^+(\alpha)$ and $\Lambda_{m\pm}(\alpha)$.

5.1 *Solution of the Riemann–Hilbert problem*

Since $p(x) = p$ we immediately have $\hat{p} = p/\mu$, $p^\circ(\xi) = \hat{p}\xi$, and the integrals $P_0^\pm(\alpha)$ defined by (4.7) and (3.11) can be computed in terms of elementary functions

$$P_0^+(\alpha) = -\frac{\hat{p}}{i\alpha}\left[\lambda e^{i\alpha\lambda} - \frac{e^{i\alpha\lambda}-1}{i\alpha}\right], \quad P_0^-(\alpha) = -\frac{\hat{p}}{i\alpha}\left[\lambda - \frac{1-e^{-i\alpha\lambda}}{i\alpha}\right]. \tag{5.1}$$

It is directly verified that

$$P_0^+(\alpha) \sim P_0^-(\alpha) \sim -\hat{p}\lambda^2/2, \quad \alpha \to 0. \tag{5.2}$$

We begin with the case $m = 0$ and aim to find the splitting (4.8) of the functions $X^-(\alpha)P_m^+(\alpha)$ and $P_m^-(\alpha)/X^+(\alpha)$ that is free of integrals (4.9). For this we need to derive the two-term asymptotic relations

$$X^\pm(\alpha) = X^\pm(0) + \alpha\frac{dX^\pm}{d\alpha}(0) + O(\alpha^2), \quad \alpha \to 0. \tag{5.3}$$

By the Sokhotski–Plemelj formulas,

$$X^\pm(0) = e^{\pm\frac{1}{2}\ln G(0)} = \gamma_0^{\pm 1}, \tag{5.4}$$

where $\gamma_0 = \sqrt{1 + 1/\hat{\mu}_s}$. To find the second term of the asymptotic expansion in (5.3), first, we represent the derivative $(d/d\alpha)X(\alpha)$ as

$$\frac{d}{d\alpha}X(\alpha) = -X(\alpha)\frac{1}{2\pi i}\frac{d}{d\alpha}\int_L \ln|\beta - \alpha|\frac{G'(\beta)}{G(\beta)}\,d\beta. \tag{5.5}$$

Here, we integrated by parts and used the fact that $G(\beta) = 1 + O(\beta^{-1})$ as $\beta \to \pm\infty$. Next, we employ the Sokhotski–Plemelj formulas again. Since $G'(\beta) = O(\beta)$ as $\beta \to 0$, we finally obtain

$$X^\pm(\alpha) = \gamma_0^{\pm 1}(1 - i\gamma_1\alpha) + O(\alpha^2), \quad \alpha \to 0, \tag{5.6}$$

where

$$\gamma_1 = \frac{1}{\pi}\int_0^\infty \frac{G'(\beta)\,d\beta}{\beta G(\beta)}. \tag{5.7}$$

Now, for the case $m = 0$, instead of (4.15) and (4.16), a new arrangement is possible, namely,

$$\begin{aligned}&\frac{\hat{\mu}_s}{2}X^+(\alpha)\Psi_0^+(\alpha) - \frac{e^{i\alpha\lambda}X^+(\alpha)}{G(\alpha)}\left[F_0^+(\alpha) + \frac{\hat{p}(i\alpha\lambda - 1)}{\alpha^2}\right] - \Omega_0^-(\alpha) + \frac{\hat{p}(i\alpha\gamma_1 - 1)}{\gamma_0\alpha^2}\\ &\quad = X^-(\alpha)\left[F_0^-(\alpha) + \frac{\hat{p}}{\alpha^2}\right] - \Omega_0^-(\alpha) + \frac{\hat{p}(i\alpha\gamma_1 - 1)}{\gamma_0\alpha^2} = 0, \quad \alpha \in \mathbb{C},\end{aligned} \tag{5.8}$$

$$\begin{aligned}&\frac{\hat{\mu}_s\Psi_0^-(\alpha)}{2X^-(\alpha)} - \frac{e^{-i\alpha\lambda}}{X^-(\alpha)G(\alpha)}\left[F_0^-(\alpha) + \frac{\hat{p}}{\alpha^2}\right] - \Omega_0^+(\alpha) + \frac{\hat{p}[1 + i\alpha(\gamma_1 - \lambda)]}{\gamma_0\alpha^2}\\ &\quad = \frac{1}{X^+(\alpha)}\left[F_0^+(\alpha) + \frac{\hat{p}(i\alpha\lambda - 1)}{\alpha^2}\right] - \Omega_0^+(\alpha) + \frac{\hat{p}[1 + i\alpha(\gamma_1 - \lambda)]}{\gamma_0\alpha^2} = 0, \quad \alpha \in \mathbb{C}.\end{aligned} \tag{5.9}$$

