DIFFRACTION OF AN OBLIQUELY INCIDENT ELECTROMAGNETIC WAVE BY AN IMPEDANCE RIGHT-ANGLED CONCAVE WEDGE

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Summary

Scattering of a plane electromagnetic wave by an anisotropic impedance right-angled concave wedge at skew incidence is analysed. A closed-form solution is derived by reducing the problem to a symmetric order-2 vector Riemann–Hilbert problem (RHP) on the real axis. The problem of matrix factorization leads to a scalar RHP on a genus-3 Riemann surface. Its solution is derived by the Weierstrass integrals. Due to a special symmetry of the problem the associated Jacobi inversion problem is solved in terms of elliptic integrals, not a genus-3 Riemann θ-function. The electric and magnetic field components are expressed through the Sommerfeld integrals, and the incident and reflected waves are recovered.

1. Introduction

In diffraction theory the study of scattering of a plane obliquely incident electromagnetic wave from an anisotropic impedance wedge is one of the key canonical problems. In the case of normal incidence when the tensor impedance has zero diagonal and non-zero off-diagonal entries, the problem was solved in (**1**). For oblique incidence, a closed-form solution is known for some special cases including the vector case for a half-plane (**2**), (**3**), (**4**) and the diagonal and triangular cases for a wedge when the impedance parameters meet certain conditions allowing for reduction of the problem to separately and consequently solved scalar Maliuzhinets equations (**5**), (**6**).

Approximate solutions by the perturbation technique are available for the cases when the wedge is almost a half-plane (**7**) or when the incidence is almost normal (**8**). Different approximate numerical solutions are available for the general case. They include those obtained by the method of integral equations (**9**), (**10**), (**11**), the method of approximate matrix factorization in conjunction with the Fredholm integral equation theory (**12**), and the probabilistic random walk method (**13**).

A scalar diffraction problem for a concave wedge was analysed in (**14**) by means of a Carleman-type boundary value problem of the theory of analytic functions. A method based on splitting the spectra into two functions, the solution of the Maliuzhinets equation and a function that is defined by a series whose coefficients solve a certain infinite system of linear algebraic equations, was recently proposed in (**15**). By this method, the diffraction field in the case of scattering by a wedge of angle $3\pi/2$ was recovered in (**16**). A method of reduction of wedge diffraction problems to functional equations was presented in (**17**). In particular, it was pointed out that the electromagnetic problem for a right-angled convex wedge could be reduced to an order-6 vector RHP.

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An exact solution for the general case of the tensor impedance and obliquely incident electromagnetic wave even for a right-angled wedge is still unavailable in the literature. Our goal in this article was to derive a closed-form solution of the problem on an impenetrable right-angled concave wedge (of angle $3\pi/2$) in the case of oblique incidence when on the faces $S^{\pm}$ of the wedge the boundary conditions are

$$\pm\mathcal{E}_z = \eta^{\pm}_{\rho\rho} Z\mathcal{H}_\rho, \quad \mp\mathcal{E}_\rho = \eta^{\pm}_{zz} Z\mathcal{H}_z, \quad (\rho,\theta)\in S^{\pm}. \tag{1.1}$$

Here, $S^- = \{\theta = 0, 0 < \rho < \infty\}$, $S^+ = \{\theta = \pi/2, 0 < \rho < \infty\}$, $\mathcal{E}_\rho$, $\mathcal{H}_\rho$ and $\mathcal{E}_z$, $\mathcal{H}_z$ are the ρ- and z-components of the electromagnetic field, Z is the intrinsic impedance of the medium, and $\eta^{\pm}_{\rho\rho}$ and $\eta^{\pm}_{zz}$ are the impedance parameters.

In Section 2, we constitute the boundary value problem for the exterior of a concave wedge of an arbitrary angle, $W_e = \{0 < \rho < \infty, 0 < \theta < \alpha\}$ $(0 < \alpha < \pi)$, with the general impedance boundary conditions

$$\pm\mathcal{E}_z = \eta^{\pm}_{\rho\rho} Z\mathcal{H}_\rho + \eta^{\pm}_{\rho z} Z\mathcal{H}_z, \quad \mp\mathcal{E}_\rho = \eta^{\pm}_{z\rho} Z\mathcal{H}_\rho + \eta^{\pm}_{zz} Z\mathcal{H}_z, \quad (\rho,\theta)\in S^{\pm}. \tag{1.2}$$

Next, by applying the Laplace transform (**17**) we reduce the problem to two sets of six functional equations. We focus on the case $\alpha = \pi/2$, and in Section 3, we transform the functional equations into two homogeneous vector RHPs with a 2×2 matrix coefficient. The residues of the unknown vectors at the geometric optics poles are not prescribed at this stage. The solution to the second RHP is derived from the solution to the RHP 1 by a certain transformation of the impedance parameters and the angle of incidence $\beta \in (0, \pi)$. The solution of the RHPs 1 and 2, vectors $\Phi^{\pm}(\eta)$ and $\hat{\Phi}^{\pm}(\eta)$, respectively, have to satisfy the symmetry conditions $\Phi^{+}(\eta) = \Phi^{-}(-\eta)$ and $\hat{\Phi}^{+}(\eta) = \hat{\Phi}^{-}(-\eta)$. It is shown that the matrix coefficient of both RHPs has the structure $G(\eta) = b_1(\eta)Q_1(\eta) + b_2(\eta)Q_2(\eta)$, where $b_1(\eta)$ and $b_2(\eta)$ are Hölder functions on the contour of the problem $L = (-\infty, +\infty)$, and $Q_1(\eta)$ and $Q_2(\eta)$ are polynomial 2×2 matrices. This case is algebraic, and the symmetric vector RHPs can be solved exactly (**18**), (**19**).

In Section 4, for simplicity, we assume that the impedance parameters $\eta^{\pm}_{\rho z}$ and $\eta^{\pm}_{z\rho}$ vanish and deal with the boundary conditions (1.1). For this case we reduce the problem of matrix factorization to a scalar RHP on a genus-3 Riemann surface. The coefficient of the RHP on the first and second sheets coincides with the first and second eigenvalues of the matrix coefficient, respectively. Because of the symmetry of the problem we manage to solve the associated Jacobi inversion problem in terms of elliptic integrals and construct the solution to the matrix factorization problem by quadratures.

In Section 5, we derive a closed-form solution to the vector RHPs. It turns out that the number of free constants in the general solution is governed by the location of the zeros η_j^- (RHP 1) and η_j^+ (RHP 2) of the polynomials

$$\delta_0(\eta) = (\eta^2 - k_0^2)\cos^2\beta - (\eta - \gamma_1^-)(\eta - \gamma_4^-), \quad \hat{\delta}_0(\eta) = (\eta^2 - k_0^2)\cos^2\beta - (\eta - \gamma_1^+)(\eta - \gamma_4^+), \tag{1.3}$$

where $\gamma_1^{\pm} = k_0(\eta^{\pm}_{\rho\rho})^{-1}\sin\beta$, $\gamma_4^{\pm} = k_0\eta^{\pm}_{zz}\sin\beta$, $k_0 = k\sin\beta$, and k is the wave number. It is shown that the two RHPs have to fulfil certain compatibility conditions. These conditions when satisfied give the solution with two free constants regardless of the location of the zeros $\eta_j^{\pm}$.

In Section 6, we derive the spectra $S_1(s)$ and $S_2(s)$ of the problem and express the Sommerfeld integrals through the solution to the two vector RHPs. On continuing analytically the functions $S_1(s)$ and $S_2(s)$ to the right and to the left, applying the steepest descent method and the Cauchy theorem, we recover the incident, reflected, and diffracted waves and fix the residues of the solution at the geometric optics poles. For the normal incidence case we find a simple closed-form solution to the RHPs 1 and 2 and verify the compatibility conditions in Section 7.

2. Formulation: diffraction by a concave wedge of an arbitrary angle

Consider diffraction of an electromagnetic wave

$$\left(\mathcal{E}_z^{inc},\, \mathcal{H}_z^{inc}\right) = \left(E_z^i,\, H_z^i\right) e^{ikz\cos\beta - i\omega t}, \tag{2.1}$$

where

$$\left(E_z^i,\, ZH_z^i\right) = \left(i_1,\, i_2\right) e^{-ik\rho\cos(\theta-\theta_0)\sin\beta}, \tag{2.2}$$

by a wedge $W = \{0 < \rho < \infty, 2\pi - \alpha < \theta < 2\pi, |z| < \infty\}$, $\alpha \in (0, \pi)$, characterized by the impedance boundary conditions

$$\begin{aligned} \pm\mathcal{E}_z &= \eta_{\rho\rho}^{\pm} Z\mathcal{H}_\rho + \eta_{\rho z}^{\pm} Z\mathcal{H}_z \quad \text{on } S^{\pm}, \\ \mp\mathcal{E}_\rho &= \eta_{z\rho}^{\pm} Z\mathcal{H}_\rho + \eta_{zz}^{\pm} Z\mathcal{H}_z \quad \text{on } S^{\pm}. \end{aligned} \tag{2.3}$$

Here, k ($\operatorname{Im} k > 0$) is the wave number, ω is the angular velocity, $Z = \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of the medium, μ_0 is the magnetic permeability, ε_0 is the electric permittivity, $\eta_{\rho\rho}^{\pm}$, $\eta_{\rho z}^{\pm}$, $\eta_{z\rho}^{\pm}$ and $\eta_{zz}^{\pm}$ are the impedance parameters. The symbols $S^{\pm}$ stand for the boundaries of the wedge, $S^- = \{\theta = 0, \rho > 0\}$, and $S^+ = \{\theta = \alpha, \rho > 0\}$. Because of the representation (2.1) for the incident waves it is natural to split the electric and magnetic fields as

$$\left(\mathcal{E},\, \mathcal{H}\right) = \left(E,\, H\right) e^{ikz\cos\beta - i\omega t}. \tag{2.4}$$

By employing the Maxwell equations we eliminate the radial components of the electric and magnetic fields

$$\begin{aligned} E_\rho &= \frac{i}{k\sin^2\beta}\left[\cos\beta\frac{\partial}{\partial\rho}E_z + \frac{Z}{\rho}\frac{\partial}{\partial\theta}H_z\right], \\ ZH_\rho &= \frac{i}{k\sin^2\beta}\left[Z\cos\beta\frac{\partial}{\partial\rho}H_z - \frac{1}{\rho}\frac{\partial}{\partial\theta}E_z\right]. \end{aligned} \tag{2.5}$$

The resulting boundary conditions, after rearrangement, are formulated in terms of the z-components, $\phi_1 = E_z$ and $\phi_2 = ZH_z$, as

$$\begin{aligned} -\frac{i}{\rho}\frac{\partial\phi_1}{\partial\theta} + i\cos\beta\frac{\partial\phi_2}{\partial\rho} \mp \gamma_1^{\pm}\phi_1 + \gamma_2^{\pm}\phi_2 = 0 \quad \text{on } S^{\pm}, \\ i\cos\beta\frac{\partial\phi_1}{\partial\rho} + \frac{i}{\rho}\frac{\partial\phi_2}{\partial\theta} + \gamma_3^{\pm}\phi_1 \pm \gamma_4^{\pm}\phi_2 = 0 \quad \text{on } S^{\pm}, \end{aligned} \tag{2.6}$$

where

$$\gamma_1^{\pm} = \frac{k\sin^2\beta}{\eta_{\rho\rho}^{\pm}}, \quad \gamma_2^{\pm} = \frac{\eta_{\rho z}^{\pm} k\sin^2\beta}{\eta_{\rho\rho}^{\pm}},$$

$$\gamma_3^{\pm} = \frac{\eta_{z\rho}^{\pm} k\sin^2\beta}{\eta_{\rho\rho}^{\pm}}, \quad \gamma_4^{\pm} = \frac{k\sin^2\beta}{\eta_{\rho\rho}^{\pm}}(\eta_{\rho\rho}^{\pm}\eta_{zz}^{\pm} - \eta_{\rho z}^{\pm}\eta_{z\rho}^{\pm}). \tag{2.7}$$

It is assumed that the functions E_z and H_z satisfy the Meixner edge condition and therefore

$$\phi_j \to c_j, \quad r \to 0, \quad c_j = \text{const}, \quad j = 1, 2. \tag{2.8}$$

Also, at infinity, they meet the radiation condition

$$\phi_1 - E_z^i - E_z^{rw} = O(e^{-\varepsilon k_0 r}), \quad Z^{-1}\phi_2 - H_z^i - H_z^{rw} = O(e^{-\varepsilon k_0 r}), \quad 0 < \theta < \alpha, \quad k_0 r \to \infty, \tag{2.9}$$

where $\varepsilon > 0$, $k_0 = k\sin\beta$, E_z^{rw} and H_z^{rw} are the reflected waves.

In the exterior of the wedge, $W_e = \{0 < \rho < \infty, 0 < \theta < \alpha\}$, the functions ϕ_1 and ϕ_2 satisfy the Helmholtz equation. To convert the boundary value problem for the Helmholtz equation to a system of functional equations, we use the scheme (**17**). On making the affine transformation of the Cartesian coordinates $u = x - y\cot\alpha$ and $v = y\csc\alpha$ we obtain that, in the new coordinates, the functions ϕ_1 and ϕ_2 are the solutions of the following differential equation in a quarter-plane:

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2\cos\alpha\frac{\partial^2}{\partial u\partial v} + k_0^2\sin^2\alpha\right)\phi_j = 0, \quad 0 < u, v < \infty, \quad j = 1, 2, \tag{2.10}$$

subject to the four boundary conditions

$$\begin{aligned}
&i\left(\csc\alpha\frac{\partial}{\partial u} - \cot\alpha\frac{\partial}{\partial v}\right)\phi_1 + i\cos\beta\frac{\partial\phi_2}{\partial v} - \gamma_1^+\phi_1 + \gamma_2^+\phi_2 = 0, \quad (u, v) \in \tilde{S}^+,\\
&i\cos\beta\frac{\partial\phi_1}{\partial v} - i\left(\csc\alpha\frac{\partial}{\partial u} - \cot\alpha\frac{\partial}{\partial v}\right)\phi_2 + \gamma_3^+\phi_1 + \gamma_4^+\phi_2 = 0, \quad (u, v) \in \tilde{S}^+,\\
&i\left(-\csc\alpha\frac{\partial}{\partial v} + \cot\alpha\frac{\partial}{\partial u}\right)\phi_1 + i\cos\beta\frac{\partial\phi_2}{\partial u} + \gamma_1^-\phi_1 + \gamma_2^-\phi_2 = 0, \quad (u, v) \in \tilde{S}^-,\\
&i\cos\beta\frac{\partial\phi_1}{\partial u} + i\left(\csc\alpha\frac{\partial}{\partial v} - \cot\alpha\frac{\partial}{\partial u}\right)\phi_2 + \gamma_3^-\phi_1 - \gamma_4^-\phi_2 = 0, \quad (u, v) \in \tilde{S}^-,
\end{aligned} \tag{2.11}$$

where $\tilde{S}^+ = \{(u, v) \in \mathbb{R}^2 | u = 0, v > 0\}$, $\tilde{S}^- = \{(u, v) \in \mathbb{R}^2 | v = 0, u > 0\}$.

By means of the Laplace transform

$$\tilde{\phi}_j(u, \eta) = \int_0^\infty e^{i\eta v}\phi_j(u, v)dv \tag{2.12}$$

the differential equation (2.10) may be put into the form

$$\left(\frac{d^2}{du^2} + 2i\eta\cos\alpha\frac{d}{du} + k_0^2\sin^2\alpha - \eta^2\right)\tilde{\phi}_j(u,\eta) = f_j(u), \quad j = 1, 2, \tag{2.13}$$

where

$$f_j(u) = \frac{\partial\phi_j}{\partial v}(u,0) - i\eta\phi_j(u,0) - 2\cos\alpha\frac{\partial\phi_j}{\partial u}(u,0). \tag{2.14}$$

The two roots of the characteristic equation of the differential operator in (2.13) are $-i\eta\cos\alpha \pm \sqrt{\eta^2 - k_0^2}\sin\alpha$. Fix a branch $\zeta = \sqrt{\eta^2 - k_0^2}$ of the two-valued function $\zeta^2 = \eta^2 - k_0^2$ by the condition $\zeta(0) = -ik_0$. The branch is a single-valued analytic function in the η-plane cut along the line joining the branch points $\pm k_0$ and passing through the infinite point. As $-\infty < \eta < +\infty$, the branch chosen possesses the property $\operatorname{Re}\zeta \geq 0$, and the general solution to equation (2.13) bounded as $u \to \infty$ has the form

$$\tilde{\phi}_j(u,\eta) = A_j(\eta)e^{-qu} - \frac{1}{2q}\int_0^\infty e^{-q|u-u_1|}f_j(u_1)du_1, \quad j = 1, 2, \tag{2.15}$$

where $q = -i\eta\cos\alpha + \zeta\sin\alpha$ ($\operatorname{Re} q \geq 0$). To derive a functional equation for $\tilde{\phi}_j$, we apply the Laplace transform with respect to u using the function q as its parameter

$$\hat{\phi}_j(iq, v) = \int_0^\infty e^{-qu}\phi_j(u, v)du. \tag{2.16}$$

On referring to (2.14), we find that

$$\int_0^\infty f_j(u)e^{-qu}du = \frac{d}{dv}\hat{\phi}_j(iq,0) - (i\eta + 2q\cos\alpha)\hat{\phi}_j(iq,0) + 2\cos\alpha\phi_j^\circ, \tag{2.17}$$

where $\phi_j^\circ = \phi_j(0,0)$. On the other hand, from (2.15)

$$\tilde{\phi}_j(0,\eta) = A_j(\eta) - \frac{1}{2q}\int_0^\infty e^{-qu}f_j(u)du,$$
$$\frac{d\tilde{\phi}_j}{du}(0,\eta) = -qA_j(\eta) - \frac{1}{2}\int_0^\infty e^{-qu}f_j(u)du. \tag{2.18}$$

By excluding $A_j(\eta)$ and the Laplace transforms of the functions $f_j(u)$ and employing the second relation in (2.18) we get the following two equations:

$$\frac{d\tilde{\phi}_j}{du}(0,\eta) + q\tilde{\phi}_j(0,\eta) + \frac{d\hat{\phi}_j}{dv}(iq,0) - (i\eta + 2q\cos\alpha)\hat{\phi}_j(iq,0) + 2\cos\alpha\phi_j^\circ = 0 \quad j = 1, 2. \tag{2.19}$$

