INTEGRO-DIFFERENTIAL EQUATION FOR A FINITE CRACK IN A STRIP WITH SURFACE EFFECTS

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Summary

We present a mathematical investigation of a nonstandard boundary value problem for Laplace's equation in an infinite strip containing a finite crack. This unusual problem arises when surface elasticity is incorporated in the description of deformation of the crack faces. We use the Fourier transform to reduce the original boundary value problem to an integral equation that is further shown to reduce to a vector Riemann–Hilbert problem. The latter is solved analytically. The solution consists of two parts. The first is found explicitly, and the second is in a series form. The series converge exponentially, and the series coefficients are obtained by solving an infinite system. It is shown that the rate of convergence of an approximate solution to the exact one is exponential. We illustrate our solution with a particular example which, among other interesting phenomena, reveals the effect of the surface mechanics on the shape of the crack face [redacted].

1. Introduction

The incorporation of surface mechanics into mathematical models describing deformation of various elastic structures has drawn an increasing amount of attention in the literature (see, for example, (**1** to **11**)). The concept is particularly significant when considering analytical models of deformation at the nanoscale where the high surface area to volume ratio means that the effects of the surface can no longer be neglected.

It is widely accepted that atoms near the surface of a solid material experience a local environment different from those in the bulk material (see, for example, (**12**, **13**)). Consequently, a more accurate and comprehensive analysis of the deformation of an elastic solid with one or more surfaces can be achieved by incorporating a description of the separate surface mechanics near each surface of the solid. In the case of a solid containing a crack, a comprehensive model would include the surface effects corresponding to the two surfaces (faces) of the crack. In the context of continuum-based analytical models, the surface model proposed in (**3**, **4**) has been used extensively in a number of

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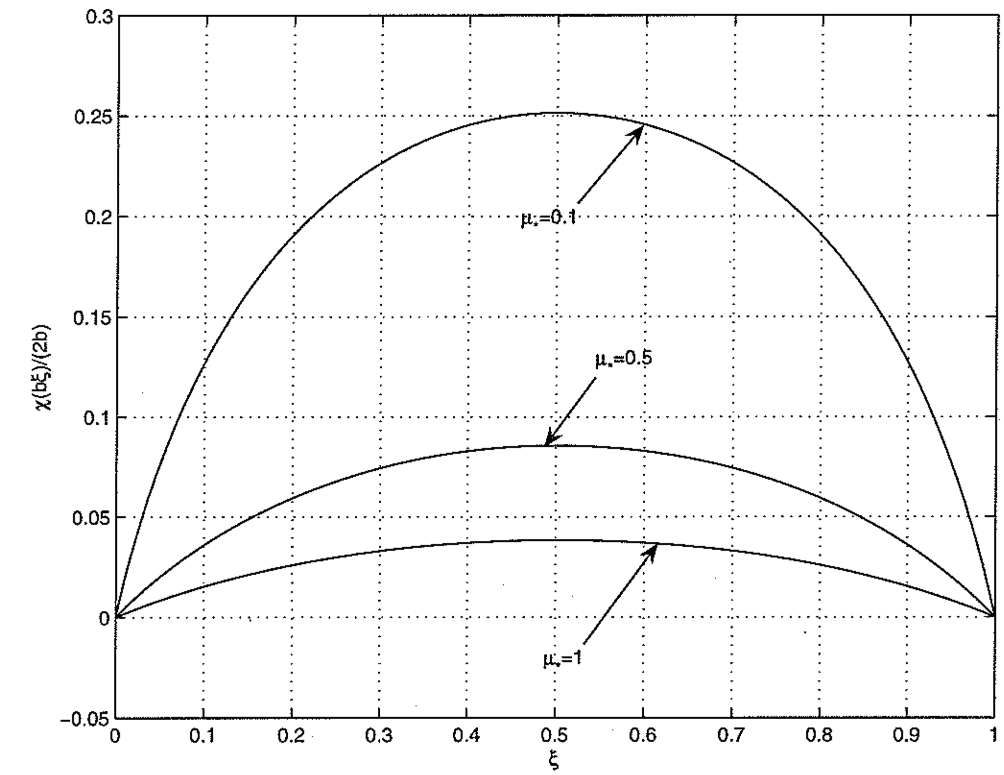

Fig. 3 The dimensionless profile of the upper side of the crack, $\frac{1}{2}\hat{\chi}(\xi)$, for some values of the surface parameter $\hat{\mu}_s = \mu_*$ when $\hat{p} = 1$ and $\lambda = 1$

shown that the problem is equivalent to a singular integro-differential equation in a finite segment $(0, \lambda)$. We have reduced this equation to a Riemann–Hilbert problem for two pairs of functions and found its solution as a sum of two functions. The first function, the principal part of the solution, was given by quadratures. The second function is of order $e^{-\beta_1\lambda}$ for large λ. Here, β_1 is the smallest positive root of the equation $1 + (\hat{\mu}_s\beta)^{-1}\tan\beta = 0$. This second part of the solution was found in a series form. The series converge exponentially, and their coefficients are the solution of a certain infinite system of linear algebraic equations. The rate of convergence of an approximate solution to the exact one is also exponential. We emphasize that for such singular integro-differential equations, the classical methods of orthogonal polynomials and collocations converge slowly.

We have determined the dimensionless profile of the crack, $\frac{1}{2}\hat{\chi}(\xi)$, and the derivative of the displacement jump, $\psi(\xi) = \hat{\chi}'(\xi)$. We have shown that the function $\psi(x/b)$ [redacted] is bounded as x approaches the crack tips, $x = 0$ and $x = a$, from the interior and the exterior of the crack, respectively. If the thickness, δ, of the surface layers surrounding the crack tends to zero, then the values $|\psi(x/b)|$ [redacted] at $x = 0$ and $x = a$ remain finite but grow. They become infinite if $\delta = 0$ as it happens in the classical theory without the surface effects.