On analyzing these equations as $\alpha \to 0$ and using the asymptotic relations (5.6), we can show that the left- and right-hand sides of (5.8) and (5.9) are not singular at the point $\alpha = 0$. As in section 4, the Liouville theorem enables us to write down the solution in the form (4.17)–(4.20) with

$$\Lambda_{0-}^{-}(\alpha) = -\frac{\hat{p}}{\alpha^2}\left[X^{-}(\alpha) + \frac{i\alpha\gamma_1 - 1}{\gamma_0}\right], \tag{5.10}$$

$$\Lambda_{0-}^{+}(\alpha) = \frac{\hat{p}}{\alpha^2}\left[\frac{e^{i\alpha\lambda}X^{+}(\alpha)}{G(\alpha)}(i\alpha\lambda - 1) - \frac{i\alpha\gamma_1 - 1}{\gamma_0}\right], \tag{5.11}$$

$$\Lambda_{0+}^{-}(\alpha) = -\frac{\hat{p}}{\alpha^2}\left[\frac{e^{-i\alpha\lambda}}{X^{-}(\alpha)G(\alpha)} - \frac{i\alpha(\gamma_1 - \lambda) + 1}{\gamma_0}\right], \tag{5.12}$$

$$\Lambda_{0+}^{+}(\alpha) = \frac{\hat{p}}{\alpha^2}\left[\frac{i\alpha\lambda - 1}{X^{+}(\alpha)} + \frac{i\alpha(\gamma_1 - \lambda) + 1}{\gamma_0}\right]. \tag{5.13}$$

We emphasize that these functions are not related to the singular integrals (4.9). The functions $\Lambda_{0\pm}^{\pm}(\alpha)$, given by (5.10)–(5.13), are analytic everywhere in the half-plane $\mathbb{C}^{\pm}$ and continuous up to the whole real axis apart from the point $\alpha = 0$. The other two functions in (5.8) and (5.9), $\Lambda_{0\mp}^{\pm}(\alpha)$, are analytic everywhere in the half-plane $\mathbb{C}^{\pm}$ apart from the points $\alpha = \pm i\beta_n$ $(n = 1, 2, \ldots)$ and continuous up to the whole real axis apart from the point $\alpha = 0$. However, the solution of the Riemann–Hilbert problem (3.13), (4.1), the functions $\Psi_0^{\pm}(\alpha)$ and $F_0^{\pm}(\alpha)$, are analytic everywhere in $\mathbb{C}^{\pm}$ and continuous up to the whole real axis.

The case $m = 1$ is treated similarly. The final formulas for the solution coincide with (4.17)–(4.20), where the following expressions for $\Lambda_{1\pm}^{+}(\alpha)$ and $\Lambda_{1\pm}^{-}(\alpha)$ need to be taken:

$$\Lambda_{1-}^{-}(\alpha) = \frac{1}{i\alpha}\left[X^{-}(\alpha) - \frac{1}{\gamma_0}\right], \qquad \Lambda_{0-}^{+}(\alpha) = \frac{1}{i\alpha}\left[\frac{e^{i\alpha\lambda}X^{+}(\alpha)}{G(\alpha)} - \frac{1}{\gamma_0}\right],$$

$$\Lambda_{0+}^{-}(\alpha) = \frac{1}{i\alpha}\left[\frac{e^{-i\alpha\lambda}}{X^{-}(\alpha)G(\alpha)} - \frac{1}{\gamma_0}\right], \qquad \Lambda_{0+}^{+}(\alpha) = \frac{1}{i\alpha}\left[\frac{1}{X^{+}(\alpha)} - \frac{1}{\gamma_0}\right].$$

The coefficients $A_{mk}^{\pm}$ solve the infinite system (4.21) and (4.22), where $d_k^{\pm}$ are defined in (4.23), while $g_{mk}^{\pm}$ should be replaced by

$$g_{0k}^{+} = -\frac{\hat{p}[1 - \beta_k(\gamma_1 - \lambda)]}{\beta_k^2\gamma_0}, \quad g_{0k}^{-} = \frac{\hat{p}(\beta_k\gamma_1 - 1)}{\beta_k^2\gamma_0}, \quad g_{1k}^{+} = \frac{1}{\beta_k\gamma_0}, \quad g_{1k}^{-} = -\frac{1}{\beta_k\gamma_0}. \tag{5.14}$$

Since all the numbers $g_{km}^{\pm}$ are real, the coefficients $A_{mk}^{\pm}$ are also real. From the analysis of the system (4.21) and (4.22), it becomes evident that the coefficients $A_{mk}^{\pm}$ decay exponentially as $k \to \infty$ (Table 1).