These two equations should be complemented by the Laplace-transformed boundary conditions. We apply the Laplace transform (2.12) to the boundary conditions (2.11) on the boundary $\tilde{S}^+$ and the transform (2.16) to the conditions on the side $\tilde{S}^-$. We have

$$i\csc\alpha\frac{d\tilde{\phi}_1}{du}(0,\eta)-(\gamma_1^+ +\eta\cot\alpha)\tilde{\phi}_1(0,\eta)+(\gamma_2^+ +\eta\cos\beta)\tilde{\phi}_2(0,\eta)+i\cot\alpha\phi_1^\circ - i\cos\beta\phi_2^\circ=0,$$

$$-i\csc\alpha\frac{d\tilde{\phi}_2}{du}(0,\eta)+(\gamma_3^+ +\eta\cos\beta)\tilde{\phi}_1(0,\eta)+(\gamma_4^+ +\eta\cot\alpha)\tilde{\phi}_2(0,\eta)-i\cos\beta\phi_1^\circ - i\cot\alpha\phi_2^\circ=0,$$

$$-i\csc\alpha\frac{d\hat{\phi}_1}{dv}(iq,0)+(\gamma_1^- +iq\cot\alpha)\hat{\phi}_1(iq,0)+(\gamma_2^- +iq\cos\beta)\hat{\phi}_2(iq,0)-i\cot\alpha\phi_1^\circ - i\cos\beta\phi_2^\circ=0,$$

$$i\csc\alpha\frac{d\hat{\phi}_2}{dv}(iq,0)+(\gamma_3^- +iq\cos\beta)\hat{\phi}_1(iq,0)-(\gamma_4^- +iq\cot\alpha)\hat{\phi}_2(iq,0)-i\cos\beta\phi_1^\circ + i\cot\alpha\phi_2^\circ=0. \tag{2.20}$$

Notice that by applying the Laplace transform (2.12) with respect to u and then the Laplace transform (2.16) with respect to v we obtain another set of six equations which coincide with (2.19) and (2.20) if we make the following transformation of the parameters and the functions:

$$\gamma_1^+ \leftrightarrow \gamma_1^-, \quad \gamma_2^+ \leftrightarrow -\gamma_2^-, \quad \gamma_3^+ \leftrightarrow -\gamma_3^-, \quad \gamma_4^+ \leftrightarrow \gamma_4^-, \quad \beta \leftrightarrow \pi-\beta,$$

$$\tilde{\phi}_j(0,\eta) \leftrightarrow \hat{\phi}_j(\eta,0), \quad \frac{d\tilde{\phi}_j}{du}(0,\eta) \leftrightarrow \frac{d\hat{\phi}_j}{dv}(\eta,0),$$

$$\hat{\phi}_j(iq,0) \leftrightarrow \tilde{\phi}_j(0,iq), \quad \frac{d\hat{\phi}_j}{dv}(iq,0) \leftrightarrow \frac{d\tilde{\phi}_j}{du}(0,iq), \quad j=1,2. \tag{2.21}$$

In the next section, we consider the case $\alpha=\pi/2$ and reduce the system of six functional equations (2.19), (2.20) and the one obtained from the system (2.19), (2.20) by the transformation (2.21) to two symmetric RHPs with a 2×2 matrix coefficient.

3. Vector RHPs in the case of a right-angled concave wedge

3.1 *Statement of the problem*

If the domain of interest is a quarter-plane (Fig. 1), then the standard Cartesian coordinates $x=u$, $y=v$ can be employed, and $q=\zeta$, $\zeta=\sqrt{\eta^2-k_0^2}$. From the boundary conditions (2.20) we express the derivatives $d\tilde{\phi}_j(0,\eta)/dx$ and $d\hat{\phi}_j(i\zeta,0)/dy$ through the functions $\tilde{\phi}_j(0,\eta)$ and $\hat{\phi}_j(i\zeta,0)$ $(j=1,2)$

$$\frac{d\hat{\phi}_1}{dy}(i\zeta,0)=-i\gamma_1^-\hat{\phi}_1(i\zeta,0)-(i\gamma_2^- -\zeta\cos\beta)\hat{\phi}_2(i\zeta,0)-\phi_2^\circ\cos\beta,$$

$$\frac{d\hat{\phi}_2}{dy}(i\zeta,0)=(i\gamma_3^- -\zeta\cos\beta)\hat{\phi}_1(i\zeta,0)-i\gamma_4^-\hat{\phi}_2(i\zeta,0)+\phi_1^\circ\cos\beta,$$

$$\frac{d\tilde{\phi}_1}{dx}(0,\eta)=-i\gamma_1^+\tilde{\phi}_1(0,\eta)+i(\gamma_2^+ +\eta\cos\beta)\tilde{\phi}_2(0,\eta)+\phi_2^\circ\cos\beta,$$

$$\frac{d\tilde{\phi}_2}{dx}(0,\eta)=-i(\gamma_3^+ +\eta\cos\beta)\tilde{\phi}_1(0,\eta)-i\gamma_4^+\tilde{\phi}_2(0,\eta)-\phi_1^\circ\cos\beta. \tag{3.1}$$

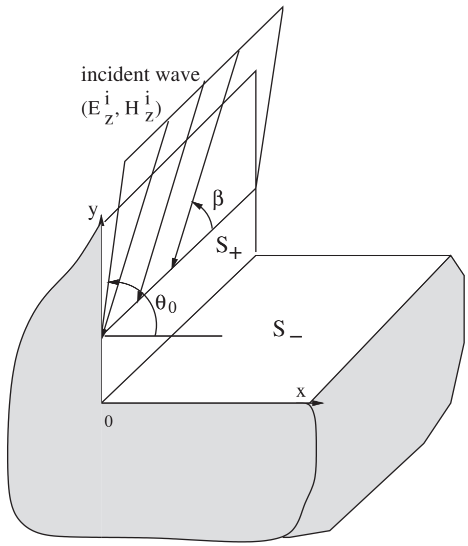

Fig. 1 Geometry of the problem on a right-angled concave wedge

Next, on replacing η by $-\eta$ in (2.19) we obtain two more equations

$$\frac{d\tilde{\phi}_j}{dx}(0,-\eta)+\zeta\tilde{\phi}_j(0,-\eta)+\frac{d\hat{\phi}_j}{dy}(i\zeta,0)+i\eta\hat{\phi}_j(i\zeta,0)=0, \quad j=1,2. \tag{3.2}$$

These two extra equations in conjunction with (2.19) allow us to express the functions $\hat{\phi}_j(i\zeta,0)$ through the functions $\tilde{\phi}_j(0,\pm\eta)$

$$\hat{\phi}_1(i\zeta,0)=-\frac{\gamma_1^+ + i\zeta}{2\eta}[\tilde{\phi}_1(0,\eta)-\tilde{\phi}_1(0,-\eta)]+\frac{\gamma_2^+ +\eta\cos\beta}{2\eta}\tilde{\phi}_2(0,\eta)+\frac{-\gamma_2^+ +\eta\cos\beta}{2\eta}\tilde{\phi}_2(0,-\eta),$$

$$\hat{\phi}_2(i\zeta,0)=-\frac{\gamma_3^+ +\eta\cos\beta}{2\eta}\tilde{\phi}_1(0,\eta)+\frac{\gamma_3^+ -\eta\cos\beta}{2\eta}\tilde{\phi}_1(0,-\eta)-\frac{\gamma_4^+ + i\zeta}{2\eta}[\tilde{\phi}_2(0,\eta)-\tilde{\phi}_2(0,-\eta)]. \tag{3.3}$$

On substituting the expressions for the functions $\hat{\phi}_j(i\zeta,0)$ and the derivatives $d\tilde{\phi}_j(0,\eta)/dx$ and $d\hat{\phi}_j(i\zeta,0)/dy$ in equations (2.19) we ultimately obtain

$$A(\eta)\Phi^+(\eta)+B(\eta)\Phi^+(-\eta)=0, \quad \eta\in L=(-\infty,+\infty), \tag{3.4}$$

where

$$\Phi^+(\eta)=\begin{pmatrix}\tilde{\phi}_1(0,\eta)\\ \tilde{\phi}_2(0,\eta)\end{pmatrix}, \quad A(\eta)=\begin{pmatrix}a_{11}(\eta) & a_{12}(\eta)\\ a_{21}(\eta) & a_{22}(\eta)\end{pmatrix}, \quad B(\eta)=\begin{pmatrix}b_{11}(\eta) & b_{12}(\eta)\\ b_{21}(\eta) & b_{22}(\eta)\end{pmatrix},$$

$$a_{11}=\delta_{1+}(\eta)[1-\delta_{1-}(\eta)]-\delta_{2-}(\eta)\delta_{3+}(\eta),\quad a_{12}=-\delta_{2+}(\eta)[1-\delta_{1-}(\eta)]-\delta_{2-}(\eta)\delta_{4+}(\eta),$$
$$a_{21}=\delta_{3+}(\eta)[1-\delta_{4-}(\eta)]+\delta_{3-}(\eta)\delta_{1+}(\eta),\quad a_{22}=\delta_{4+}(\eta)[1-\delta_{4-}(\eta)]-\delta_{3-}(\eta)\delta_{2+}(\eta),$$
$$b_{11}=\delta_{1-}(\eta)\delta_{1+}(-\eta)+\delta_{2-}(\eta)\delta_{3+}(-\eta),\quad b_{12}=-\delta_{1-}(\eta)\delta_{2+}(-\eta)+\delta_{2-}(\eta)\delta_{4+}(-\eta),$$
$$b_{21}=-\delta_{3-}(\eta)\delta_{1+}(-\eta)+\delta_{4-}(\eta)\delta_{3+}(-\eta),\quad b_{22}=\delta_{3-}(\eta)\delta_{2+}(-\eta)+\delta_{4-}(\eta)\delta_{4+}(-\eta),$$

$$\delta_{j+}(\eta)=\gamma_j^++i\zeta,\quad \delta_{j-}(\eta)=\frac{\gamma_j^-+\eta}{2\eta},\quad j=1,4,$$

$$\delta_{j+}(\eta)=\gamma_j^++\eta\cos\beta,\quad \delta_{j-}(\eta)=\frac{\gamma_j^-+i\zeta\cos\beta}{2\eta},\quad j=2,3. \tag{3.5}$$

It can be directly verified that $A(\eta)=B(-\eta)$ for all η, and the replacement of η by $-\eta$ does not change the problem (3.4). We now summarize the results. Denote $\eta_0=k_0\sin\theta_0$, $\hat{\eta}_0=k_0\cos\theta_0$.

THEOREM 3.1. Suppose that $\alpha=\pi/2$, and the incident wave is (E_z^i,H_z^i). Then the diffraction problem (2.10), (2.11), (2.8), (2.9) is equivalent to the following symmetric vector RHP provided its solution recovers the incident and reflected waves.

RHP 1. Find two vectors $\Phi^\pm(\eta)$ analytic everywhere in the half-planes $\mathbb{C}^\pm=\{\pm\,\mathrm{Im}\,\eta>0\}$ except at the simple poles $\eta=\pm\eta_0$ and Hölder-continuous up to the real axis L. At the contour L, their limit values satisfy the boundary condition

$$\Phi^+(\eta)=G(\eta)\Phi^-(\eta),\quad \eta\in L, \tag{3.6}$$

where $G(\eta)=-[A(\eta)]^{-1}B(\eta)$. In the plane, the vectors meet the symmetry condition

$$\Phi^-(\eta)=\Phi^+(-\eta),\quad \eta\in\mathbb{C}^-, \tag{3.7}$$

and at infinity, they vanish.

On examining the coefficient of the vector RHP we discover that it is a non-singular matrix admitting the representation

$$G(\eta)=\frac{1}{d(\eta)}[G^0(\eta)+\zeta G^1(\eta)], \tag{3.8}$$

where $d(\eta)$ is a scalar, G^0 and G^1 are polynomial matrices. Any non-singular matrix G of such a structure can be factorized (**18**) by reducing the vector RHP to a scalar RHP on a Riemann surface of the associated algebraic function. To simplify our derivations, it will be helpful to assume that either $\gamma_2^\pm=\gamma_3^\pm=0$ and the angle β is arbitrary $(0<\beta<\pi)$, or that all the four impedance parameters $\gamma_j^\pm$ are arbitrary and $\beta=\pi/2$. In what follows we consider the former case, namely, $\gamma_2^\pm=\gamma_3^\pm=0$ and $\beta\in(0,\pi)$.

We also notice that the transformation (2.21) maps the diffraction problem to another vector RHP.

RHP 2. Find two vectors $\hat{\Phi}^\pm(\eta)=(\hat{\phi}_1(\pm\eta,0),\hat{\phi}_2(\pm\eta,0))^T$ analytic everywhere in the half-planes $\mathbb{C}^\pm$ except at the simple poles $\eta=\pm\hat{\eta}_0$ and Hölder-continuous up to the real axis L. At the contour L, their limit values satisfy the boundary condition

$$\hat{\Phi}^+(\eta)=\hat{G}(\eta)\hat{\Phi}^-(\eta),\quad \eta\in L, \tag{3.9}$$

where $\hat{G}(\eta)$ is the matrix $G(\eta)$ whose parameters $\gamma_j^{\pm}$ $(j = 1, \ldots, 4)$ and β are transformed by (2.21). At infinity, the vectors $\hat{\Phi}^+(\eta)$ and $\hat{\Phi}^-(\eta)$ vanish, and in the plane, they meet the symmetry condition $\hat{\Phi}^-(\eta) = \hat{\Phi}^+(-\eta)$, $\eta \in \mathbb{C}^-$.

Undoubtedly, by the symmetry transformation (2.21) any result obtained for the first vector RHP is valid for the second problem and vice versa. Since both problems recover the same vectors, the values of the Laplace-transformed functions ϕ_1 and ϕ_2 on the faces of the wedge, we solve only one problem, the vector RHP 1 stated in Theorem 3.1 (we will call it the vector RHP).

3.2 *Reflected waves*

Before we start the matrix factorization procedure we note that the choice of the residues of the functions $\Phi^{\pm}(\eta)$ and $\hat{\Phi}^{\pm}(\eta)$ at the geometric optics poles $\eta = \pm\eta_0$ and $\pm\hat{\eta}_0$, respectively, have to bring us to the incident and reflected waves. To recover them, split the incident waves E_z^i and H_z^i into two waves

$$\left(E_z^{i+}, ZH_z^{i+}\right) = \left(i_1, i_2\right) e^{-ik_0\rho\cos(\theta-\theta_0)}\omega(\theta; \theta_0, \pi/2),$$
$$\left(E_z^{i-}, ZH_z^{i-}\right) = \left(i_1, i_2\right) e^{-ik_0\rho\cos(\theta-\theta_0)}\omega(\theta; 0, \theta_0), \tag{3.10}$$

where

$$\omega(\theta; a, b) = \begin{cases} 1, & a < \theta < b, \\ 0 & \text{otherwise}. \end{cases} \tag{3.11}$$

The first wave (E_z^{i+}, H_z^{i+}) reflects from the vertical wall $\{x = 0, 0 < y < \infty\}$, falls on the horizontal wall $\{0 < x < \infty, y = 0\}$, reflects from it and runs away to infinity. We denote the first and second reflected waves as (E_{z+}^r, H_{z+}^r) and (E_{z+}^R, H_{z+}^R), respectively, which are

$$\left(E_{z+}^r, ZH_{z+}^r\right) = \left(r_1^+, r_2^+\right) e^{i\hat{\eta}_0 x - i\eta_0 y},$$
$$\left(E_{z+}^R, ZH_{z+}^R\right) = \left(R_1^+, R_2^+\right) e^{i\hat{\eta}_0 x + i\eta_0 y}\omega(\theta; 0, \theta_0). \tag{3.12}$$

Likewise, the second wave (E_z^{i-}, H_z^{i-}) impinges upon the horizontal face of the wedge, reflects from it, strikes the vertical wall and then goes to infinity. The reflection waves are described by

$$\left(E_{z-}^r, ZH_{z-}^r\right) = \left(r_1^-, r_2^-\right) e^{-i\hat{\eta}_0 x + i\eta_0 y},$$
$$\left(E_{z-}^R, ZH_{z-}^R\right) = \left(R_1^-, R_2^-\right) e^{i\hat{\eta}_0 x + i\eta_0 y}\omega(\theta; \theta_0, \pi/2). \tag{3.13}$$

The reflection coefficients are recovered by substituting the incident and reflected waves (3.10), (3.12) and (3.13) into (2.11)

$$r_1^+ = K_1^- i_1 + K_2 i_2, \quad r_2^+ = -K_2 i_1 + K_1^+ i_2,$$
$$R_1^+ = \hat{K}_1^- r_1^+ + \hat{K}_2 r_2^+, \quad R_2^+ = -\hat{K}_2 r_1^+ + \hat{K}_1^+ r_2^+,$$
$$r_1^- = \hat{K}_1^- i_1 - \hat{K}_2 i_2, \quad r_2^- = \hat{K}_2 i_1 + \hat{K}_1^+ i_2,$$
$$R_1^- = K_1^- r_1^- - K_2 r_2^-, \quad R_2^- = K_2 r_1^- + K_1^+ r_2^-, \tag{3.14}$$

where

$$K_1^{\pm} = \frac{(\hat\eta_0 \pm \gamma_1^+)(\hat\eta_0 \mp \gamma_4^+) - \eta_0^2\cos^2\beta}{\Delta_0}, \quad K_2 = \frac{2\eta_0\hat\eta_0\cos\beta}{\Delta_0},$$

$$\hat K_1^{\pm} = \frac{(\eta_0 \pm \gamma_1^-)(\eta_0 \mp \gamma_4^-) - \hat\eta_0^2\cos^2\beta}{\hat\Delta_0}, \quad \hat K_2 = \frac{2\eta_0\hat\eta_0\cos\beta}{\hat\Delta_0},$$

$$\Delta_0 = (\hat\eta_0 + \gamma_1^+)(\hat\eta_0 + \gamma_4^+) + \eta_0^2\cos^2\beta, \quad \hat\Delta_0 = (\eta_0 + \gamma_1^-)(\eta_0 + \gamma_4^-) + \hat\eta_0^2\cos^2\beta. \tag{3.15}$$

3.3 *Analysis of the matrix $G(\eta)$*

The matrix $G(\eta)$ is continuous and non-singular everywhere in the contour L, and as $\eta \to \pm\infty$, $G(\eta) \sim \operatorname{diag}\{-1, -1\}$. Compute first the determinant of G

$$\det G(\eta) = \frac{\delta_0(-\eta)}{\delta_0(\eta)}, \tag{3.16}$$

where

$$\delta_0(\eta) = (\eta^2 - k_0^2)\cos^2\beta - (\eta - \gamma_1^-)(\eta - \gamma_4^-), \tag{3.17}$$

and determine its index

$$2\kappa = \frac{1}{2\pi}[\arg\det G(\eta)]_{-\infty}^{\infty}. \tag{3.18}$$

We distinguish the following three cases:

(i) the two zeros of the quadratic polynomial $\delta_0(\eta)$ lie in the lower half-plane, $\delta_0(\eta) = -(\eta+\tau_1)(\eta+\tau_2)\sin^2\beta$, $\operatorname{Im}\tau_j > 0, j = 1, 2,$

(ii) the zeros lie in the opposite half-planes, $\delta_0(\eta) = -(\eta-\tau_1)(\eta+\tau_2)\sin^2\beta$, $\operatorname{Im}\tau_j > 0, j = 1, 2,$ and

(iii) both zeros lie in the upper half-plane, $\delta_0(\eta) = -(\eta-\tau_1)(\eta-\tau_2)\sin^2\beta$, $\operatorname{Im}\tau_j > 0, j = 1, 2.$

By the argument principle, κ is an integer number, and in the case (i) $\kappa = 1$ (in the upper half-plane, the function $\delta_0(-\eta)$ has two zeros, while the function $\delta_0(\eta)$ has no zeros in $\mathbb{C}^+$). In the case (ii) $\kappa = 0$, and in the last case $\kappa = -1$.