5.2 *Constant C*

The solution derived possesses the constant C that has to be fixed by satisfying the condition (4.25). To compute the value $\Psi_0^{+}(\alpha)$ at $\alpha = 0$, we need the three-term asymptotic expansion of the function $X(\alpha)$ as $\alpha \to 0$, $\alpha \in \mathbb{C}^{+}$

$$X^{+}(\alpha) = X^{+}(0) + \alpha\frac{dX^{+}(0)}{d\alpha} + \frac{\alpha^2}{2}\frac{d^2X^{+}(0)}{d\alpha^2} + O(\alpha^3), \quad \alpha \to 0. \tag{5.15}$$

Table 1 The values of the coefficients A^-_{0k}, A^+_{0k}, A^-_{1k}, and A^+_{1k} for $k = 1, 2, 3, 4$ in the case $\hat{\mu}_s = 0.1$, $\lambda = \hat{p} = 1$

k	A^-_{0k}	A^+_{0k}	A^-_{1k}	A^+_{1k}
1	-1.01190×10^{-2}	-4.08808×10^{-3}	6.03091×10^{-3}	-6.03091×10^{-3}
2	-3.0265×10^{-4}	-1.00407×10^{-4}	2.02248×10^{-4}	-2.02248×10^{-4}
3	-9.74957×10^{-6}	-2.95462×10^{-6}	6.79495×10^{-6}	-6.79495×10^{-6}
4	-3.17480×10^{-7}	-9.13090×10^{-8}	2.26171×10^{-7}	-2.26171×10^{-7}

Notice that the first two terms are defined by (5.6). To recover the third, we integrate by parts to obtain

$$\frac{d^2X(\alpha)}{d\alpha^2} = X(\alpha)\left[\left(\frac{1}{2\pi i}\int_L \frac{G'(\beta)\,d\beta}{G(\beta)(\beta-\alpha)}\right)^2 + \frac{1}{2\pi i}\int_L \left(\frac{G'(\beta)}{G(\beta)}\right)'\frac{d\beta}{\beta-\alpha}\right]. \tag{5.16}$$

Here,

$$\left(\frac{G'(\beta)}{G(\beta)}\right)' = \frac{G''(\beta)}{G(\beta)} - \left(\frac{G'(\beta)}{G(\beta)}\right)^2. \tag{5.17}$$

It is directly verified that $[G'(\beta)/G(\beta)]'$ is an even function bounded at the point $\beta = 0$:

$$\left(\frac{G'(\beta)}{G(\beta)}\right)' \sim -\frac{2}{3(\hat{\mu}_s+1)}, \quad \beta \to 0, \tag{5.18}$$

and it vanishes as constant $\times |\beta|^{-3}$ when $\beta \to \pm\infty$. Therefore, on employing the Sokhotski-Plemelj formulas, we have

$$\frac{d^2X^+(0)}{d\alpha^2} = \gamma_0\left(-\gamma_1^2 - \frac{1}{3(\hat{\mu}_s+1)}\right). \tag{5.19}$$

Now we can evaluate $\Psi_0^+(0)$. It follows from (4.17)–(4.20) and (5.10)–(5.13) that

$$\Psi_0^+(0) = \frac{2}{\mu_s\gamma_0}\left[\Omega_0^-(0) + \Omega_0^+(0) - \hat{p}(I_0 + I_1)\right], \tag{5.20}$$

where

$$I_0 = \lim_{\alpha\to 0}\frac{1}{\alpha^2}\left(\frac{i\alpha\gamma_1 - 1}{\gamma_0} - \frac{e^{i\alpha\lambda}X^+(\alpha)(i\alpha\lambda - 1)}{G(\alpha)}\right),$$

$$I_1 = \lim_{\alpha\to 0}\frac{1}{\alpha^2}\left(\frac{i\alpha(\gamma_1 - \lambda) + 1}{\gamma_0} + \frac{i\alpha\lambda - 1}{X^+(\alpha)}\right).$$

These limits can be computed if we use the following asymptotic expansions as $\alpha \to 0$:

$$e^{i\alpha\lambda} = 1 + i\alpha\lambda - \frac{\alpha^2\lambda^2}{2} + O(\alpha^3), \quad G(\alpha) = \gamma_0^2 - \frac{\alpha^2}{3\hat{\mu}_s} + O(\alpha^4),$$

$$X^+(\alpha) = \gamma_0 \left[1 - i\alpha\gamma_1 + \frac{\alpha^2}{2}\left(-\gamma_1^2 - \frac{1}{3(\hat{\mu}_s + 1)}\right)\right] + O(\alpha^3).$$

Finally, this gives

$$I_0 = \frac{1}{2\gamma_0}\left(\lambda^2 - \gamma_1^2 + \frac{1}{3(\hat{\mu}_s + 1)}\right), \quad I_1 = \frac{1}{2\gamma_0}\left(\gamma_1^2 - 2\lambda\gamma_1 - \frac{1}{3(\hat{\mu}_s + 1)}\right). \tag{5.21}$$

Hence the value $\Psi_0^+(0)$ needed becomes

$$\Psi_0^+(0) = \frac{2}{\hat{\mu}_s\gamma_0}\left[\Omega_0^-(0) + \Omega_0^+(0) - \frac{\hat{p}\lambda\lambda_0}{2}\right], \tag{5.22}$$

where $\lambda_0 = (\lambda - 2\gamma_1)/\gamma_0$. A similar procedure applied to the function $\Psi_1^+(\alpha)$ yields

$$\Psi_1^+(0) = \frac{2}{\hat{\mu}_s\gamma_0}[\Omega_1^-(0) + \Omega_1^+(0) + \lambda_0]. \tag{5.23}$$

The condition (4.25) determines the constant C as

$$C = -\frac{\Omega_0^-(0) + \Omega_0^+(0) - \frac{1}{2}\hat{p}\lambda\lambda_0}{\Omega_1^-(0) + \Omega_1^+(0) + \lambda_0}. \tag{5.24}$$

Numerical results based on this solution and implemented for different sets $\{\hat{p}, \lambda, \hat{\mu}_s\}$ of the dimensionless parameters of the problem show that the constant C is practically independent of the surface parameter $\hat{\mu}_s$ and with a good accuracy depends on the other two parameters in a very simple way, namely, $C \approx \frac{1}{2}\hat{p}\lambda$.

5.3 *Function $\psi(\xi)$ and the crack shape*

The recovery of the solution $\psi(\xi) = \psi_0(\xi) + C\psi_1(\xi)$ of the integral equation (3.17) is standard. It is based on the inversion of the Fourier transform and application of the Cauchy theorem. For this, it is useful to rearrange the expression for the function $\psi_0(\xi)$ as

$$\psi_0(\xi) = \frac{1}{\pi\hat{\mu}_s}[J_-(\xi) + J_+(\xi)], \tag{5.25}$$

where

$$J_-(\xi) = \int_L \left[\Omega_0^-(\alpha) - \frac{\hat{p}(i\alpha\gamma_1 - 1)}{\alpha^2\gamma_0}\right] \frac{e^{-i\alpha\xi}}{G(\alpha)X^-(\alpha)}\, d\alpha,$$

$$J_+(\xi) = \int_L \left[\Omega_0^+(\alpha) - \frac{\hat{p}[i\alpha(\gamma_1 - \lambda) + 1]}{\alpha^2\gamma_0}\right] \frac{X^+(\alpha)e^{i\alpha(\lambda-\xi)}}{G(\alpha)}\, d\alpha.$$

On computing the residues of the integrands in $J_{\pm}(\xi)$ at the poles $\pm i\beta_k$ $(k = 1, 2, \ldots)$, we derive a series representation of the function $\psi_0(\xi)$. The same procedure can be applied to the function $\psi_1(\xi)$. On introducing new notations

$$\omega^{\pm}_{mk} = \sum_{n=1}^{\infty} \frac{A^{\pm}_{mn}}{\beta_n + \beta_k}, \quad m = 0, 1, \tag{5.26}$$

we may write

$$\psi_m(\xi) = -\frac{2}{\hat{\mu}_s} \sum_{k=1}^{\infty} \frac{X_k}{v_k} \left[(-\omega^{-}_{mk} + g^{-}_{mk}) e^{-\beta_k \xi} + (\omega^{+}_{mk} - g^{+}_{mk}) e^{-\beta_k(\lambda - \xi)} \right], \quad 0 \leqslant \xi \leqslant \lambda. \tag{5.27}$$

Here, X_k, v_k and $g^{\pm}_{mk}$ are the coefficients given by (4.23), (4.24) and (5.14). Since $A_{mk} = O(e^{-\beta_k \lambda})$ as $k \to \infty$, the series (5.26) are rapidly convergent. Next, show that the series (5.27) converge uniformly in the segment $[0, \lambda]$. To do this, we need to estimate the coefficients v_k as $k \to \infty$. Since $i\beta_k$ are roots of (4.11), we have the identity

$$\beta_k = \pi \left(k - \frac{1}{2} \right) + \frac{1}{2i} \ln \frac{1 - (\hat{\mu}_s i \beta_k)^{-1}}{1 + (\hat{\mu}_s i \beta_k)^{-1}}, \quad k = 1, 2, \ldots, \tag{5.28}$$

where $\ln 1 = 0$. This formula implies

$$\beta_k \sim \pi \left(k - \frac{1}{2} \right) + \frac{1}{\pi \hat{\mu}_s k}, \quad k \to \infty. \tag{5.29}$$

Analysis of the coefficient v_k from (5.14) shows that

$$v_k = \pi \hat{\mu}_s k + O(k^{-1}), \quad k \to \infty. \tag{5.30}$$

The coefficient X_k are bounded, $X_k \sim 1$, $\omega^{\pm}_{mk} = O(k^{-1})$, and $g^{\pm}_{mk} = O(k^{-1})$ as $k \to \infty$. Therefore, at the end points, $\xi = 0$ and $\xi = \lambda$, the series coefficients in (5.27) are of order k^{-2} as $k \to \infty$, and the series are convergent. In the interval $(0, \lambda)$, the series coefficients decay exponentially.