Our next step is to rearrange the representation (3.8) as

$$G(\eta) = \frac{\zeta}{\delta_0(\eta)\delta_1(\eta)} G^1(\eta)\left[I + \frac{1}{\zeta d_1(\eta)}\begin{pmatrix} r_1(\eta) & r_2(\eta) \\ r_3(\eta) & r_4(\eta)\end{pmatrix}\right], \tag{3.19}$$

where I is the unit 2×2 matrix, $r_j(\eta)$ $(j = 1, \dots, 4)$ are polynomials,

$$\delta_1(\eta) = (\gamma_1^+ + i\zeta)(\gamma_4^+ + i\zeta) + \eta^2\cos^2\beta, \tag{3.20}$$

the entries $g_{st}^1(\eta)$ of the polynomial matrix $G^1(\eta)$ are

$$g_{jj}^1(\eta) = -i(\gamma_1^+ + \gamma_4^+)[k_0^2\cos^2\beta - \eta^2\sin^2\beta + (-1)^j\eta(\gamma_1^- - \gamma_4^-) + \gamma_1^-\gamma_4^-], \quad j = 1, 2,$$

$$g_{12}^1(\eta) = -2i\eta\cos\beta[\gamma_1^-\gamma_4^- - (\gamma_4^+)^2 - k_0^2\sin^2\beta],$$

$$g_{21}^1(\eta) = 2i\eta\cos\beta[\gamma_1^-\gamma_4^- - (\gamma_1^+)^2 - k_0^2\sin^2\beta], \tag{3.21}$$

and $d_1(\eta) = \det G^1(\eta)$ is a degree-4 even polynomial

$$d_1(\eta) = \alpha_0 + \alpha_1\eta^2 + \alpha_2\eta^4 \tag{3.22}$$

whose coefficients are

$$\alpha_0 = -(\gamma_1^+ + \gamma_4^+)^2(\gamma_1^-\gamma_4^- + k_0^2\cos^2\beta)^2, \quad \alpha_2 = -(\gamma_1^+ + \gamma_4^+)^2\sin^4\beta,$$
$$\alpha_1 = (\gamma_1^+ + \gamma_4^+)^2[(\gamma_1^-)^2 + (\gamma_4^-)^2 - 2k_0^2\cos^4\beta] + \{-4(\gamma_1^-\gamma_4^-)^2 - (2\gamma_1^+\gamma_4^+ - k_0^2)^2$$
$$+ 2\gamma_1^-\gamma_4^-[(\gamma_1^+ - \gamma_4^+)^2 + 2k_0^2] + 2k_0^2[(\gamma_1^+)^2 - 2\gamma_1^-\gamma_4^- + (\gamma_4^+)^2 + k_0^2]\cos 2\beta - k_0^4\cos^2 2\beta\}\cos^2\beta. \tag{3.23}$$

It is an easy matter now to represent the matrix $G(\eta)$ in the form

$$G(\eta) = \frac{\delta_*\sqrt{\Delta(\eta)}}{\delta_0(\eta)} G^1(\eta)\Gamma(\eta), \tag{3.24}$$

where δ_* is a constant to be determined,

$$\Gamma(\eta) = \begin{pmatrix} b_0(\eta) + c_0(\eta)l(\eta) & c_0(\eta)m(\eta) \\ c_0(\eta)n(\eta) & b_0(\eta) - c_0(\eta)l(\eta) \end{pmatrix},$$
$$b_0 = \frac{b}{\sqrt{\Delta}}, \quad c_0 = \frac{c}{\sqrt{\Delta}}, \quad \Delta = b^2 - c^2 f, \quad f = l^2 + mn,$$
$$b = \frac{\zeta}{\delta_*\delta_1}\left(1 + \frac{r}{\zeta d_1}\right), \quad c = \frac{i\hat{\gamma}\eta\cos\beta}{\delta_*\delta_1 d_1},$$
$$\hat{\gamma} = \gamma_1^-\gamma_4^- + \gamma_1^+\gamma_4^+ - k_0^2\sin^2\beta, \tag{3.25}$$

and l, m, n and r are polynomials,

$$l(\eta) = \eta\cos\beta[\eta^2(\gamma_1^+ - \gamma_4^+) - 2\eta(\gamma_1^- - \gamma_4^-)(\gamma_1^+ + \gamma_4^+)$$
$$+ (\gamma_1^+ - \gamma_4^+)(2\gamma_1^-\gamma_4^- - 3k_0^2) - (\gamma_1^+ - \gamma_4^+)(\eta^2 - k_0^2)\cos 2\beta],$$
$$m(\eta) = -\frac{3}{4}\eta^4 + \eta^3(\gamma_1^- - \gamma_4^-) + 2\eta(\gamma_1^- - \gamma_4^-)[(\gamma_4^+)^2 - k_0^2]$$
$$+ [(\gamma_4^+)^2 - k_0^2](2\gamma_1^-\gamma_4^- + k_0^2) + \frac{3}{4}\eta^2[4\gamma_1^-\gamma_4^- - 4(\gamma_4^+)^2 + k_0^2] - \frac{1}{4}\eta^2(\eta^2 - k_0^2)\cos 4\beta$$
$$+ \{\eta^4 - \eta^3(\gamma_1^- - \gamma_4^-) + \eta^2[\gamma_1^-\gamma_4^- - (\gamma_4^+)^2 - k_0^2] + k_0^2[(\gamma_4^+)^2 - k_0^2]\}\cos 2\beta, \tag{3.26}$$

the polynomial $n(\eta)$ coincides with $-m(\eta)$ if γ_1^- and γ_1^+ are replaced by γ_4^- and γ_4^+, respectively, and

$$r(\eta) = \frac{i}{16}(\gamma_1^+ + \gamma_4^+)[-\eta^2 + 2(\gamma_1^+\gamma_4^+ + k_0^2) + \eta^2\cos 2\beta]$$
$$\times [3\eta^4 - 8\eta^2(\gamma_1^-)^2 - 8\eta^2\gamma_1^-\gamma_4^- - 8\eta^2(\gamma_4^-)^2 + 8(\gamma_1^-)^2(\gamma_4^-)^2 + 2\eta^2k_0^2$$
$$+ 8\gamma_1^-\gamma_4^-k_0^2 + 3k_0^4 - 4(\eta^2 - k_0^2)(\eta^2 + 2\gamma_1^-\gamma_4^- + k_0^2)\cos 2\beta + (\eta^2 - k_0^2)^2\cos 4\beta]. \tag{3.27}$$

In (3.25) we divided the coefficients of the matrix $\Gamma(\eta)$ by $\sqrt{\Delta(\eta)}$ ($\sqrt{\Delta(\eta)}$ is a fixed branch of the function $\Delta^{1/2}(\eta)$) and in (3.24) we multiplied the matrix $\Gamma(\eta)$ by the same factor. This transformation implies $\det\Gamma(\eta)=1$ for all $\eta\in L$ and simplifies the procedure of matrix factorization. Notice that from (3.24),

$$\det G(\eta)=\frac{\delta_*^2\Delta(\eta)d_1(\eta)}{\delta_0^2(\eta)}. \tag{3.28}$$

On comparing (3.28) and (3.16), we obtain that $\Delta(\eta)$ is a rational even function

$$\Delta(\eta)=\frac{\delta_0(\eta)\delta_0(-\eta)}{\delta_*^2 d_1(\eta)}. \tag{3.29}$$

4. Scalar RHP on a Riemann surface

4.1 *Statement of the problem*

Since $G^1(\eta)$ is a polynomial matrix, the problem of factorization of the matrix $G(\eta)$ reduces to the one for the matrix $\Gamma(\eta)$. Its factorization can be expressed through the solution of the associated scalar RHP on the genus-3 Riemann surface $\mathfrak{R}$ of the algebraic function $w^2=f(\eta)$. The function $f(\eta)$ is an even degree-8 polynomial which has the form

$$f(\eta)=h_0+h_1\eta^2+h_2\eta^4+h_3\eta^6+h_4\eta^8, \tag{4.1}$$

where

$$\begin{aligned}
h_0&=-4u_1u_2u_3^2,\quad h_4=-4\sin^8\beta,\\
h_1&=4(\gamma_1^- -\gamma_4^-)^2u_1u_2+(\gamma_1^+-\gamma_4^+)^2(2\gamma_1^-\gamma_4^- -3k_0^2+k_0^2\cos 2\beta)^2\cos^2\beta-2u_3\\
&\quad\times\{u_2[(\gamma_1^-\gamma_4^- -(\gamma_1^+)^2)(\cos 2\beta+3)+2k_0^2\sin^4\beta]+u_1[(\gamma_1^-\gamma_4^- -(\gamma_4^+)^2)(\cos 2\beta+3)\\
&\quad+2k_0^2\sin^4\beta]\},\\
h_2&=4u_4^+u_4^- +8(\gamma_1^-\gamma_4^-u_4^+ +\gamma_1^+\gamma_4^+u_4^-)\cos^2\beta+8\gamma_1^-\gamma_1^+\gamma_4^-\gamma_4^+\cos^2\beta(\cos 2\beta-3)\\
&\quad-[(\gamma_1^-\gamma_4^-)^2+(\gamma_1^+\gamma_4^+)^2](\cos 2\beta+3)^2+2k_0^2\sin^2\beta\{-u_4^+(3\cos 2\beta+1)\\
&\quad+2\cos^2\beta[-2\gamma_1^+\gamma_4^+(\cos 2\beta-3)+\gamma_1^-\gamma_4^-(\cos 2\beta+3)]-4u_4^- -2k_0^2\sin^2\beta(1+\cos^4\beta)\},\\
h_3&=4\sin^4\beta[u_4^- -u_4^+ +2(\gamma_1^-\gamma_4^- -\gamma_1^+\gamma_4^+)\cos^2\beta+2k_0^2\sin^4\beta],\\
u_1&=(\gamma_1^+)^2-k_0^2,\quad u_2=(\gamma_4^+)^2-k_0^2,\quad u_3=\gamma_1^-\gamma_4^- +k_0^2\cos^2\beta,\quad u_4^\pm=(\gamma_1^\pm)^2+(\gamma_4^\pm)^2.
\end{aligned} \tag{4.2}$$

Let $\pm a_j$ $(j=1,2,3,4)$ be the eight branch point of the function $\sqrt{f(\eta)}$ (they are determined explicitly by radicals) such that $0<\operatorname{Im}a_1\le\operatorname{Im}a_2\le\operatorname{Im}a_3\le\operatorname{Im}a_4$. Cut the plane along the segments $l_1^\pm$ and $l_2^\pm$ with the starting points $\pm a_1$ and $\pm a_3$ and the terminal points $\pm a_2$ and $\pm a_4$, respectively. Fix the single branch of the function $\sqrt{f(\eta)}$ as follows:

$$\sqrt{f(\eta)}=2i\sin^4\beta\prod_{j=1}^4\sqrt{\rho_j^+\rho_j^-}\,e^{i(\varphi_j^+ +\varphi_j^-)/2}, \tag{4.3}$$

where $\rho_j^{\pm} = |\eta \mp a_j|$, $\varphi_j^{\pm} = \arg(\eta \mp a_j)$, $\varphi_j^{+} \in (-\pi, 0)$, $\varphi_j^{-} \in (0, \pi)$, $\eta \in L$, $j = 1, 2, 3, 4$.

The genus-3 hyperelliptic surface $\mathfrak{R}$ is formed by two copies, $\mathbb{C}_1$ and $\mathbb{C}_2$, of the extended η-plane $\mathbb{C} \cup \{\infty\}$ cut along the segments $l_1^{\pm}$ and $l_2^{\pm}$. The two sheets are glued according to the rule

$$w = \begin{cases} \sqrt{f(\eta)}, & \eta \in \mathbb{C}_1, \\ -\sqrt{f(\eta)}, & \eta \in \mathbb{C}_2. \end{cases} \tag{4.4}$$

Let $\lambda(\eta, w)$ be a new function defined on the surface $\mathfrak{R}$ as

$$\lambda(\eta, w) = \begin{cases} \lambda_1(\eta), & \eta \in \mathbb{C}_1, \\ \lambda_2(\eta), & \eta \in \mathbb{C}_2, \end{cases} \tag{4.5}$$

where $\lambda_1(\eta) = b_0(\eta) + c_0(\eta)\sqrt{f(\eta)}$ and $\lambda_1(\eta) = b_0(\eta) - c_0(\eta)\sqrt{f(\eta)}$ are the eigenvalues of the matrix $\Gamma(\eta)$. We find it convenient to introduce two matrices, Y and Q, as

$$Y(\eta, w) = \frac{1}{2}\left[I + \frac{1}{w}Q(\eta)\right], \quad Q(\eta) = \begin{pmatrix} l(\eta) & m(\eta) \\ n(\eta) & -l(\eta) \end{pmatrix}. \tag{4.6}$$

Then the Wiener-Hopf matrix factors of the matrix $\Gamma(\eta)$ may be expressed through a non-trivial solution to the associated scalar RHP on the surface $\mathfrak{R}$ (**20**), (**21**), (**22**).

THEOREM 4.1. Let $\chi(\eta, w)$ be a non-trivial solution to the following RHP.

Find a function $\chi(\eta, w)$ piece-wise analytic on the surface $\mathfrak{R} \setminus \mathfrak{L}$, $\mathfrak{L} = (L \subset \mathbb{C}_1) \cup (L \subset \mathbb{C}_2)$, except for at most a finite number of poles, Hölder-continuous up to the contour $\mathfrak{L}$, bounded at both infinite points of the surface $\mathfrak{R}$ and satisfying the boundary condition

$$\chi^{+}(\eta, w) = \lambda(\eta, w)\chi^{-}(\eta, w), \quad (\eta, w) \in \mathfrak{L}. \tag{4.7}$$

Then the matrix $X(\eta)$ and its inverse

$$X(\eta) = \chi(\eta, w)Y(\eta, w) + \chi(\eta, -w)Y(\eta, -w), \quad \eta \in \mathbb{C}^{\pm},$$
$$[X(\eta)]^{-1} = \frac{Y(\eta, w)}{\chi(\eta, w)} + \frac{Y(\eta, -w)}{\chi(\eta, -w)}, \quad \eta \in \mathbb{C}^{\pm}, \tag{4.8}$$

provide a piece-wise meromorphic solution of the matrix factorization problem

$$\Gamma(\eta) = X^{+}(\eta)[X^{-}(\eta)]^{-1}, \quad \eta \in L. \tag{4.9}$$

4.2 *Solution with an essential singularity at infinity*

The chief difficulty in the procedure of matrix factorization based on the use of the solution of the scalar RHP on a Riemann surface (4.7) arises in the necessity of constructing a meromorphic solution with explicitly determined poles and zeros. In order to derive such a solution, we consider the function

$$\chi_0(\eta, w) = \exp\{\psi_0(\eta, w)\}, \quad (\eta, w) \in \mathfrak{R}, \tag{4.10}$$

where $\psi_0(\eta, w)$ is the Weierstrass integral

$$\psi_0(\eta, w) = \frac{1}{2\pi i}\int_{\mathfrak{L}} \log\lambda(t,\xi)\frac{w+\xi}{2\xi}\frac{dt}{t-\eta}, \tag{4.11}$$

and $\xi = w(t)$, $t \in \mathfrak{L}$. The integral over the contour $\mathcal{L}$ on the surface $\mathfrak{R}$ can be transformed into two integrals over the contour L in the η-plane as

$$\psi_0(\eta, w) = \frac{1}{4\pi i}\int_L [\log\lambda_1(t) + \log\lambda_2(t)]\frac{dt}{t-\eta} + \frac{w}{4\pi i}\int_L [\log\lambda_1(t) - \log\lambda_2(t)]\frac{dt}{\sqrt{f(t)}(t-\eta)}. \tag{4.12}$$

To simplify this representation, we study the eigenvalues λ_j of the matrix $\Gamma(\eta)$. Referring to (3.25), we obtain that $b(\eta) = b(-\eta)$, $c(\eta) = -c(-\eta)$ and also

$$\lambda_1(\eta)\lambda_2(\eta) = \det\Gamma(\eta) = 1, \quad \frac{\lambda_1(\eta)}{\lambda_2(\eta)} = \frac{b(\eta) + c(\eta)\sqrt{f(\eta)}}{b(\eta) - c(\eta)\sqrt{f(\eta)}} = \frac{\lambda_2(-\eta)}{\lambda_1(-\eta)}. \tag{4.13}$$

As $\eta \to \infty$,

$$b(\eta) = -\frac{i}{\delta_*(\gamma_1^+ + \gamma_4^+)} + O(\eta^{-1}), \quad c(\eta) \sim -\frac{\hat{\gamma}\cos\beta}{\eta^5(\gamma_1^+ + \gamma_4^+)\sin^6\beta}, \quad \eta \to \infty. \tag{4.14}$$

On putting $\delta_* = -i(\gamma_1^+ + \gamma_4^+)^{-1}$, we have $b(\eta) = 1 + O(\eta^{-1})$, $\eta \to \infty$. At zero and at infinity, the function $\lambda_1(\eta)/\lambda_2(\eta)$ has the following properties:

$$\frac{\lambda_1(0)}{\lambda_2(0)} = 1, \quad \frac{\lambda_1(\eta)}{\lambda_2(\eta)} = 1 + O(\eta^{-1}), \quad \eta \to \infty. \tag{4.15}$$

It has also been established that $\lambda_1(\eta)$ and $\lambda_2(\eta)$ are bounded and do not vanish on the contour L. On fixing the branches of the functions $\log\lambda_1(\eta)$ and $\lambda_2(\eta)$ as $\log\lambda_1(\infty) = \log\lambda_2(\infty) = 0$ we obtain that $\log\lambda_2(\eta) = -\log\lambda_1(\eta)$ for all $\eta \in L$ and therefore

$$\psi_0(\eta, w) = \frac{w}{4\pi i}\int_L \frac{\varepsilon(t)dt}{\sqrt{f(t)}(t-\eta)}, \tag{4.16}$$

where

$$\varepsilon(\eta) = \log l(\eta), \quad l(\eta) = \frac{\lambda_1(\eta)}{\lambda_2(\eta)}, \tag{4.17}$$

and the branch of the logarithmic function is fixed by the condition $\varepsilon(\infty) = 0$. To establish whether the function $\varepsilon(\eta)$ is continuous in the contour L, we study next the increment of the argument of the function $l(\eta)$ when η traverses the contour L in the positive direction. Introduce the index of the function $l(\eta)$

$$\operatorname{ind} l(\eta) = \frac{1}{2\pi}[\arg l(\eta)]_{-\infty}^{\infty}. \tag{4.18}$$

Due to the fact that $\lambda_2(-\eta) = \lambda_1(\eta)$, the function $\varepsilon(\eta)$ is odd, and the index of the function $l(\eta)$ is an even number, ind $l(\eta) = 2\kappa_0$. If $\kappa_0 = 0$, then the function $\varepsilon(\eta)$ is continuous at the point $\eta = 0$, otherwise it is discontinuous at $\eta = 0$. Notice also that

$$\kappa_0 = \frac{1}{2\pi}[\arg \lambda_1(\eta)]_{-\infty}^{\infty} = \frac{1}{2\pi}\left[\arg \frac{\lambda_1(\eta)}{\lambda_2(\eta)}\right]_0^{\infty}. \tag{4.19}$$

Numerical computations implemented for different sets of the problem parameters k_0, β, $\gamma_1^{\pm}$ and $\gamma_4^{\pm}$ show that κ_0 may be 0, 1, or -1. Figure 2 shows samples of the graph of the function $l(\eta)$ for $0 \le \eta < +\infty$ in the plane $(\operatorname{Re} l(\eta), \operatorname{Im} l(\eta))$ for some values of the problem parameters in the case (i) when both zeros of the polynomial $\delta_0(\eta)$ lie in the lower half-plane. The graphs of the function $l(\eta)$ for some values of the problem parameters in the case (ii), when the two zeros of $\delta_0(\eta)$ lie in the opposite half-planes, are given in Figure 3a and 3b. Figure 3c and 3d illustrate the case (iii) (both

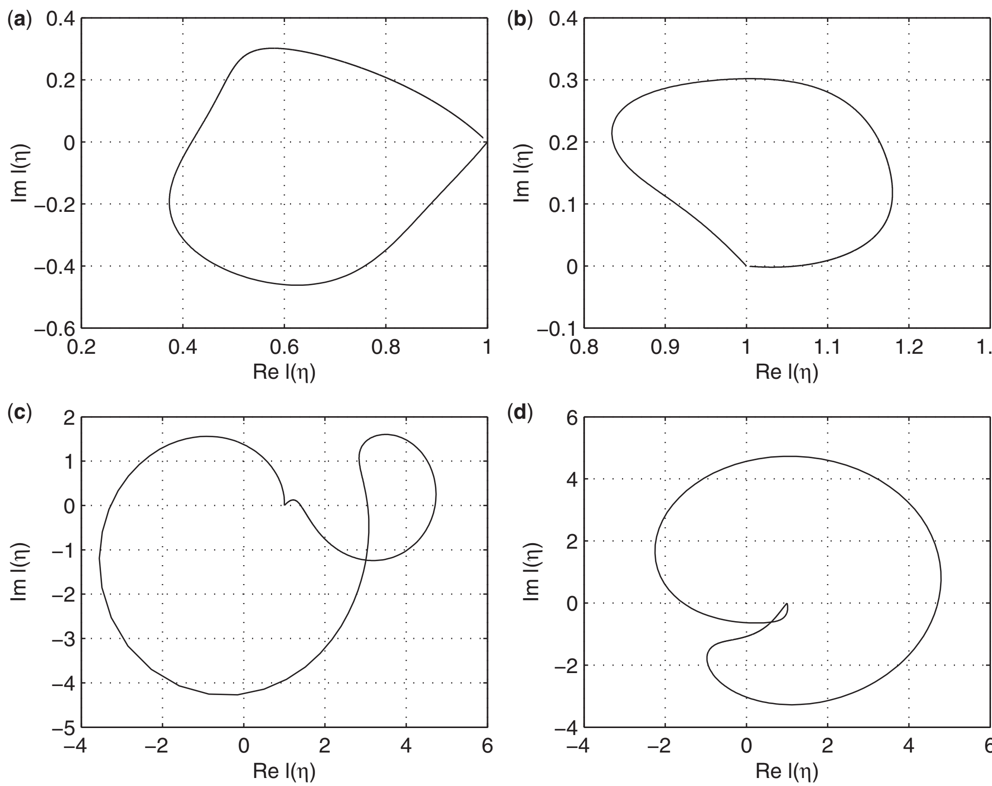

Fig. 2 Parametrically defined function $l(\eta)$, $0 \le \eta < +\infty$, in the case (i, section 3) when $k_0 = 1 + 0.1i$, $\beta = \pi/4$. Case a: $\kappa_0 = 0$ ($\gamma_1^+ = 1 - i$, $\gamma_4^+ = 1 + 2i$, $\gamma_1^- = 1 - 2i$, $\gamma_4^- = 1 - 3i$). Case b: $\kappa_0 = 0$ ($\gamma_1^+ = 2 + i$, $\gamma_4^+ = 1 + 2i$, $\gamma_1^- = 1 - i$, $\gamma_4^- = 1 - 2i$). Case c: $\kappa_0 = 1$ (the anticlockwise direction, $\gamma_1^+ = -1 + i$, $\gamma_4^+ = -1 + 2i$, $\gamma_1^- = -1 - 0.1i$, $\gamma_4^- = -1 - 0.2i$). Case d: $\kappa_0 = -1$ (the clockwise direction, $\gamma_1^+ = 1 + i$, $\gamma_4^+ = 1 - i$, $\gamma_1^- = 1 - 0.3i$, $\gamma_4^- = 1 - i$)

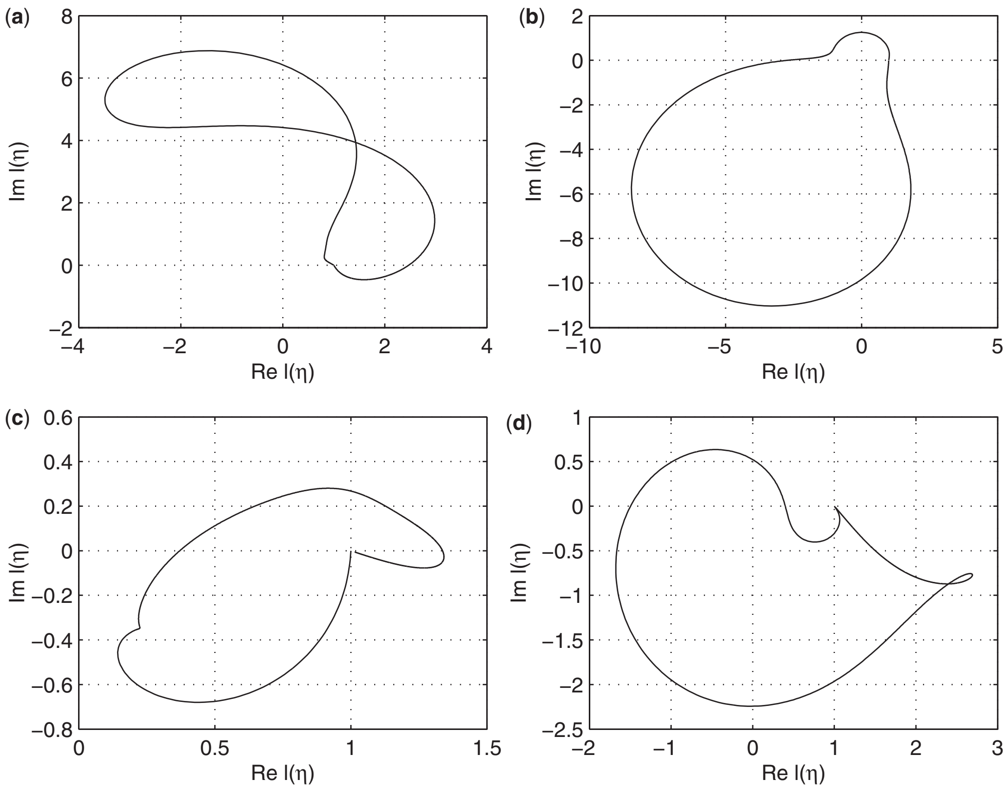

Fig. 3 Parametrically defined function $l(\eta)$, $0 \le \eta < +\infty$, in cases (ii, section 3): a, b and (iii): c, d, when $k_0 = 1 + 0.1i$, $\beta = \pi/4$. Case a: $\kappa_0 = 0$ ($\gamma_1^+ = 1.5 + 0.5i$, $\gamma_4^+ = 1 + i$, $\gamma_1^- = 1 - 0.5i$, $\gamma_4^- = 1 + i$). Case b: $\kappa_0 = 1$ (the anticlockwise direction, $\gamma_1^+ = 1 + i$, $\gamma_4^+ = 1 - i$, $\gamma_1^- = 1 - i$, $\gamma_4^- = 1 + i$). Case c: $\kappa_0 = 0$ ($\gamma_1^+ = 1 + 3i$, $\gamma_4^+ = 1 + 4i$, $\gamma_1^- = 1 + i$, $\gamma_4^- = 1 + 2i$). Case d: $\kappa_0 = -1$ (the clockwise direction, $\gamma_1^+ = 1 + 0.5i$, $\gamma_4^+ = 1 + i$, $\gamma_1^- = 2 + 0.5i$, $\gamma_4^- = 1 + 2i$)

zeros lie in the upper half-plane) when index κ_0 vanishes and equals -1, respectively. In the last case η traverses the contour around the origin in the negative (clock-wise) direction.

Due to the choice of the branch of the logarithmic function $\log l(\eta)$, if $\kappa_0 = 1$, then $\varepsilon(0^\pm) = \mp 2\pi i$, $\varepsilon(0^\pm) = \pm 2\pi i$ when $\kappa_0 = -1$, and $\varepsilon(\pm 0) = 0$ in the case $\kappa_0 = 0$. Transform now the integral (4.16) into the form

$$\psi_0(\eta, w) = \frac{w}{2\pi i} \int_0^\infty \frac{\varepsilon(t) t dt}{\sqrt{f(t)}(t^2 - \eta^2)}. \tag{4.20}$$

This integral is bounded as $\eta \to 0$ when $\kappa_0 = 0$ and has a logarithmic singularity otherwise,

$$\psi_0(\eta, w) = (-1)^{j-1} \kappa_0 \log \eta + O(1), \quad \eta \to 0, \quad (\eta, w) \in \mathbb{C}_j, \quad j = 1, 2. \tag{4.21}$$

It can be directly verified that the function

$$\hat{\psi}_0(\eta, w) = \psi_0(\eta, w) + \kappa_0 \sum_{j=1}^{2} (-1)^{j-1} \int_{q_{0j}}^{q_{1j}} \frac{w+\xi}{2\xi} \frac{dt}{t-\eta} \tag{4.22}$$

is bounded at the point $\eta = 0$. Here, $q_{0j} = (0, (-1)^{j-1}\sqrt{f(0)}) \in \mathbb{C}_j$ are the two zero points of the surface, and $q_{1j} = ((-1)^{j-1}\rho_0, (-1)^{j-1}\sqrt{f(\rho_0)}) \in \mathbb{C}_j$, where ρ_0 is an arbitrary fixed point in the upper half-plane which coincides with none of the branch points a_j ($j = 1, 2, 3, 4$), the point η_0, and the roots of the polynomials $d_1(\eta)$ and $\delta_0(\eta)\delta_0(-\eta)$ (the final solution of the vector RHP is invariant with respect to the choice of the point ρ_0). Because of the symmetry of the points q_{11} and q_{12} we can rewrite the function $\hat{\psi}_0(\eta, w)$ as

$$\hat{\psi}_0(\eta, w) = \psi_0(\eta, w) + \kappa_0 \left[\int_0^{\rho_0} \frac{\eta dt}{t^2 - \eta^2} + w \int_0^{\rho_0} \frac{tdt}{\sqrt{f(t)}(t^2 - \eta^2)} \right]. \tag{4.23}$$

The function $\exp\{\hat{\psi}_0(\eta, w\}$ satisfies the boundary condition of the RHP (4.7). However, at the infinite points of the surface it has essential singularities due to the second order pole at infinity of the function (4.23). In order to remove this singularity, we cut the surface $\mathfrak{R}$ along two pairs of loops $\mathbf{a}_-$, $\mathbf{b}_-$ and $\mathbf{a}_+$, $\mathbf{b}_+$ (Fig. 4). The loops $\mathbf{a}_\pm$ intersect the loops $\mathbf{b}_\pm$ at the points $\pm a_2$, respectively. The cross-sections $\mathbf{a}_-$ and $\mathbf{a}_+$ and $\mathbf{b}_-$ and $\mathbf{b}_+$ are symmetric with respect to the origin. The curves $\mathbf{a}_-$ and $\mathbf{a}_+$ are closed contours that join the branch points $-a_1$ with $-a_2$ and a_1 with a_2, respectively, and lie on both sheets of the surface. The loops $\mathbf{b}_\pm$ join the points $\pm a_2$ with $\pm a_3$ and pass through the infinite points, where they touch each other but do no intersect. Both parts of the loop $\mathbf{b}_-$ lie on the lower half-planes of both sheets, while the contour $\mathbf{b}_+$ lies on the upper half-planes. The solid lines in Fig. 4 correspond to the parts of the loops $\mathbf{b}_\pm$ on the sheet $\mathbb{C}_1$, while the broken lines show the parts of the loops on the second sheet. Notice that the system of loops $\mathbf{a}_\pm$, $\mathbf{b}_\pm$ does not form a system of canonical cross-section of the genus-3 surface $\mathfrak{R}$ which comprises three pairs of loops. Due to the special symmetry of the problem, to remove the essential singularity at infinity, it suffices to use the

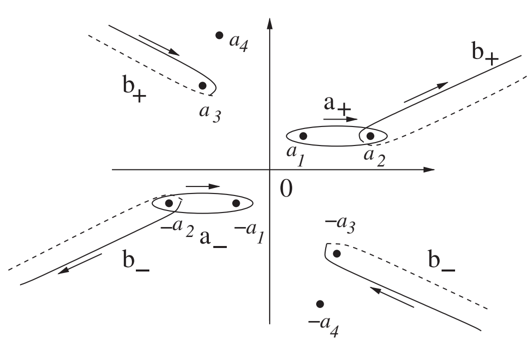

Fig. 4 The loops $\mathbf{a}_\pm$ and $\mathbf{b}_\pm$

two cross-sections $\mathbf{a}_\pm$ and $\mathbf{b}_\pm$. Analyze now the function

$$\tilde{\psi}_0(\eta, w) = \left[\int_{c_-} + \int_{c_+} + m_0\left(\oint_{\mathbf{a}_-} + \oint_{\mathbf{a}_+}\right) + n_0\left(\oint_{\mathbf{b}_-} + \oint_{\mathbf{b}_+}\right)\right] \frac{w+\xi}{2\xi}\frac{dt}{t-\eta}. \tag{4.24}$$

Here, the contours c_- and c_+ are smooth simple curves on the surface which do not cross the loops $\mathbf{a}_\pm$ and $\mathbf{b}_\pm$. The starting and terminal points of the contours $c_\pm$ are $\pm p_0$ and $\pm p_1$, respectively. The point $p_0 = (\sigma_0, \sqrt{f(\sigma_0)})$ is an arbitrary fixed point in the sheet $\mathbb{C}_1$ (the final solution of the vector RHP is independent of this point), while the point $p_1 = (\sigma_1, w(\sigma_1))$ may lie on either sheet of the surface and has to be determined. The numbers m_0 and n_0 are integers to be fixed. The function $\exp\{\tilde{\psi}_0(\eta, w)\}$ is meromorphic in any finite part of the surface $\mathfrak{R}$ and continues through the contours $c_\pm$, $\mathbf{a}_\pm$ and $\mathbf{b}_\pm$. Add the function $\tilde{\psi}_0(\eta, w)$ to $\hat{\psi}_0(\eta, w)$. A simple alteration of this sum, $\psi(\eta, w)$, implies

$$\psi(\eta, w) = \frac{w}{2\pi i}\int_0^\infty \frac{\varepsilon(t)tdt}{\sqrt{f(t)}(t^2-\eta^2)} + \kappa_0\int_0^{\rho_0}\left(\eta + \frac{wt}{\sqrt{f(t)}}\right)\frac{dt}{t^2-\eta^2}$$
$$+ \left(\int_{p_0}^{p_1} + m_0\oint_{\mathbf{a}_+} + n_0\oint_{\mathbf{b}_+}\right)\frac{w+\xi}{\xi}\frac{tdt}{t^2-\eta^2}. \tag{4.25}$$

The function $\chi(\eta, w) = \exp\{\psi(\eta, w)\}$ is meromorphic in each finite part of the Riemann surface and satisfies the boundary condition (4.7). However, due to the pole of the Weierstrass kernel at infinity, it has essential singularities at both infinite points of the surface, (∞, ∞_1) and (∞, ∞_2).

4.3 *Jacobi inversion problem*

The analysis of the solution at the points (∞, ∞_1) and (∞, ∞_2) shows that both functions, $\psi(\eta, w)$ and $\chi(\eta, w) = \exp\{\psi(\eta, w)\}$, are bounded as $\eta \to \infty$ if and only if the following condition holds:

$$\frac{1}{2\pi i}\int_0^\infty \frac{\varepsilon(t)tdt}{\sqrt{f(t)}} + \kappa_0\int_0^{\rho_0}\frac{tdt}{\sqrt{f(t)}} + \int_{p_0}^{p_1}\frac{tdt}{\xi} + m_0\oint_{\mathbf{a}_+}\frac{tdt}{\xi} + n_0\oint_{\mathbf{b}_+}\frac{tdt}{\xi} = 0. \tag{4.26}$$

In this section we aim to reduce the condition (4.26) to a genus-1 Jacobi inversion problem and solve it by inversion of an elliptic integral. Generically, a scalar RHP on a genus-ρ surface requires solving the associated Jacobi inversion problem in terms of the zeros of the genus-ρ Riemann θ-function (**23**). However, due to the symmetry properties of the problem, the genus of the θ-function can be decreased. In our case we have managed to avoid the genus-3 θ-function associated with the surface $\mathfrak{R}$ and decrease it genus to $\rho = 1$ because

(i) the degree-8 characteristic polynomial $f(\eta)$ is even and
(ii) the function $\varepsilon(\eta)$ is odd.