The function $\psi(\xi)$ is plotted in Fig. 1 for $\hat{\mu}_s = 0.1$, 0.5, and 1 (in all these cases, $\hat{p} = \lambda = 1$). It is seen that the function is monotonically decaying and bounded at the ends. Figure 2 shows the effect on the value $\psi(\xi)$ at the point $\xi = 0$ ($\psi(\lambda) = -\psi(0)$) of the parameter $\hat{\mu}_s$. It is seen that as the surface parameter $\hat{\mu}_s \to 0$, $\psi(0) \to +\infty$. In the case $\hat{\mu}_s = 0$, the integral equation coincides with the classical equation of fracture mechanics (when the surface effect is ignored)

$$\frac{1}{2\pi} \int_0^{\lambda} \ln \coth \frac{\pi |\tau - \xi|}{4} \psi(\tau)\, d\tau = -p^{\circ}(\xi) + C, \quad 0 < \xi < \lambda, \tag{5.31}$$

whose solution has the square root singularity as $\xi \to 0^+$ and $\lambda - 0$. The surface tension effect becomes stronger for longer cracks.

Asymptotics of the function $X^+(\alpha)$ as $\alpha \to \infty$, $\alpha \in \mathbb{C}^+$ (**22**)

$$X^+(\alpha) \sim 1 - \frac{\ln \alpha}{\pi i \alpha \hat{\mu}_s} + O\left(\frac{1}{\alpha}\right), \tag{5.32}$$

enable us to derive asymptotic relations for the functions $\Psi^+_m(\alpha)$ for large α. If $|\alpha - i\beta_n| > \varepsilon$, $n = 1, 2, \ldots$, $\varepsilon > 0$, $\alpha \to \infty$, and $\alpha \in \mathbb{C}^+$, then

$$\Psi^+_m(\alpha) \sim \frac{2}{\hat{\mu}_s} \left(1 + \frac{\ln \alpha}{\pi i \alpha \hat{\mu}_s} \right) \frac{i e^+_m}{\alpha}, \tag{5.33}$$

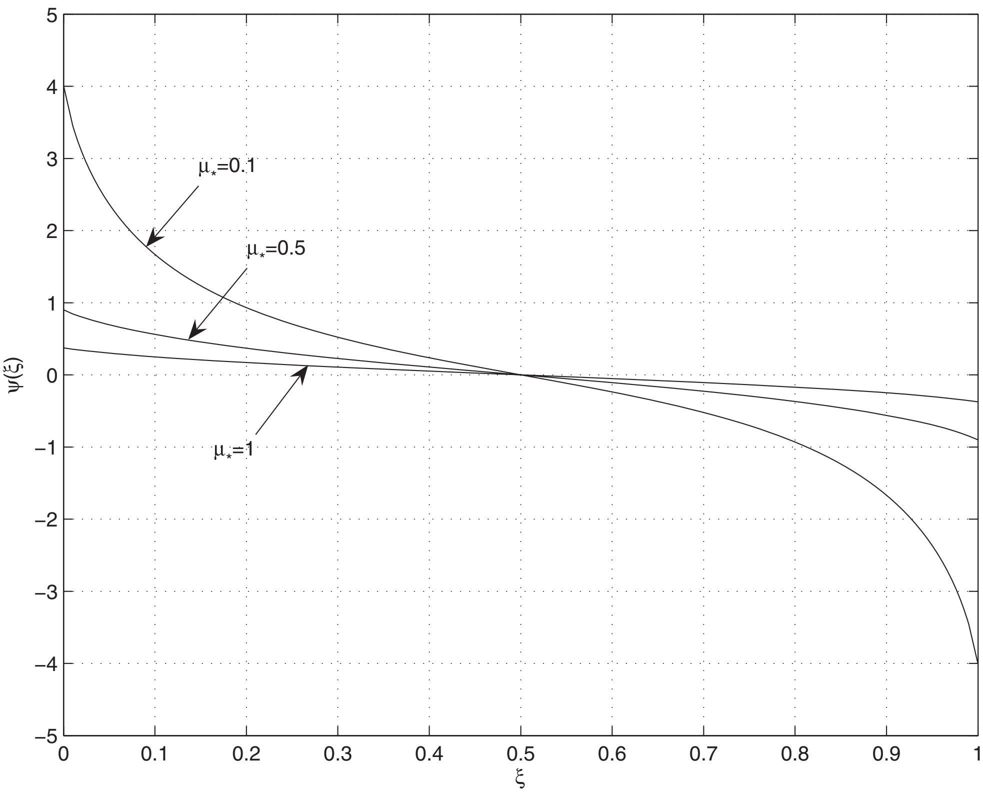

Fig. 1 The function $\psi(\xi) = \hat{\chi}'(\xi)$ for some values of the surface parameter $\hat{\mu}_s = \mu_*$ when $\hat{p} = 1$ and $\lambda = 1$

where

$$e_m^+ = \sum_{n=1}^{\infty} A_{mn}^- + \omega_m^+, \quad \omega_0^+ = -\frac{\hat{p}\gamma_1}{\gamma_0}, \quad \omega_1^+ = \frac{1}{\gamma_0}. \tag{5.34}$$