To solve the non-linear equation (4.26) that is to define the point $p_1 \in \mathfrak{R}$ and the integers m_0 and n_0, first, we express the integrals over the loops $\mathbf{a}_+$ and $\mathbf{b}_+$

$$\oint_{\mathbf{a}_+}\frac{tdt}{\xi} = \int_{a_1^2}^{a_2^2}\frac{dt_*}{\sqrt{f_*(t_*)}},$$

$$\oint_{\mathbf{b}_+} \frac{tdt}{\xi} = \int_{a_2^2}^{\infty} \frac{dt_*}{\sqrt{f_*(t_*)}} + \int_{\infty}^{a_3^2} \frac{dt_*}{\sqrt{f_*(t_*)}} \tag{4.27}$$

through elliptic integrals. Here, $f(t) = f_*(t^2)$, and the branch $\sqrt{f_*(\eta_*)}$ is fixed by the condition $\sqrt{f_*(\eta_*)} \sim 2i\eta_*^2 \sin^4\beta$, $\eta_* \to \infty$, in the η_*-plane ($\eta_* = \eta^2$) cut along the two lines joining the branch points, a_1^2 with a_2^2 and a_3^2 with a_4^2. By making the homographic transformation (**24**) $t_* = (\beta_1 + \beta_2\tau)(1 - \mu\tau)^{-1}$ we fix the parameters β_1, β_2 and μ such that the points a_1^2, a_2^2, a_3^2, and a_4^2 are mapped into the points 1, $1/\kappa$, $-1/\kappa$ and -1, respectively (κ is to be determined). It directly follows that

$$\frac{t_* - a_4^2}{a_1^2 - a_4^2} = \frac{(1-\mu)(1+\tau)}{2(1-\mu\tau)}, \quad \frac{t_* - a_1^2}{a_4^2 - a_1^2} = \frac{(1+\mu)(1-\tau)}{2(1-\mu\tau)}, \tag{4.28}$$

and

$$\frac{t_* - a_3^2}{a_2^2 - a_3^2} = \frac{(1-\mu/\kappa)(1+\kappa\tau)}{2(1-\mu\tau)}, \quad \frac{t_* - a_2^2}{a_3^2 - a_2^2} = \frac{(1+\mu/\kappa)(1-\kappa\tau)}{2(1-\mu\tau)}. \tag{4.29}$$

Equations (4.28) imply

$$\frac{t_* - a_4^2}{t_* - a_1^2} = \frac{(\mu-1)(1+\tau)}{(\mu+1)(1-\tau)}. \tag{4.30}$$

On putting in this equation $t_* = a_2^2$, $\tau = 1/\kappa$ and $t_* = a_3^2$, $\tau = -1/\kappa$ and denoting $\mu_* = (\mu - 1)/(\mu + 1)$, $\kappa_* = (\kappa - 1)/(\kappa + 1)$ we obtain the system of equations

$$\frac{\mu_*}{\kappa_*} = \frac{a_2^2 - a_4^2}{a_2^2 - a_1^2}, \quad \mu_*\kappa_* = \frac{a_3^2 - a_4^2}{a_3^2 - a_1^2}. \tag{4.31}$$

It has two sets of solutions defined by

$$\kappa_*^2 = \frac{(a_2^2 - a_1^2)(a_3^2 - a_4^2)}{(a_2^2 - a_4^2)(a_3^2 - a_1^2)}, \quad \mu_* = \frac{(a_2^2 - a_4^2)\kappa_*}{a_2^2 - a_1^2}. \tag{4.32}$$

Each of them determines the parameters μ and κ,

$$\mu = \frac{1+\mu_*}{1-\mu_*}, \quad \kappa = \frac{1+\kappa_*}{1-\kappa_*}. \tag{4.33}$$

By expressing t_* from the two relations (4.28) and adding the formulas obtained we have the homographic transformation sought

$$t_* = \frac{a_1^2 + a_4^2}{2} + \frac{(a_1^2 - a_4^2)(\tau - \mu)}{2(1-\mu\tau)}. \tag{4.34}$$

We next differentiate the two equations (4.29) and find

$$dt_* = \frac{\sqrt{(1-\mu^2)(\kappa - \mu^2/\kappa)(a_2^2 - a_3^2)(a_1^2 - a_4^2)}}{2(1-\mu\tau)^2} d\tau. \tag{4.35}$$

On multiplying the four relations in (4.28) and (4.29) and using (4.35) we deduce

$$\int \frac{dt_*}{\sqrt{f_*(t_*)}} = h \int \frac{d\tau}{\sqrt{(1-\tau^2)(1-\kappa^2\tau^2)}}, \tag{4.36}$$

where

$$h = \frac{\epsilon}{i \sin^4 \beta} \sqrt{\frac{\kappa}{(a_2^2 - a_3^2)(a_1^2 - a_4^2)}}, \quad \epsilon = \pm 1, \tag{4.37}$$

and $v(\tau) = \sqrt{(1-\tau^2)(1-\kappa^2\tau^2)}$ is the branch fixed by the condition $v(0) = 1$ of the two-valued function $v^2 = (1-\tau^2)(1-\kappa^2\tau^2)$ in the τ-plane cut along the segments $\hat{l}_1$ and $\hat{l}_2$ with the starting and terminal points $-1/\kappa$ and -1 for $\hat{l}_1$ and 1 and $1/\kappa$ for $\hat{l}_2$. We can now compute the integrals over the loops $\mathbf{a}_+$ and $\mathbf{b}_+$ in (4.27)

$$\oint_{\mathbf{a}_+} \frac{tdt}{\xi} = h \int_1^{1/\kappa} \frac{d\tau}{v(\tau)} = hi\mathbf{K}',$$
$$\oint_{\mathbf{b}_+} \frac{tdt}{\xi} = 2h \int_{1/\kappa}^{\infty} \frac{d\tau}{v(\tau)} = -2h\mathbf{K}, \tag{4.38}$$

where $\mathbf{K}' = \mathbf{K}(\sqrt{1-\kappa^2})$, and $\mathbf{K} = \mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind. Denote next the inverse to (4.34) transformation as

$$\tau = u(t_*) = \frac{t_* - a_4^2 - \mu_*(t_* - a_1^2)}{t_* - a_4^2 + \mu_*(t_* - a_1^2)} \tag{4.39}$$

and let $\hat{\sigma}_j = u(\sigma_j^2)$, $j = 0, 1$. On making this transformation in the third integral in (4.26) we derive

$$\int_{p_0}^{p_1} \frac{tdt}{\xi} = \frac{h}{2} \left(\int_{\hat{\sigma}_0}^{0} \frac{d\tau}{v(\tau)} + \int_0^{\hat{\sigma}_1} \frac{d\tau}{v(\tau)} \right) \tag{4.40}$$

in the case $p_1 \in \mathbb{C}_1$ and

$$\int_{p_0}^{p_1} \frac{tdt}{\xi} = \frac{h}{2} \left(\int_{\hat{\sigma}_0}^{0} \frac{d\tau}{v(\tau)} - 2\int_0^1 \frac{d\tau}{v(\tau)} - \int_0^{\hat{\sigma}_1} \frac{d\tau}{v(\tau)} \right) \tag{4.41}$$

in the case $p_1 \in \mathbb{C}_2$. The transformation u maps the points $0, \infty$ and ρ_0^2 of the t_*-plane into the points $u_0 = (a_4^2 - \mu_* a_1^2)(a_4^2 + \mu_* a_1^2)^{-1}$, $1/\mu$ and $u_1 = u(\rho_0^2)$ of the τ-plane, respectively. It is helpful to denote

$$\hat{d} = -\frac{1}{2\pi i} \int_{u_0}^{1/\mu} \frac{\hat{\varepsilon}(\tau) d\tau}{v(\tau)} - \kappa_0 \int_{u_0}^{u_1} \frac{d\tau}{v(\tau)} + \int_0^{\hat{\sigma}_0} \frac{d\tau}{v(\tau)}, \quad \hat{\varepsilon}(\tau) = \varepsilon(\sqrt{t_*}). \tag{4.42}$$

Now the Jacobi inversion problem (4.26) may be put in a more convenient form. Let first $p_1 \in \mathbb{C}_1$. Then

$$\int_0^{\hat{\sigma}_1} \frac{d\tau}{v(\tau)} = \hat{d} + 4n_0\mathbf{K} - 2im_0\mathbf{K}'. \tag{4.43}$$

By inversion of the elliptic integral in (4.43) we find $\hat{\sigma}_1 = \operatorname{sn}\hat{d}$ and therefore the affix σ_1 of the point p_1 is given by

$$\sigma_1 = \pm\sqrt{\frac{a_1^2+a_4^2}{2} + \frac{(a_1^2-a_4^2)(\hat{\sigma}_1-\mu)}{2(1-\mu\hat{\sigma}_1)}}. \tag{4.44}$$

Because of the symmetry, either sign leads to the same solution. The integers m_0 and n_0 can be directly found from (4.43)

$$m_0 = -\frac{\operatorname{Im}[(I_* - \hat{d})\overline{\mathbf{K}}]}{2\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}, \quad n_0 = \frac{\operatorname{Re}[(I_* - \hat{d})\overline{\mathbf{K}'}]}{4\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}, \tag{4.45}$$

where I_* is an elliptic integral of the first kind

$$I_* = \int_0^{\hat{\sigma}_1} \frac{d\tau}{v(\tau)} = F(\sin^{-1}(\operatorname{sn}\hat{d}), \kappa). \tag{4.46}$$

If at least one of the numbers m_0, n_0 is not integer, then $p_1 \in \mathbb{C}_2$, and n_0, m_0 are both necessarily integer. In this case the Jacobi inversion problem (4.26) becomes

$$-\int_0^{\hat{\sigma}_1} \frac{d\tau}{v(\tau)} = \hat{d} + 4\left(n_0 + \frac{1}{2}\right)\mathbf{K} - 2im_0\mathbf{K}'. \tag{4.47}$$

The affix of the point p_1 is the same as in the previous case, while the integers m_0 and n_0 are different,

$$m_0 = \frac{\operatorname{Im}[(I_* + \hat{d})\overline{\mathbf{K}}]}{2\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}, \quad n_0 = -\frac{1}{2} - \frac{\operatorname{Re}[(I_* + \hat{d})\overline{\mathbf{K}'}]}{4\operatorname{Re}[\mathbf{K}\overline{\mathbf{K}'}]}. \tag{4.48}$$

This completes the solution of the Jacobi problem. Summarize the results of Section 4.

THEOREM 4.2. Let $p_0 = (\sigma_0, \sqrt{f(\sigma_0)}) \in \mathbb{C}_1$, $\rho_0 \in \mathbb{C}^+$ be arbitrary fixed points and none of them coincide with the branch points of $f^{1/2}(\eta)$. Let κ_0 be the integer (4.19) and σ_1 be either complex number in (4.44). Choose the point p_1 as $(\sigma_1, \sqrt{f(\sigma_1)}) \in \mathbb{C}_1$ if both numbers in (4.45) are integers and as $(\sigma_1, -\sqrt{f(\sigma_1)}) \in \mathbb{C}_2$ otherwise (in this case both numbers in (4.48) are necessarily integers). Define $\psi(\eta)$ by (4.25). Then the function

$$\chi(\eta, w) = e^{\psi(\eta, w)}, \quad (\eta, w) \in \mathfrak{R}, \tag{4.49}$$

is bounded at the two infinite points of the surface $\mathfrak{R}$, piece-wise meromorphic on $\mathfrak{R}$ with the discontinuity line $\mathfrak{L}$ in which it satisfies the boundary condition (4.7) of the scalar RHP.

5. Solution to the vector RHP

5.1 *Matrix factorization and its analysis*

As a matter of utility, it will be desirable to have the solution to the RHP (4.7) on the Riemann surface $\mathfrak{R}$ expressed in terms of two functions defined on the η-plane

$$\chi(\eta, w) = \exp\{\psi_1(\eta) + w\psi_2(\eta)\}, \tag{5.1}$$

where

$$\psi_1(\eta) = \frac{\kappa_0}{2}\log\frac{\eta-\rho_0}{\eta+\rho_0} + \frac{1}{2}\log\frac{\eta^2-\sigma_1^2}{\eta^2-\sigma_0^2},$$

$$\psi_2(\eta) = \frac{1}{2\pi i}\int_0^{\infty}\frac{\varepsilon(t)tdt}{\sqrt{f(t)}(t^2-\eta^2)} + \kappa_0\int_0^{\rho_0}\frac{tdt}{\sqrt{f(t)}(t^2-\eta^2)}$$
$$+\left(\int_{p_0}^{p_1} + m_0\oint_{\mathbf{a}_+} + n_0\oint_{\mathbf{b}_+}\right)\frac{tdt}{\xi(t^2-\eta^2)}. \tag{5.2}$$

Because the solution to the Jacobi problem, the point σ_1 and the integers m_0 and n_0, satisfy the condition (4.26), it is possible to alter formula (5.2) for the function $\psi_2(\eta)$ as

$$\psi_2(\eta) = \frac{1}{\eta^2}\left\{\frac{1}{2\pi i}\int_0^{\infty}\frac{\varepsilon(t)t^3dt}{\sqrt{f(t)}(t^2-\eta^2)} + \kappa_0\int_0^{\rho_0}\frac{t^3dt}{\sqrt{f(t)}(t^2-\eta^2)}\right.$$
$$\left.+\left(\int_{p_0}^{p_1} + m_0\oint_{\mathbf{a}_+} + n_0\oint_{\mathbf{b}_+}\right)\frac{t^3dt}{\xi(t^2-\eta^2)}\right\} \tag{5.3}$$

which can be conveniently used for large η. The analysis of this formula shows that $\psi_1(\eta)+w\psi_2(\eta)$ is bounded as $\eta\to\infty$, and therefore the essential singularity of the solution $\chi(\eta,w)$ has been removed.

With the solution to the RHP on the surface $\mathfrak{R}$ at hand, we may write down formulas (4.8) for the factor-matrices $X^{\pm}(\eta)$ and their inverses in terms of functions defined on the η-plane

$$X^{\pm}(\eta) = e^{\psi_1^{\pm}(\eta)}\left[\cosh(f^{1/2}(\eta)\psi_2^{\pm}(\eta))I + \frac{1}{f^{1/2}(\eta)}\sinh(f^{1/2}(\eta)\psi_2^{\pm}(\eta))Q(\eta)\right],$$
$$[X^{\pm}(\eta)]^{-1} = e^{-\psi_1^{\pm}(\eta)}\left[\cosh(f^{1/2}(\eta)\psi_2^{\pm}(\eta))I - \frac{1}{f^{1/2}(\eta)}\sinh(f^{1/2}(\eta)\psi_2^{\pm}(\eta))Q(\eta)\right], \quad \eta\in\mathbb{C}^{\pm}. \tag{5.4}$$

It is essential for the solution of the vector RHP to study the behavior of the matrix $X(\eta)=X^{\pm}(\eta)$, $\eta\in\mathbb{C}^{\pm}$, at the points $p_0=(\sigma_0,\sqrt{f(\sigma_0)})\in\mathbb{C}_1$ and $p_1=(\sigma_1,w(\sigma_1))\in\mathfrak{R}$. For this purpose we use the representation (4.8). From (5.1), (5.2),

$$\chi(\eta,w)\sim\frac{E_{01}^{\pm}}{\eta\mp\sigma_0}, \quad \eta\to\pm\sigma_0, \quad (\eta,w)\in\mathbb{C}_1,$$
$$\chi(\eta,w)\sim E_{02}^{\pm}, \quad \eta\to\pm\sigma_0, \quad (\eta,w)\in\mathbb{C}_2, \tag{5.5}$$

where $E_{0j}^{\pm}$ are non-zero constants. This yields

$$X(\eta)\sim\frac{E_{01}^{\pm}}{\eta\mp\sigma_0}Y^{\pm}, \quad \eta\to\pm\sigma_0, \tag{5.6}$$

where $Y^{\pm}=Y(\pm\sigma_0,\sqrt{f(\sigma_0)})$ are rank-1 2×2 matrices.

If p_1 is a point of the first sheet of the surface $\mathfrak{R}$, then

$$\chi(\eta, w) \sim E_{11}^{\pm}(\eta \mp \sigma_1), \quad \chi(\eta, -w) \sim E_{12}^{\pm}, \quad \eta \to \pm\sigma_1, \quad (\eta, w) \in \mathbb{C}_1. \tag{5.7}$$

In the case $p_1 \in \mathbb{C}_2$,

$$\chi(\eta, w) \sim E_{21}^{\pm}, \quad \chi(\eta, -w) \sim E_{22}^{\pm}(\eta \mp \sigma_1), \quad \eta \to \pm\sigma_1, \quad (\eta, w) \in \mathbb{C}_1. \tag{5.8}$$

Here, $E_{sj}^{\pm}$ $(s, j = 1, 2)$ are non-zero constants. Then, regardless of whether the point p_1 belongs to the first sheet or the second one, the inverse matrix $[X(\eta)]^{-1}$ has poles at the points $\pm\sigma_1$. Assume now that $z(\eta)$ is an order-2 vector whose components have certain non-zero limits at the points $\eta = \pm\sigma_1$. Then since $Y(\eta, w)$ is a rank-1 matrix, we have

$$[X(\eta)]^{-1} z(\eta) \sim \frac{E_j^{\pm}}{\eta \mp \sigma_1} \begin{pmatrix} 1 \\ s_j^{\pm} \end{pmatrix}, \quad \eta \to \pm\sigma_1, \quad p_1 \in \mathbb{C}_j, \quad j = 1, 2, \tag{5.9}$$

where $E_j^{\pm}$ =const, $s_j^{\pm} = n(\pm\sigma_1)[l(\pm\sigma_1) + w_j]^{-1}$, and $w_j = (-1)^{j-1}\sqrt{f(\sigma_1)}$ for $p_1 \in \mathbb{C}_j, j = 1, 2$. At the same time, if $\hat{z}(\eta)$ is an order-2 vector such that

$$\hat{z}(\eta) \sim \frac{\hat{E}_j^{\pm}}{\eta \mp \sigma_1} \begin{pmatrix} 1 \\ s_j^{\pm} \end{pmatrix}, \quad \eta \to \pm\sigma_1, \tag{5.10}$$

where $\hat{E}_j^{\pm}$ are non-zero constants, then

$$X(\eta)\hat{z}(\eta) = Y(\pm\sigma_1, -w_j) \frac{\hat{E}_j^{\pm}}{\eta \mp \sigma_1} \begin{pmatrix} 1 \\ s_j^{\pm} \end{pmatrix} + \tilde{z}(\eta) = \tilde{z}(\eta), \quad \eta \to \pm\sigma_1, \tag{5.11}$$

where $\tilde{z}(\eta)$ is an order-2 vector bounded as $\eta \to \pm\sigma_1$.

We next examine the matrix $X(\eta)$ and its inverse as $\eta \to \pm\rho_0$. If $\kappa_0 = 0$, then both functions, $\psi_1(\eta)$ and $\psi_2(\eta)$, are bounded at the points $\pm\rho_0$, and therefore the matrices $X(\eta)$ and $[X(\eta)]^{-1}$ are bounded as well. If $|\kappa_0| = 1$, then

$$\begin{aligned} &\chi(\eta, w) \sim D_1^{+}(\eta - \rho_0)^{\kappa_0}, \quad \eta \to \rho_0, \quad (\eta, w) \in \mathbb{C}_1, \\ &\chi(\eta, w) \sim D_2^{+}, \quad \eta \to \rho_0, \quad (\eta, w) \in \mathbb{C}_2. \end{aligned} \tag{5.12}$$

At the points with affix $-\rho_0$,

$$\begin{aligned} &\chi(\eta, w) \sim D_1^{-}, \quad \eta \to -\rho_0, \quad (\eta, w) \in \mathbb{C}_1, \\ &\chi(\eta, w) \sim D_2^{-}(\eta + \rho_0)^{-\kappa_0}, \quad \eta \to -\rho_0, \quad (\eta, w) \in \mathbb{C}_2. \end{aligned} \tag{5.13}$$

Here, $D_j^{\pm}$ are non-zero constants. At the points $\eta = \pm\kappa_0\rho_0$ $(\kappa_0 = \pm 1)$, the matrix $X(\eta)$ and its inverse accordingly behave

$$\begin{aligned} &X(\eta) = O(1), \quad [X(\eta)]^{-1} \sim D'_{\pm}(\eta - \kappa_0\rho_0)^{-1}\tilde{Y}_{\kappa_0}^{+}, \quad \eta \to \kappa_0\rho_0, \\ &X(\eta) \sim D''_{\pm}(\eta + \kappa_0\rho_0)^{-1}\tilde{Y}_{\kappa_0}^{-}, \quad [X(\eta)]^{-1} = O(1), \quad \eta \to -\kappa_0\rho_0, \end{aligned} \tag{5.14}$$

where $\tilde{Y}_{\kappa_0}^{\pm} = Y(\pm\kappa_0\rho_0, \pm\kappa_0\sqrt{f(\rho_0)})$ are rank-1 2×2 matrices, and $D'_{\pm}$, $D''_{\pm}$ are non-zero constant.

Before proceeding with the solution of the vector RHP, we need to factorize the function $\sqrt{\Delta(\eta)}$ with $\Delta(\eta)$ being a rational function given by (3.29). Since the function $\Delta(\eta)$ admits the splitting

$$\Delta(\eta)=\frac{(\eta^2-\tau_1^2)(\eta^2-\tau_2^2)}{(\eta^2-t_1^2)(\eta^2-t_2^2)}, \tag{5.15}$$

where $\operatorname{Im} t_j>0$, $\operatorname{Im}\tau_j>0$, $j=1,2$, we immediately obtain

$$\sqrt{\Delta(\eta)}=\frac{\rho^+(\eta)}{\rho^-(\eta)},\quad \rho^\pm(\eta)=\left\{\frac{(\eta\pm\tau_1)(\eta\pm\tau_2)}{(\eta\pm t_1)(\eta\pm t_2)}\right\}^{\pm1/2},\quad \eta\in\mathbb{C}^\pm. \tag{5.16}$$

The branches of the functions $\rho^\pm(\eta)$ are chosen such that $\rho^\pm(\eta)\to 1$, $\eta\to\infty$, and the branch cuts join the branch points $\mp t_1$ with $\mp t_2$ and $\mp\tau_1$ with $\mp\tau_2$. Referring now to the representations (3.24) and (4.9) we split the matrix $G(\eta)$ as

$$G(\eta)=\frac{\delta_* G^1(\eta)}{\delta_0(\eta)}\rho^+(\eta)X^+(\eta)[\rho^-(\eta)X^-(\eta)]^{-1},\quad \eta\in L. \tag{5.17}$$

We remind that $\delta_0(\eta)$ and $G^1(\eta)$ are the polynomial and the polynomial matrix given by (3.17) and (3.21), respectively.

5.2 *Vectors* $\Phi^+(\eta)$ *and* $\Phi^-(\eta)$

To determine the vectors $\Phi^+(\eta)$ and $\Phi^-(\eta)$, we insert the splitting (5.17) into the boundary condition (3.6) of the vector RHP and use the formulas

$$\hat G^1(\eta)=d_1(\eta)[G^1(\eta)]^{-1},\quad \hat G^1(\eta)=\begin{pmatrix} g^1_{22}(\eta) & -g^1_{12}(\eta)\\ -g^1_{21}(\eta) & g^1_{11}(\eta)\end{pmatrix}, \tag{5.18}$$

where g^1_{sj} $(s,j=1,2)$ are given by (3.21). On factorizing the polynomial $d_1(\eta)$, we discover

$$\frac{\delta_0(\eta)[X^+(\eta)]^{-1}\hat G^1(\eta)\Phi^+(\eta)}{\delta_*\alpha_2(\eta+t_1)(\eta+t_2)\rho^+(\eta)}=\frac{(\eta-t_1)(\eta-t_2)}{\rho^-(\eta)}[X^-(\eta)]^{-1}\Phi^-(\eta),\quad \eta\in L. \tag{5.19}$$

Here, $\pm t_1$ and $\pm t_2$ are the four zeros of the polynomial $d_1(\eta)=\alpha_2(\eta^2-t_1^2)(\eta^2-t_2^2)$, and $\operatorname{Im} t_j>0$, $j=1,2$.

We begin with the case (i) when the two zeros of the quadratic polynomial $\delta_0(\eta)$ lie in the lower half-plane, $\delta_0(\eta)=-(\eta+\tau_1)(\eta+\tau_2)\sin^2\beta$, $\operatorname{Im}\tau_j>0$, $j=1,2$. On applying the principle of analytic continuation and the Liouville theorem, we have

$$\frac{\nu(\eta+\tau_1)(\eta+\tau_2)[X^+(\eta)]^{-1}\hat G^1(\eta)\Phi^+(\eta)}{(\eta+t_1)(\eta+t_2)\rho^+(\eta)}=\frac{(\eta-t_1)(\eta-t_2)}{\rho^-(\eta)}[X^-(\eta)]^{-1}\Phi^-(\eta)=R(\eta),\quad \eta\in\mathbb{C}, \tag{5.20}$$

where $\nu=-\delta_*\csc^2\beta$, and $R(\eta)$ is a rational vector-function. Note that due to (5.9) the vector $R(\eta)$ has simple poles at the points $\pm\sigma_1$. Also, because of (5.14) the vector $R(\eta)$ has simple poles at the points $\eta = \kappa_0\rho_0$ in the case $\kappa_0 = \pm 1$ and is bounded if $\kappa_0 = 0$. In addition, it has to have simple poles (the geometric optics poles) at the points $\eta = \pm\eta_0$. Since the vectors $\Phi^{\pm}(\eta)$ vanish, the matrices $[X^{\pm}(\eta)]^{-1}$ are bounded at infinity, and the elements of the matrix $G^1(\eta)$ are degree-2 polynomials, the vector $R(\eta)$ has a pole at the infinite point of multiplicity 1 if $\kappa_0 = 0$ and multiplicity 2 if $\kappa_0 = \pm 1$. The most general form of the vector $R(\eta)$ in the former case is given by

$$R(\eta) = \frac{P(\eta)}{(\eta^2 - \sigma_1^2)(\eta^2 - \eta_0^2)}, \tag{5.21}$$

where $P(\eta)$ is an order-2 vector whose components, $P_1(\eta)$ and $P_2(\eta)$, are degree-5 polynomials. In the case $\kappa_0 = \pm 1$,

$$R(\eta) = \frac{P(\eta)}{(\eta^2 - \sigma_1^2)(\eta^2 - \eta_0^2)(\eta - \kappa_0\rho_0)}, \tag{5.22}$$

where the components $P_1(\eta)$ and $P_2(\eta)$ of the vector $P(\eta)$ are degree-6 polynomials. The vectors $\Phi^{\pm}(\eta)$ can be deduced from (5.20)

$$\Phi^{+}(\eta) = \frac{\nu\rho^{+}(\eta)G^1(\eta)X^{+}(\eta)R(\eta)}{(\eta+\tau_1)(\eta+\tau_2)(\eta-t_1)(\eta-t_2)}, \quad \eta \in \mathbb{C}^{+},$$

$$\Phi^{-}(\eta) = \frac{\rho^{-}(\eta)X^{-}(\eta)R(\eta)}{(\eta-t_1)(\eta-t_2)}, \quad \eta \in \mathbb{C}^{-}. \tag{5.23}$$

The solution has twelve arbitrary constants, the coefficients of the degree-5 polynomials $P_1(\eta)$ and $P_2(\eta)$, in the case $\kappa_0 = 0$ and fourteen constants if $\kappa_0 = \pm 1$.

In the case (ii), when one of the zeros of the polynomial $\delta_0(\eta)$, τ_1, lies in the upper half-plane and the other, τ_2, is in the lower half-plane, $\delta_0(\eta) = -(\eta - \tau_1)(\eta + \tau_2)\sin^2\beta$, and the boundary condition of the RHP gives

$$\frac{\nu(\eta+\tau_2)[X^{+}(\eta)]^{-1}\hat{G}^1(\eta)\Phi^{+}(\eta)}{(\eta+t_1)(\eta+t_2)\rho^{+}(\eta)} = \frac{(\eta-t_1)(\eta-t_2)}{(\eta-\tau_1)\rho^{-}(\eta)}[X^{-}(\eta)]^{-1}\Phi^{-}(\eta) = R(\eta), \quad \eta \in \mathbb{C}. \tag{5.24}$$

The rational vector $R(\eta)$ is given by (5.21), where $P_1(\eta)$ and $P_2(\eta)$ are polynomials of the fourth degree in the case $\eta_0 = 0$ and by (5.22) with the fifth degree polynomials in the case $\kappa_0 = \pm 1$. The rearrangement of the factors of the polynomial $\delta_0(\eta)$ produces the new solution

$$\Phi^{+}(\eta) = \frac{\nu\rho^{+}(\eta)G^1(\eta)X^{+}(\eta)R(\eta)}{(\eta+\tau_2)(\eta-t_1)(\eta-t_2)}, \quad \eta \in \mathbb{C}^{+},$$

$$\Phi^{-}(\eta) = \frac{(\eta-\tau_1)\rho^{-}(\eta)X^{-}(\eta)R(\eta)}{(\eta-t_1)(\eta-t_2)}, \quad \eta \in \mathbb{C}^{-}, \tag{5.25}$$

which has either 10, or 12 arbitrary constants depending whether $\kappa_0 = 0$, or $\kappa_0 = \pm 1$.

In the third case, when both zeros of the polynomial $\delta_0(\eta)$ lie in the upper half-plane, $\delta_0(\eta) = -(\eta - \tau_1)(\eta - \tau_2)\sin^2\beta$, we have

$$\frac{\nu[X^{+}(\eta)]^{-1}\hat{G}^1(\eta)\Phi^{+}(\eta)}{(\eta+t_1)(\eta+t_2)\rho^{+}(\eta)} = \frac{(\eta-t_1)(\eta-t_2)}{(\eta-\tau_1)(\eta-\tau_2)\rho^{-}(\eta)}[X^{-}(\eta)]^{-1}\Phi^{-}(\eta) = R(\eta), \quad \eta \in \mathbb{C}. \tag{5.26}$$

As before, the rational vector $R(\eta)$ has the form (5.21) for $\kappa_0 = 0$ and (5.22) for $\kappa_0 = \pm 1$. However, the components of the vector $P(\eta)$ are degree-3 polynomials if $\kappa_0 = 0$ and degree-4 polynomials if $\kappa_0 = \pm 1$. The solution has eight arbitrary constants in the former case and ten constants in the second case. It may be written as

$$\Phi^+(\eta) = \frac{\nu\rho^+(\eta)G^1(\eta)X^+(\eta)R(\eta)}{(\eta - t_1)(\eta - t_2)}, \quad \eta \in \mathbb{C}^+,$$
$$\Phi^-(\eta) = \frac{(\eta - \tau_1)(\eta - \tau_2)\rho^-(\eta)X^-(\eta)R(\eta)}{(\eta - t_1)(\eta - t_2)}, \quad \eta \in \mathbb{C}^-. \tag{5.27}$$

5.3 *Symmetry conditions*

In general, the solution derived in the previous section does not meet the symmetry condition $\Phi^+(\eta) = \Phi^-(-\eta)$, $\eta \in \mathbb{C}$. We begin with the case (i). In order to satisfy the symmetry condition, we rewrite it as

$$R(-\eta) = \frac{\nu(\eta + t_1)(\eta + t_2)[\rho^+(\eta)]^2}{(\eta + \tau_1)(\eta + \tau_2)(\eta - t_1)(\eta - t_2)}[X^-(-\eta)]^{-1}G^1(\eta)X^+(\eta)R(\eta), \quad \eta \in \mathbb{C}^+. \tag{5.28}$$

Here, we used the relation $\rho^+(\eta) = [\rho^-(-\eta)]^{-1}$, $\eta \in \mathbb{C}^+$. We wish now to simplify the matrix $U(\eta) = [X^-(-\eta)]^{-1}G^1(\eta)X^+(\eta)$. It will be convenient to represent the function $\psi_1(\eta)$ as the sum of the even and odd functions

$$\psi_1(\eta) = \psi_{1o}(\eta) + \psi_{1e}(\eta), \quad \psi_{1o}(\eta) = \frac{\kappa_0}{2}\log\frac{\eta - \rho_0}{\eta + \rho_0}, \quad \psi_{1e}(\eta) = \frac{1}{2}\log\frac{\eta^2 - \sigma_1^2}{\eta^2 - \sigma_0^2}, \tag{5.29}$$

(the function $\psi_{1o}(\eta) \equiv 0$ if $\kappa_0 = 0$) and use the following notations ($\psi_2(\eta)$ is an even function):

$$c = \cosh[f^{1/2}(\eta)\psi_2(\eta)], \quad s = \sinh[f^{1/2}(\eta)\psi_2(\eta)],$$
$$Q(\pm\eta) = \begin{pmatrix} l_\pm & m_\pm \\ n_\pm & -l_\pm \end{pmatrix}. \tag{5.30}$$

By referring now to (5.4), we transform the matrix $U(\eta)$ as

$$U(\eta) = e^{2\psi_{1o}(\eta)}\begin{pmatrix} c - sl_-/\sqrt{f} & -sm_-/\sqrt{f} \\ -sn_-/\sqrt{f} & c + sl_-/\sqrt{f} \end{pmatrix}\begin{pmatrix} g_{11}^1 & g_{12}^1 \\ g_{21}^1 & g_{22}^1 \end{pmatrix}\begin{pmatrix} c + sl_+/\sqrt{f} & sm_+/\sqrt{f} \\ sn_+/\sqrt{f} & c - sl_+/\sqrt{f} \end{pmatrix}. \tag{5.31}$$

On using next the directly verified identities

$$(l_+ - l_-)g_{11}^1 + n_+g_{12}^1 - m_-g_{21}^1 = 0, \quad m_+g_{11}^1 - (l_+ + l_-)g_{12}^1 - m_-g_{22}^1 = 0,$$
$$-n_-g_{11}^1 + (l_+ + l_-)g_{21}^1 + n_+g_{22}^1 = 0, \quad -n_-g_{12}^1 + m_+g_{21}^1 - (l_+ - l_-)g_{22}^1 = 0,$$
$$c^2g_{11}^1 + \frac{s^2}{f}(-l_-l_+g_{11}^1 - l_-n_+g_{12}^1 - l_+m_-g_{21}^1 - m_-n_+g_{22}^1) = g_{11}^1,$$
$$c^2g_{12}^1 + \frac{s^2}{f}(-l_-m_+g_{11}^1 + l_-l_+g_{12}^1 - m_-m_+g_{21}^1 + l_+m_-g_{22}^1) = g_{12}^1,$$

$$c^2 g^1_{21} + \frac{s^2}{f}(-l_+ n_- g^1_{11} - n_- n_+ g^1_{12} + l_- l_+ g^1_{21} + l_- n_+ g^1_{22}) = g^1_{21},$$

$$c^2 g^1_{22} + \frac{s^2}{f}(-m_+ n_- g^1_{11} + l_+ n_- g^1_{12} + l_- m_+ g^1_{21} - l_- l_+ g^1_{22}) = g^1_{22}, \quad (5.32)$$

we ultimately deduce that $\exp\{-2\psi_{1o}(\eta)\}U(\eta)$ is a polynomial matrix that coincides with $G^1(\eta)$. Furthermore,

$$[X^-(-\eta)]^{-1} G^1(\eta) X^+(\eta) = \left(\frac{\eta - \rho_0}{\eta + \rho_0}\right)^{\kappa_0} G^1(\eta), \quad \eta \in \mathbb{C}^+. \quad (5.33)$$

Next, for the present purpose, we transform the symmetry condition (5.28) as

$$R(-\eta) = \frac{\nu}{(\eta - t_1)(\eta - t_2)} \left(\frac{\eta - \rho_0}{\eta + \rho_0}\right)^{\kappa_0} G^1(\eta) R(\eta) \quad (5.34)$$

or, equivalently,

$$P(-\eta) = \frac{(-1)^{\kappa_0} \nu G^1(\eta) P(\eta)}{(\eta - t_1)(\eta - t_2)}, \quad (5.35)$$

where the components of the vector $P(\eta)$, $P_1(\eta)$ and $P_2(\eta)$, are the polynomials

$$P_1(\eta) = \sum_{j=0}^{|\kappa_0|+5} a_j \eta^j, \quad P_2(\eta) = \sum_{j=0}^{|\kappa_0|+5} b_j \eta^j \quad (5.36)$$

whose coefficients are to be determined. On replacing η by $-\eta$ we reduce the condition (5.35) to

$$P(-\eta) = \frac{(\eta + t_1)(\eta + t_2)}{(-1)^{\kappa_0} \nu} [G^1(-\eta)]^{-1} P(\eta) \quad (5.37)$$

which is equivalent to the condition (5.35) since the matrix $G^1(\eta)$ possesses the following property:

$$G^1(\eta) G^1(-\eta) = d_1(\eta) I. \quad (5.38)$$

The symmetry condition (5.35) asserts that the coefficients of the polynomials $P_1(\eta)$ and $P_2(\eta)$ can be determined from the two equations

$$Q_1(\eta) = 0, \quad Q_2(\eta) = 0, \quad (5.39)$$

where

$$Q_j(\eta) = \sin^2 \beta (\eta - t_1)(\eta - t_2) P_j(-\eta) + (-1)^{\kappa_0} \delta_* [g^1_{j1}(\eta) P_1(\eta) + g^1_{j2}(\eta) P_2(\eta)], \quad j = 1, 2, \quad (5.40)$$

are polynomials of degree $6 + |\kappa_0|$ (the terms $\eta^{7+|\kappa_0|}$ in both polynomials have zero-coefficients).