A similar relation can be obtained for the functions $\Psi_m^-(\alpha)$. On employing the Abelian theorems for the Fourier transform we derive the asymptotic relations at the ending points of the solution to the integro-differential equation (3.18)

$$\psi(\xi) \sim K + K_0\xi \ln \xi, \qquad \xi \to 0^+,$$

$$\psi(\xi) \sim -K + K_\lambda(\lambda - \xi)\ln(\lambda - \xi), \qquad \xi \to \lambda - 0,$$

where $K = \psi_0(0) + C\psi_1(0)$, $\psi_0(0)$ and $\psi_1(0)$ are defined by (5.27), and K_0 and K_λ are some real nonzero constants.

The use of (5.27) and integration of the function $\psi(\xi) = \psi_0(\xi) + C\psi_1(\xi)$ enables us to find the displacement jump, $\hat{\chi}(\xi)$

$$\hat{\chi}(\xi) = \theta_0(\xi) + C\theta_1(\xi) - \theta_0(0) - C\theta_1(0), \quad 0 \leqslant \xi \leqslant \lambda, \tag{5.35}$$

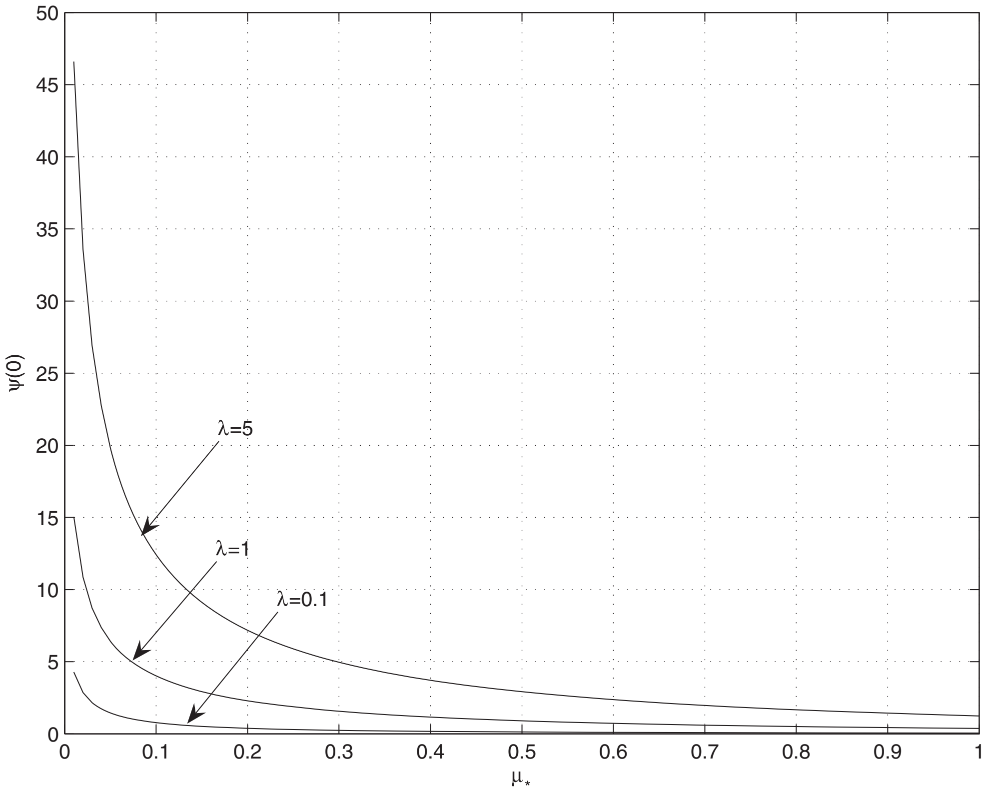

Fig. 2 The value $\psi(0)$ $(\psi(\lambda) = -\psi(0))$ as a function of the surface parameter $\hat{\mu}_s = \mu_*$ for some values of λ when $\hat{p} = 1$

where

$$\theta_m(\xi) = -\frac{2}{\hat{\mu}_s}\sum_{k=1}^{\infty}\frac{X_k}{v_k\beta_k}\left[(\omega_{mk}^- - g_{mk}^-)e^{-\beta_k\xi} + (\omega_{mk}^+ - g_{mk}^+)e^{-\beta_k(\lambda-\xi)}\right], \quad m = 0, 1. \tag{5.36}$$

The function $\frac{1}{2}\hat{\chi}(\xi)$, the upper dimensionless profile of the crack, is plotted in Fig. 3 for $\hat{\mu}_s = 0.1$, 0.5 and 1 when $\hat{p} = \lambda = 1$. It is seen that the surface tension decreases the gap between the upper and the lower sides of the crack.