Consider first the case $\kappa_0 = 0$. To define the coefficients, we compute the derivatives $Q_j^{(m)}(\eta)$, $m = 0, 1, \dots, 6; j = 1, 2$. The first six equations $Q_1^{(6-m)}(\eta) = 0$ $(m = 0, 1, \dots, 5)$ express the coefficients b_{5-m} through the coefficients of the polynomial $P_1(\eta)$

$$b_{2j} = \nu_0(-a_{2j}\nu_1^- + a_{2j+1}\nu_2^+),$$
$$b_{2j+1} = \nu_0(-2a_{2j}\sin^2\beta - a_{2j+1}\nu_1^+ + a_{2j+2}\nu_2^-), \quad j = 0, 1, 2, \tag{5.41}$$

where

$$\nu_0 = \frac{\gamma_1^+ + \gamma_4^+}{2\cos\beta[(\gamma_4^+)^2 - \gamma_1^-\gamma_4^- + k^2\sin^2\beta]}, \quad \nu_1^\pm = \gamma_1^- - \gamma_4^- \pm (t_1 + t_2)\sin^2\beta,$$
$$\nu_2^\pm = \gamma_1^-\gamma_4^- + k^2\cos^2\beta \pm t_1 t_2 \sin^2\beta, \quad a_6 = 0. \tag{5.42}$$

The equation $Q_1(\eta) = 0$ yields

$$a_0[2\gamma_1^-\gamma_4^- + k^2\cos^2\beta - t_1 t_2 \sin^2\beta] = 0. \tag{5.43}$$

It is directly verified that $2\gamma_1^-\gamma_4^- + k^2\cos^2\beta - t_1 t_2\sin^2\beta = 0$ and therefore a_0 is free. The other seven equations $Q_2^{(j)} = 0$ $(j = 0, 1, \dots, 6)$ are also identically satisfied provided the coefficients b_j $(j = 0, 1, \dots, 5)$ are chosen as in (5.41), and t_1, t_2 are the two zeros in the upper-half plane of the even degree-4 polynomial $d_1(\eta)$.

A similar result holds for the case $|\kappa_0| = 1$. The equations $Q_1^{(7-m)}(\eta) = 0$ $(m = 0, 1, \dots, 6)$ express the coefficients b_{6-m} through the coefficients of the polynomial $P_1(\eta)$

$$b_{2j} = \nu_0(-2a_{2j-1}\sin^2\beta - a_{2j}\nu_1^+ + a_{2j+1}\nu_2^-), \quad j = 0, 1, 2, 3, \quad a_{-1} = a_7 = 0,$$
$$b_{2j+1} = \nu_0(-a_{2j+1}\nu_1^- + a_{2j+2}\nu_2^+), \quad j = 0, 1, 2. \tag{5.44}$$

In the case $\kappa_0 = \pm 1$, we have $2\gamma_1^-\gamma_4^- + k^2\cos^2\beta + t_1 t_2\sin^2\beta = 0$. Therefore, the equation $Q_1(\eta) = 0$, which is equivalent to

$$a_0[2\gamma_1^-\gamma_4^- + k^2\cos^2\beta + t_1 t_2\sin^2\beta] = 0, \tag{5.45}$$

is satisfied for any a_0. If the coefficients b_j are defined by (5.44), then $Q_2(\eta) \equiv 0$. Consequently, in the case (i), the solution to the vector RHP possesses $6 + |\kappa_0|$ arbitrary constants a_j $(j = 0, 1, \dots, 5 + |\kappa_0|)$ and satisfies the symmetry condition $\Phi^+(\eta) = \Phi^-(-\eta)$.

Consider next the case (ii) when the two zeros of the polynomial $\delta_0(\eta)$ lie in the opposite half-planes. Since the relation (5.33) is invariant with respect to the location of the zeros of the polynomial $\delta_0(\eta)$, we employ it again and deduce from (5.25)

$$R(-\eta) = -\frac{\nu}{(\eta - t_1)(\eta - t_2)}\left(\frac{\eta - \rho_0}{\eta + \rho_0}\right)^{\kappa_0} G^1(\eta)R(\eta). \tag{5.46}$$

The rational vector $R(\eta)$ is given by (5.21) if $\kappa_0 = 0$ and by (5.22) if $\kappa_0 = \pm 1$, and

$$P_1(\eta) = \sum_{j=0}^{|\kappa_0|+4} a_j\eta^j, \quad P_2(\eta) = \sum_{j=0}^{|\kappa_0|+4} b_j\eta^j. \tag{5.47}$$

The symmetry condition (5.46) reads for the polynomials $P_1(\eta)$ and $P_2(\eta)$

$$P(-\eta) = \frac{(-1)^{\kappa_0+1}\nu G^1(\eta)P(\eta)}{(\eta - t_1)(\eta - t_2)}. \tag{5.48}$$

As in our earlier derivations described for the case (i), we obtain for $\kappa_0 = 0$

$$\begin{aligned} b_{2j} &= \nu_0(-2a_{2j-1}\sin^2\beta - a_{2j}\nu_1^+ + a_{2j+1}\nu_2^-), \quad j = 0, 1, 2, \quad a_{-1} = a_5 = 0, \\ b_{2j+1} &= \nu_0(-a_{2j+1}\nu_1^- + a_{2j+2}\nu_2^+), \quad j = 0, 1. \end{aligned} \tag{5.49}$$

If $|\kappa_0| = 1$, then the coefficients b_{2j} and b_{2j+1} coincide with the corresponding coefficients in the case (i), $\kappa = 0$, and are defined by (5.41). In the case (ii), the solution is given by (5.25), (5.21), (5.22), (5.47), (5.49) and (5.41). It satisfies the symmetry conditions and possesses $5 + |\kappa_0|$ arbitrary constants $a_0, a_1, \ldots, a_{4+|\kappa_0|}$.

Finally, in the case (iii), when the two zeros of the polynomial $\delta_0(\eta)$ lie in the upper half-plane, the symmetry relation coincides with the one derived in the case (i) and is given by (5.34). The polynomials $P_1(\eta)$ and $P_2(\eta)$ in the representation of the solution (5.27) have the form

$$P_1(\eta) = \sum_{j=0}^{|\kappa_0|+3} a_j\eta^j, \quad P_2(\eta) = \sum_{j=0}^{|\kappa_0|+3} b_j\eta^j, \tag{5.50}$$

where the constants a_j ($j = 0, 1, \ldots, 3 + |\kappa_0|$) are free and if $\kappa_0 = 0$,

$$\begin{aligned} b_{2j} &= \nu_0(-a_{2j}\nu_1^- + a_{2j+1}\nu_2^+), \\ b_{2j+1} &= \nu_0(-2a_{2j}\sin^2\beta - a_{2j+1}\nu_1^+ + a_{2j+2}\nu_2^-), \quad j = 0, 1, \quad a_4 = 0. \end{aligned} \tag{5.51}$$

If $|\kappa_0| = 1$, the coefficients are defined by (5.49). The solution derived has $4 + |\kappa_0|$ arbitrary constants $a_0, a_1, \ldots, a_{3+|\kappa_0|}$.

5.4 *Additional mathematical conditions*

Since the vector polynomial $P(\eta)$ meets the condition (5.35) in the cases (i) and (iii) and the condition (5.48) in the case (ii) for all η, and the left-hand side in (5.35) and (5.48) is bounded at the points t_1 and t_2, due to (5.23), (5.25) and (5.27), the vector $\Phi^+(\eta)$, regardless of which case is considered, (i), (ii), or (iii), has removable singularities at the points t_1 and t_2 ($\mathrm{Im}\, t_j > 0$).

Now, the property (5.9) of the matrix $[X(\eta)]^{-1}$ asserts that at the point σ_1 the polynomials $P_1(\eta)$ and $P_2(\eta)$ satisfy the relation

$$s_j^+ P_1(\sigma_1) = P_2(\sigma_1), \tag{5.52}$$

where $s_j^+ = n(\sigma_1)[l(\sigma_1) + (-1)^{j-1}\sqrt{f(\sigma_1)}]^{-1}$, $j = 1$ if $p_1 \in \mathbb{C}_1$, and $j = 2$ if $p_1 \in \mathbb{C}_2$. This relation is necessary and sufficient for the removal of the simple pole of the solution at the point σ_1. Because of the symmetry condition $\Phi^+(\eta) = \Phi^-(-\eta)$, $\eta \in \mathbb{C}^+$, equation (5.52) removes the singularity of the vector $\Phi^-(\eta)$ at the point $-\sigma_1$ as well.

According to the property (5.6) the solution to the vector RHP has simple poles at the points $\pm\sigma_0$. We may select σ_0, the affix of the arbitrary fixed point p_0 on the first sheet of the surface $\mathfrak{R}$, such that $\operatorname{Im}\sigma_0 > 0$. Since Y^+ in (5.6) is a rank-1 matrix, the point σ_0 is a removable point of the vector $\Phi^+(\eta)$ if and only if the following condition holds:

$$Y_{11}^+ P_1(\sigma_0) + Y_{12}^+ P_2(\sigma_0) = 0, \tag{5.53}$$

where (Y_{11}^+, Y_{12}^+) is the first row of the matrix $Y(\sigma_0, \sqrt{f(\sigma_0)})$. Again, due to the symmetry, this condition implies that the point $-\sigma_0$ is also a removable point of the solution.

When $\kappa_0 = 1$, according to (5.14) and (5.22), the vector $\Phi^+(\eta)$ has inadmissible pole at the point $\eta = \rho_0 \in \mathbb{C}^+$ (the point ρ_0 was chosen to be in the upper half-plane) due to the pole of the vector $R(\eta)$. The vector $\Phi^-(\eta)$ on the other hand, due to (5.14), has a simple pole at the point $\eta = -\rho_0$. Since $\tilde{Y}^-$ in (5.14) is a rank-1 matrix and the solution is symmetric, for the points $\eta = \pm\rho_0 \in \mathbb{C}^\pm$ being removable points of $\Phi^\pm(\eta)$ it is necessary and sufficient that the following condition holds:

$$\tilde{Y}_{11}^- P_1(-\rho_0) + \tilde{Y}_{12}^- P_2(-\rho_0) = 0, \tag{5.54}$$

where $\{\tilde{Y}_{11}^-, \tilde{Y}_{12}^-\}$ is the first row of the matrix $Y(-\rho_0, -\sqrt{f(\rho_0)})$.

In the case $\kappa_0 = -1$, the matrix $X(\eta)$ has a simple pole at the point $\eta = \rho_0$, and the vector $R(\eta)$ has a simple pole at $\eta = -\rho_0$. Therefore, the solution $\Phi^\pm(\eta)$ has simple poles at the points $\pm\rho_0$. As in the previous case, they can be removed by a single condition. It has the form

$$\tilde{Y}_{11}^+ P_1(\rho_0) + \tilde{Y}_{12}^+ P_2(\rho_0) = 0, \tag{5.55}$$

where $\{\tilde{Y}_{11}^+, \tilde{Y}_{12}^+\}$ is the first row of the matrix $Y(\rho_0, \sqrt{f(\rho_0)})$. The solution derived in the previous section has $s + |\kappa_0|$ arbitrary constants. On the other hand, if $\kappa_0 = 0$, the points $\pm\rho_0$ are regular points, and if $\kappa_0 = \pm 1$, they are poles removed by one condition. This asserts that the integer κ_0 does not effect the number of solutions of the vector RHP. This completes the solution procedure for the vector RHP.

Since the structure of the matrix coefficient $\hat{G}(\eta)$ for the RHP 2 is the same as for the RHP 1 (the five parameters of the problem, $\gamma_1^\pm$, $\gamma_4^\pm$ and β, need to be changed according to the transformation (2.21)), the same results are valid for the RHP 2. Recall that $\det G(\eta) = \delta_0(-\eta)/\delta_0(\eta)$. Then $\det \hat{G}(\eta) = \hat{\delta}_0(-\eta)/\hat{\delta}_0(\eta)$, where

$$\hat{\delta}_0(\eta) = (\eta^2 - k_0^2)\cos^2\beta - (\eta - \gamma_1^+)(\eta - \gamma_4^+). \tag{5.56}$$

Denote the zeros of the polynomials $\delta(\eta)$ and $\hat{\delta}(\eta)$ as η_j and $\hat{\eta}_j$ $(j = 1, 2)$, respectively,

$$\eta_j = \frac{\gamma_1^- + \gamma_4^- + (-1)^{j-1}\sqrt{\Delta_-}}{2\sin^2\beta}, \quad \hat{\eta}_j = \frac{\gamma_1^+ + \gamma_4^+ + (-1)^{j-1}\sqrt{\Delta_+}}{2\sin^2\beta}, \quad j = 1, 2, \tag{5.57}$$

where

$$\Delta_\pm = (\gamma_1^\pm - \gamma_4^\pm)^2 + 4(\gamma_1^\pm\gamma_4^\pm - k_0^2\sin^2\beta)\cos^2\beta. \tag{5.58}$$

Now we summarize the results.

THEOREM 5.1. Let

$$2\kappa = \frac{1}{2\pi}[\arg\det G(\eta)]|_{-\infty}^{+\infty}, \quad 2\hat{\kappa} = \frac{1}{2\pi}[\arg\det \hat{G}(\eta)]|_{-\infty}^{+\infty} \tag{5.59}$$

and

$$\kappa_j = \begin{cases} 1, \ \operatorname{Im}\eta_j < 0, \\ 0, \ \operatorname{Im}\eta_j > 0, \end{cases} \quad \hat{\kappa}_j = \begin{cases} 1, \ \operatorname{Im}\hat{\eta}_j < 0, \\ 0, \ \operatorname{Im}\hat{\eta}_j > 0. \end{cases} \tag{5.60}$$

Then

(i) $\kappa = \kappa_1 + \kappa_2 - 1$, $\hat{\kappa} = \hat{\kappa}_1 + \hat{\kappa}_2 - 1$,
(ii) the solution to the vector RHPs 1 and 2 exists and possesses $\kappa + 3$ and $\hat{\kappa} + 3$ arbitrary constants, respectively.

Now, the solutions to the vector RHPs 1 and 2 need to be compatible. Indeed, from equations (3.3) ($\gamma_2^+ = \gamma_3^+ = 0$) we can express $\hat{\phi}_j(i\zeta, 0)$ through the solution to the vector RHP 1

$$\hat{\phi}_j(i\zeta, 0) = -\frac{\gamma_{j+} + i\zeta}{2\eta}[\Phi_j^+(\eta) - \Phi_j^-(\eta)] - \frac{(-1)^j\cos\beta}{2}[\Phi_{3-j}^+(\eta) + \Phi_{3-j}^-(\eta)], \quad j = 1, 2, \tag{5.61}$$

where $\gamma_{1+} = \gamma_1^+$, $\gamma_{2+} = \gamma_4^+$. On replacing $i\zeta$ by η we obtain the functions $\hat{\Phi}_j^{\pm}(\eta)$. They constitute the solution of the vector RHP 2 but have a set of $\kappa + 3$ free constants of the RHP 1. On the other hand, the functions $\hat{\Phi}_j^{\pm}(\eta)$ can be derived directly by solving the RHP 2. These new expressions will have their own set of $\hat{\kappa} + 3$ free constants. The two solutions have to be the same, and we require

$$-\frac{\gamma_{j+}+\eta}{2\hat{\zeta}}[\Phi_j^+(\hat{\zeta}) - \Phi_j^-(\hat{\zeta})] - \frac{(-1)^j\cos\beta}{2}[\Phi_{3-j}^+(\hat{\zeta}) + \Phi_{3-j}^-(\hat{\zeta})]\Big|_{RHP1} = \hat{\Phi}_j^+(\eta)\Big|_{RHP2}, \quad j = 1, 2. \tag{5.62}$$

Here, $\hat{\zeta}$ is a branch of the two-valued function $\hat{\zeta}^2 = k_0^2 - \eta^2$. We fix it by the condition $\hat{\zeta}(0) = k_0$ in the η-plane cut along the lines passing through the infinite points and joining the branch points $\pm k_0$ (formulas (5.62) are invariant with respect to the choice of the branch $\hat{\zeta}$). We conjecture that the compatibility conditions (5.62) comprise $\kappa + \hat{\kappa} + 4$ equations for $\kappa + \hat{\kappa} + 6$ constants. Furthermore, in the case of normal incidence,

(i) if $\kappa_1 = \kappa_2 = \hat{\kappa}_1 = \hat{\kappa}_2 = 1$, then each equation in (5.62) yields three conditions that eliminate six constants,
(ii) if $\kappa_1 = \kappa_2 = \hat{\kappa}_1 = 1$, $\hat{\kappa}_2 = 0$, then the first and second equations give three and two conditions, respectively,
(iii) if $\kappa_1 = \kappa_2 = 1$, $\hat{\kappa}_1 = \hat{\kappa}_2 = 0$, then each equation in (5.62) eliminate two constants,
(iv) if $\kappa_1 = 1$, $\kappa_2 = 0$, $\hat{\kappa}_1 = 1$, $\hat{\kappa}_2 = 0$, then the first equation gives three conditions for the unknown constants, while the second one gives only one,
(v) if $\kappa_1 = 1$, $\kappa_2 = \hat{\kappa}_1 = \hat{\kappa}_2 = 0$, then the first and second equations give two and one conditions, respectively,

(vi) if $\kappa_1 = \kappa_2 = \hat{\kappa}_1 = \hat{\kappa}_2 = 0$, then there are two arbitrary constants say, D_j (RHP 1) and $\hat{D}_j$ (RHP 2), in the solution to each RHP, and $D_j = c_j \hat{D}_j$, $j = 1, 2$. Each equation in (5.62) brings us just one equation defining the scale factor c_j. Therefore, the compatibility conditions can be discarded.

We will verify this statement in Section 7 by using the method of undetermined coefficients. It is possible that the same statement is true in the general case. To apply this method for the oblique incidence, it it is required to implement some tedious computations. For numerical purposes, it is sufficient to solve equations (5.62) at arbitrary distinct fixed points. The number of points and equations is equal to $\kappa + \hat{\kappa} + 4$, and the equations need to be chosen such that the determinant of the system does not vanish.

We also note that the vectors $\Phi^+(\eta_0)$ and $\hat{\Phi}^+(\eta)$ found have simple poles at the geometric optics poles $\pm\eta_0$ and $\pm\hat{\eta}_0$, respectively. The residues of the solution at these points have not been specified. In the next section, we fix the residues and recover the incident and reflected waves. Since the compatibility conditions have been satisfied, both RHPs have two free constants, and it is sufficient to work with the solution to the RHP 1 only.