6. Conclusions

We have analyzed the problem of a finite mode-III crack in a strip that incorporates surface effects. By using the asymptotic analysis and averaging the solution for an infinite strip, we have given an alternative derivation of the Gurtin–Murdoch interface boundary conditions. This procedure may be naturally extended to plane-strain and three-dimensional problems with surface effects. It has been

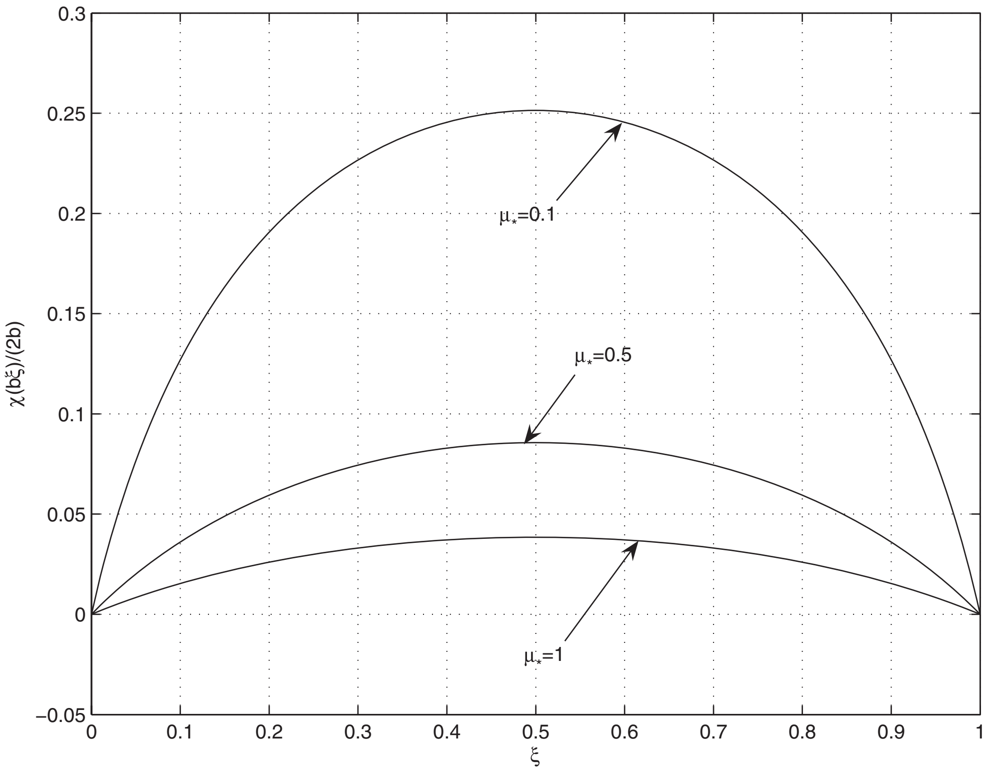

Fig. 3 The dimensionless profile of the upper side of the crack, $\frac{1}{2}\hat{\chi}(\xi)$, for some values of the surface parameter $\hat{\mu}_s = \mu_*$ when $\hat{p} = 1$ and $\lambda = 1$

shown that the problem is equivalent to a singular integro-differential equation in a finite segment $(0, \lambda)$. We have reduced this equation to a Riemann–Hilbert problem for two pairs of functions and found its solution as a sum of two functions. The first function, the principal part of the solution, was given by quadratures. The second function is of order $e^{-\beta_1\lambda}$ for large λ. Here, β_1 is the smallest positive root of the equation $1 + (\hat{\mu}_s\beta)^{-1}\tan\beta = 0$. This second part of the solution was found in a series form. The series converge exponentially, and their coefficients are the solution of a certain infinite system of linear algebraic equations. The rate of convergence of an approximate solution to the exact one is also exponential. We emphasize that for such singular integro-differential equations, the classical methods of orthogonal polynomials and collocations converge slowly.

We have determined the dimensionless profile of the crack, $\frac{1}{2}\hat{\chi}(\xi)$, and the derivative of the displacement jump, $\psi(\xi) = \hat{\chi}'(\xi)$. We have shown that the function $\psi(x/b)$ and, therefore, the stress $\tau_{yz}(x, 0)$ are bounded as x approaches the crack tips, $x = 0$ and $x = a$, from the interior and the exterior of the crack, respectively. If the thickness, δ, of the surface layers surrounding the crack tends to zero, then the values $|\psi(x/b)|$ and $|\tau_{yz}(x, 0)|$ at $x = 0$ and $x = a$ remain finite but grow. They become infinite if $\delta = 0$ as it happens in the classical theory without the surface effects.

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APPENDIX

An alternative derivation of the integral equation

The integral equation (3.17) was obtained in section 3 by using the Fourier integral transformation with respect to x under the assumption

$$\int_{-\infty}^{\xi} \frac{\partial \hat{w}}{\partial \eta}(\xi_1, 0^{\pm})\, d\xi_1 \to 0, \quad \xi \to +\infty. \tag{A.1}$$

It turns out that the method used naturally leads to the Riemann–Hilbert problem solved in section 4. Here, we propose an alternative derivation of the integral equation based on the finite integral transformation

$$\hat{w}_n(\xi) = \int_{-1}^{1} \hat{w}(\xi, \eta) \cos \frac{\pi n}{2}(\eta + 1)\, d\eta, \tag{A.2}$$

$$\hat{w}(\zeta, \eta) = \frac{\hat{w}_0(\xi)}{2} + \sum_{n=0}^{\infty} \hat{w}_n(\xi) \cos \frac{\pi n}{2}(\eta + 1), \tag{A.3}$$

that does not employ the condition (A.1). On applying the transform (A.2) to the discontinuous boundary value problem (2.14)–(2.17) we obtain

$$\hat{w}_n(\xi) = -\frac{1}{2} \sin \frac{\pi n}{2} \int_0^{\lambda} e^{-\frac{1}{2}\pi n|\xi - \tau|} \hat{\chi}(\tau)\, d\tau. \tag{A.4}$$

By integrating by parts this can be rewritten as

$$\hat{w}_n(\xi) = \frac{1}{\pi n} \sin \frac{\pi n}{2} \int_0^{\lambda} e^{-\frac{1}{2}\pi n|\xi - \tau|} \operatorname{sgn}(\xi - \tau) \hat{\chi}'(\tau)\, d\tau. \tag{A.5}$$

Now, the use of the inverse integral transform (A.3) and (A.4) gives

$$\hat{w}(\xi, \eta) = \frac{1}{2} \int_0^{\lambda} [\Sigma_-(\xi - \tau, \eta) + \Sigma_+(\xi - \tau, \eta)] \hat{\chi}(\tau)\, d\tau, \tag{A.6}$$

where

$$2\Sigma_{\pm}(\zeta, \eta) = \frac{\sin \frac{\pi}{2}\eta}{e^{\frac{\pi}{2}|\zeta|} \pm 2 \cos \frac{\pi}{2}\eta + e^{-\frac{\pi}{2}|\zeta|}}. \tag{A.7}$$

In particular, on expanding $\Sigma_{\pm}(\zeta, \eta)$ for small ζ and η, we have $\Sigma_+(\zeta, \eta) \sim 0$ and $\Sigma_-(\zeta, \eta) \sim \pm\delta(\tau - \xi)$ as $\eta \to 0^{\pm}$, and therefore

$$\frac{\partial}{\partial \xi} \hat{w}(\xi, 0^{\pm}) \sim \pm\frac{1}{2} \hat{\chi}'(\xi), \quad 0 < \xi < \lambda. \tag{A.8}$$

On the other hand, if we use (A.5) instead, then by inverting the integral transform, differentiating with respect to η and integrating with respect to ξ, we obtain

$$\int_{-\infty}^{\xi} \frac{\partial \hat{w}}{\partial \eta}(\xi_1, \eta)\, d\xi_1 = \frac{1}{\pi} \int_0^{\lambda} \left(\sum_{k=0}^{\infty} \frac{\cos \frac{\pi}{2}(2k+1)\eta}{2k+1} e^{-\frac{\pi}{2}(2k+1)|\xi-\tau|} \right) \hat{\chi}'(\tau)\, d\tau + \phi(\eta), \tag{A.9}$$

where $\phi(\eta)$ is an arbitrary function. The expression in the right-hand side in (A.9) can be simplified as

$$\int_{-\infty}^{\xi} \frac{\partial \hat{w}}{\partial \eta}(\xi_1, \eta)\, d\xi_1 = \frac{1}{4\pi} \int_0^{\lambda} \ln \frac{\Sigma_+(\xi - \tau, \eta)}{\Sigma_-(\xi - \tau, \eta)} \hat{\chi}'(\tau)\, d\tau + \phi(\eta). \tag{A.10}$$

Finally, by substituting (A.8) and (A.10) as $\eta = 0^{\pm}$ into (2.16), we derive

$$\frac{\hat{\mu}_s}{2} \hat{\chi}'(\xi) + \frac{1}{2\pi} \int_0^{\lambda} \ln \coth \frac{\pi |\tau - \xi|}{4} \hat{\chi}'(\tau)\, d\tau + p^{\circ}(\xi) = C, \quad 0 < \xi < \lambda, \tag{A.11}$$

where C is a an arbitrary constant to be fixed by the condition (2.19). By recalling that $\hat{\chi}'(\xi) = \psi(\xi)$ we see that (A.11) coincides with (3.17). On defining this equation in the whole real axis by means of the function $f(\xi)$ introduced in (3.2) and employing the Fourier transform and the convolution theorem, we obtain the Riemann–Hilbert problem (3.13), (4.1).