6. Sommerfeld integral representation: solution to the diffraction problem

The electric and magnetic field can be given in terms of the Sommerfeld integrals as

$$\begin{pmatrix} E_z(\rho, \theta) \\ ZH_z(\rho, \theta) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mathcal{T}} e^{-ik_0 \rho \cos s} \begin{pmatrix} S_1(s+\theta) \\ S_2(s+\theta) \end{pmatrix} ds, \tag{6.1}$$

where $\mathcal{T}$ is the Sommerfeld double loop with the asymptotes $\mathrm{Re}\, s = \pi/2$ and $\mathrm{Re}\, s = -3\pi/2$ for the upper loop $\mathcal{T}_+$ lying in the upper half-plane. The starting point of the contour is chosen to be $s = \pi/2 + i\infty$. The second loop $\mathcal{T}_-$ is symmetric to $\mathcal{T}_+$ with respect to the origin.

6.1 *Spectra $S_1(s)$ and $S_2(s)$*

The purpose of this section is to derive expressions for the functions $S_1(s)$ and $S_2(s)$. They are analytic everywhere in the strip $|\mathrm{Re}\, s| < \pi/2 + \epsilon$ ($\epsilon > 0$ and small) except for the simple poles $s = \pm\theta_0$, and

$$\operatorname*{res}_{s=\theta_0} S_1(s) = i_1, \quad \operatorname*{res}_{s=\theta_0} S_2(s) = i_2. \tag{6.2}$$

The knowledge of these functions enables us to fix the residues of the solution to the RHPs 1 and 2 at the geometric optics poles and recover the incident, reflected, surface and diffracted fields. On applying the inverse Maliuzhinets transform (**25**),

$$S_j(\theta + s) - S_j(\theta - s) = ik_0 \sin s \int_0^\infty e^{ik_0 \rho \cos s} \phi_j(\rho, \theta) d\rho, \tag{6.3}$$

and fixing first $\theta = 0$ and then $\theta = \pi/2$, we obtain the following two equations:

$$\begin{aligned} S_j(s) - S_j(-s) &= F_{j-}(s), \\ S_j\left(\frac{\pi}{2} + s\right) - S_j\left(\frac{\pi}{2} - s\right) &= F_{j+}(s), \quad j = 1, 2, \end{aligned} \tag{6.4}$$

where

$$F_{j-}(s) = ik_0 \sin s \int_0^\infty e^{ik_0 x \cos s}\phi_j(x,0)dx, \quad F_{j+}(s) = ik_0 \sin s \int_0^\infty e^{ik_0 y \cos s}\phi_j(0,y)dy. \tag{6.5}$$

We will adopt the Fourier transform in the form (**25**) and use the following notations for the transforms and their inverses:

$$\hat{S}_j(t) = -\frac{1}{2\pi}\int_{-i\infty}^{i\infty} S_j(s)e^{ist}ds, \quad S_j(s) = \int_{-i\infty}^{i\infty}\hat{S}_j(t)e^{-ist}dt,$$
$$\hat{F}_{j\pm}(t) = -\frac{1}{2\pi}\int_{-i\infty}^{i\infty} F_{j\pm}(s)e^{ist}ds, \quad F_{j\pm}(s) = \int_{-i\infty}^{i\infty}\hat{F}_{j\pm}(t)e^{-ist}dt. \tag{6.6}$$

When applied to the functions $S_j(\pi/2 \pm s)$, this transformation yields

$$-\frac{1}{2\pi}\int_{-i\infty}^{i\infty} S_j\left(\frac{\pi}{2} \pm s\right)e^{ist}ds = -\frac{e^{\mp i\pi t/2}}{2\pi}\int_{\pi/2-i\infty}^{\pi/2+i\infty} S_j(s)e^{\pm ist}ds$$
$$= \hat{S}_j(\pm t)e^{\mp i\pi t/2} - ii_j e^{\pm i(\theta_0-\pi/2)t}. \tag{6.7}$$

Here we used the fact that the functions $S_j(s)$ are analytic everywhere in the strip $|\operatorname{Re} s| < \pi/2 + \epsilon$ apart from a simple pole at the point $s = \theta_0$ with the residue given by (6.2). From these we find that corresponding to (6.4) we have the pair of equations

$$\hat{S}_j(t) - \hat{S}_j(-t) = \hat{F}_{j-}(t),$$
$$\hat{S}_j(t)e^{-i\pi t/2} - \hat{S}_j(-t)e^{i\pi t/2} = \hat{F}_{j+}(t) + 2i_j \sin\left(\frac{\pi}{2} - \theta_0\right)t. \tag{6.8}$$

Because of the symmetry of these equations it suffices to determine one of the functions $\hat{S}_j(\pm t)$ say, $\hat{S}_j(t)$,

$$\hat{S}_j(t) = \frac{\hat{F}_{j-}(t)e^{i\pi t/2} - \hat{F}_{j+}(t)}{2i\sin\pi t/2} + \frac{ii_j\sin(\pi/2-\theta_0)t}{\sin\pi t/2}. \tag{6.9}$$

It only remains to apply the inverse transform (6.6) and use the integral

$$\int_{-i\infty}^{i\infty}\frac{e^{i\sigma t}dt}{\sin\pi t/2} = -2\tan\sigma, \quad -\frac{\pi}{2} < \operatorname{Re}\sigma < \frac{\pi}{2}, \tag{6.10}$$

to obtain an integral representation for the function $S_j(s)$. We can fashion a similar formula for the function associated with the magnetic field. When combined the expressions become

$$S_j(s) = S_j^{(1)}(s) + S_j^{(2)}(s) + i_j[\cot(s-\theta_0) - \cot(s+\theta_0)], \quad 0 < \operatorname{Re} s < \frac{\pi}{2}, \tag{6.11}$$

where

$$S_j^{(1)}(s) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_{j-}(\sigma)\cot(\sigma - s)d\sigma, \quad 0 < \operatorname{Re} s < \pi,$$

$$S_j^{(2)}(s) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_{j+}(\sigma)\tan(\sigma - s)d\sigma, \quad -\frac{\pi}{2} < \operatorname{Re} s < \frac{\pi}{2}, \tag{6.12}$$

The functions $S_1(s)$ and $S_2(s)$ are analytic everywhere in the strip $0 < \operatorname{Re} s < \pi/2$ except for the point $s = \theta_0$, where they have a simple pole with the residue equal to i_1 and i_2, respectively. The integrands in (6.12) can be expressed through the solution to the vector RHP on employing (6.5) and (3.3)

$$\begin{aligned}
F_{1-}(s) &= -\frac{i}{2}(\gamma_1^+ + k_0\cos s)[\Phi_1^+(k_0\sin s) - \Phi_1^-(k_0\sin s)] \\
&\quad + \frac{i}{2}k_0\cos\beta\sin s[\Phi_2^+(k_0\sin s) + \Phi_2^-(k_0\sin s)], \\
F_{2-}(s) &= -\frac{i}{2}(\gamma_4^+ + k_0\cos s)[\Phi_2^+(k_0\sin s) - \Phi_2^-(k_0\sin s)] \\
&\quad - \frac{i}{2}k_0\cos\beta\sin s[\Phi_1^+(k_0\sin s) + \Phi_1^-(k_0\sin s)], \\
F_{j+}(s) &= ik_0\sin s\,\Phi_j^+(k_0\cos s), \quad j = 1, 2.
\end{aligned} \tag{6.13}$$

Formula (6.11) is remarkable for the functions $S_j(s)$ being determined in the strip $0 < \operatorname{Re} s < \pi/2$ independently of the residues of the solution of the RHP at the geometric optics poles. In the next section we will show that in order to recover the reflected waves, these residues need to be specified.

6.2 *Analytic continuation of the spectra*

Let

$$\Phi_j^+(\eta) \sim \frac{iC_j}{\eta - \eta_0}, \quad \eta \to \eta_0, \tag{6.14}$$

where C_j are some complex constants to be determined. Since the functions $\Phi_j^\pm(\eta)$ have simple poles at the points $\pm\eta_0$, the functions $\Phi_j^\pm(k_0\cos s)$ have simple poles at the points $s = \pm(\pi/2 - \theta_0) + m\pi$, $s = \pm(\pi/2 + \theta_0) + m\pi$, $m \in \mathbb{Z}$, while the functions $\Phi_j^\pm(k_0\sin s)$ have simple poles $s = \pm\theta_0 + m\pi$, $m \in \mathbb{Z}$. At the poles of interest, because of the symmetry conditions $\Phi_j^+(\eta) = \Phi_j^-(-\eta)$, we have

$$\begin{aligned}
&\operatorname*{res}_{s=\pm(\pi/2-\theta_0)} \Phi_j^+(k_0\cos s) = -\operatorname*{res}_{s=\pm(\pi/2+\theta_0)} \Phi_j^-(k_0\cos s) = \mp\frac{iC_j}{\hat\eta_0}, \\
&\pm\operatorname*{res}_{s=\pm\theta_0} \Phi_j^\pm(k_0\sin s) = -\operatorname*{res}_{s=\pm\pi-\theta_0} \Phi_j^+(k_0\sin s) = \operatorname*{res}_{s=\pm\pi+\theta_0} \Phi_j^-(k_0\sin s) = \frac{iC_j}{\hat\eta_0}, \quad j = 1, 2.
\end{aligned} \tag{6.15}$$

To determine the second group of residues associated with the reflected waves, in addition to the symmetry conditions we employ the boundary condition of the RHP to continue analytically the function $\Phi^-(\eta)$ into the upper half-plane

$$\Phi^-(\eta) = [G(\eta)]^{-1}\Phi^+(\eta), \quad \eta \in \mathbb{C}^+. \tag{6.16}$$

This enables us to write

$$\operatorname*{res}_{s=\pm(\pi/2+\theta_0)} \Phi^+(k_0\cos s) = -\operatorname*{res}_{s=\pm(\pi/2-\theta_0)} \Phi^-(k_0\cos s) = \pm\frac{i}{\hat\eta_0}[G(\eta_0)]^{-1}C,$$

$$\pm \operatorname*{res}_{s=\mp\theta_0} \Phi^{\pm}(k_0 \sin s) = -\operatorname*{res}_{s=\pm\pi+\theta_0} \Phi^{+}(k_0 \sin s) = \operatorname*{res}_{s=\pm\pi-\theta_0} \Phi^{-}(k_0 \sin s) = -\frac{i}{\hat{\eta}_0}[G(\eta_0)]^{-1}C, \tag{6.17}$$

where $C = (C_1, C_2)^T$.

Next we continue analytically the functions $S_j^{(1)}(s)$ from the strip $0 < \operatorname{Re} s < \pi$ to the strips $\pi < \operatorname{Re} s < 3\pi/2$ and $-\pi < \operatorname{Re} s < 0$ and the functions $S_j^{(2)}(s)$ from the strip $-\pi/2 < \operatorname{Re} s < \pi/2$ to the strips $\pi/2 < \operatorname{Re} s < 3\pi/2$ and $-\pi < \operatorname{Re} s < -\pi/2$. This procedure requires computing the residues of the functions $F_{j-}(s)$ at the poles $\pm\theta_0$ and $\pm(\pi - \theta_0)$ and the functions $F_{j+}(s)$ at the poles $\pm(\pi/2 - \theta_0)$ and $\pm(\pi/2 + \theta_0)$. Denote

$$[G(\eta_0)]^{-1} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}. \tag{6.18}$$

On employing (6.15) and (6.17) we derive

$$\operatorname*{res}_{s=\pm\theta_0} F_{j-}(s) = \Lambda_{j-}, \qquad \operatorname*{res}_{s=\pm(\pi-\theta_0)} F_{j-}(s) = M_{j-},$$
$$\operatorname*{res}_{s=\pm(\pi/2-\theta_0)} F_{j+}(s) = \Lambda_{j+}, \qquad \operatorname*{res}_{s=\pm(\pi/2+\theta_0)} F_{j+}(s) = M_{j+}, \tag{6.19}$$

where

$$\Lambda_{1+} = C_1, \quad \Lambda_{2+} = C_1, \quad \Lambda_{1-} = p_1^+ C_1 - q_1^+ C_2, \quad \Lambda_{2-} = -q_2^+ C_1 + p_2^+ C_2,$$
$$M_{1+} = -\mu_{11} C_1 - \mu_{12} C_2, \quad M_{2+} = -\mu_{21} C_1 - \mu_{22} C_2,$$
$$M_{1-} = -p_1^- C_1 + q_1^- C_2, \quad M_{2-} = q_2^- C_1 - p_2^- C_2.$$
$$p_1^{\pm} = \frac{1}{2\hat{\eta}_0}[(\gamma_1^+ \pm \hat{\eta}_0)(1-\mu_{11}) - \eta_0\mu_{21}\cos\beta], \quad q_1^{\pm} = \frac{1}{2\hat{\eta}_0}[(\gamma_1^+ \pm \hat{\eta}_0)\mu_{12} + \eta_0(1+\mu_{22})\cos\beta],$$
$$p_2^{\pm} = \frac{1}{2\hat{\eta}_0}[(\gamma_4^+ \pm \hat{\eta}_0)(1-\mu_{22}) + \eta_0\mu_{12}\cos\beta], \quad q_2^{\pm} = \frac{1}{2\hat{\eta}_0}[(\gamma_4^+ \pm \hat{\eta}_0)\mu_{21} - \eta_0(1+\mu_{11})\cos\beta]. \tag{6.20}$$

In addition, the functions $F_{j\pm}(s)$ may have some poles which generate surface waves. Indeed, when $\operatorname{Im}(k_0 \cos s) < 0$, the functions $\Phi_j^+(k_0 \cos s)$ need to be continued analytically from the domain $\operatorname{Im}(k_0 \cos s) > 0$. Conversely, if $\operatorname{Im}(k_0 \cos s) > 0$, the functions $\Phi_j^-(k_0 \cos s)$ need to be continued analytically from the domain $\operatorname{Im}(k_0 \cos s) < 0$. This can be done by the relations

$$\Phi^+(k_0 \cos s) = \frac{1}{\delta_0(k_0 \cos s)\delta_1(k_0 \cos s)} G^*(k_0 \cos s)\Phi^-(k_0 \cos s),$$
$$\Phi^-(k_0 \cos s) = \frac{1}{\delta_0(-k_0 \cos s)\delta_1(k_0 \cos s)} G^{**}(k_0 \cos s)\Phi^+(k_0 \cos s), \tag{6.21}$$

where

$$G^*(k_0 \cos s) = \begin{pmatrix} g_{11}^* & g_{12}^* \\ g_{21}^* & g_{22}^* \end{pmatrix}, \quad G^{**}(k_0 \cos s) = \begin{pmatrix} g_{22}^* & -g_{12}^* \\ -g_{21}^* & g_{11}^* \end{pmatrix},$$

$$\delta_0(k_0\cos s) = -(\gamma_1^- - k_0\cos s)(\gamma_4^- - k_0\cos s) - k_0^2\cos^2\beta\sin^2 s,$$
$$\delta_1(k_0\cos s) = (\gamma_1^+ + k_0\sin s)(\gamma_4^+ + k_0\sin s) + k_0^2\cos^2\beta\cos^2 s, \tag{6.22}$$

and g^*_{sj} are entire functions expressed through polynomials of $\cos s$ and $\sin s$. The same argument is applied to the functions $\Phi_j^{\pm}(k_0 \sin s)$. Since the surface waves do not affect the constants C_j, we do not specify the surface waves poles.

Before we write the analytic continuation of the spectra introduce the integrals

$$I^c_{j\pm}(s,a) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_{j\pm}(\sigma + a)\cot(\sigma - s)d\sigma, \quad a < \operatorname{Re} s < \pi + a,$$
$$I^t_{j\pm}(s,a) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_{j\pm}(\sigma + a)\tan(\sigma - s)d\sigma, \quad -\pi/2 + a < \operatorname{Re} s < \pi/2 + a. \tag{6.23}$$

We come finally to the analytic continuation of the functions $S_1(s)$ and $S_2(s)$ to the left and to the right that we need to recover the incident and reflected waves

$$S_j(s) = I^c_{j-}(s,0) + I^t_{j+}(s,0) + i_j\cot(s-\theta_0) - i_j\cot(s+\theta_0)], \quad 0 < \operatorname{Re} s < \frac{\pi}{2},$$
$$S_j(s) = I^c_{1-}(s,0) - I^c_{j+}(s,\pi/2) + i_j\cot(s-\theta_0) - (i_j - \Lambda_{j+})\cot(s+\theta_0)$$
$$+ S^s_j(s;\pi/2,\pi), \quad \frac{\pi}{2} < \operatorname{Re} s < \pi,$$
$$S_j(s) = -I^t_{j-}(s,\pi/2) + I^t_{j+}(s,\pi) + (i_j - \Lambda_{j-} + M_{j+})\cot(s-\theta_0) - (i_j - \Lambda_{j+})\cot(s+\theta_0)$$
$$+ S^s_j(s;\pi,3\pi/2), \quad \pi < \operatorname{Re} s < \frac{3\pi}{2},$$
$$S_j(s) = -I^t_{j-}(s,-\pi/2) + I^t_{j+}(s,0) + i_j\cot(s-\theta_0) - (i_j - \Lambda_{j-})\cot(s+\theta_0)$$
$$+ S^s_j(s;-\pi/2,0), \quad -\frac{\pi}{2} < \operatorname{Re} s < 0,$$
$$S_j(s) = I^c_{j-}(s,-\pi) - I^c_{j+}(s,-\pi/2) + (i_j - \Lambda_{j+} + M_{j-})\cot(s-\theta_0) - (i_j - \Lambda_{j-})\cot(s+\theta_0)$$
$$+ S^s_j(s;-\pi,-\pi/2), \quad -\pi < \operatorname{Re} s < -\frac{\pi}{2}, \tag{6.24}$$

where $S^s_j(s;a,b)$ are meromorphic functions whose poles in the strip $a < \operatorname{Re} s < b$ may generate the surface waves.

6.3 *Additional physical conditions*

Let $\mathcal{C}$ be a closed contour that comprises the Sommerfeld contours $\mathcal{T}_+$ and $\mathcal{T}_-$ and two contours $\mathcal{G}_L$ and $\mathcal{G}_R$. The starting points of the contours $\mathcal{G}_L$ and $\mathcal{G}_R$ are $-3\pi/2 + i\infty$ and $3\pi/2 - i\infty$, and they pass through the points $s = -\pi$ and $s = \pi$, respectively. The contour $\mathcal{G}_R$ is described by the equation $\operatorname{Re} s = \pi - \operatorname{gd}(\operatorname{Im} s)\operatorname{sgn}(\operatorname{Im} s)$, and the contour $\mathcal{G}_L$ is symmetric to $\mathcal{G}_R$ with respect to the origin. Here, $\operatorname{gd} x$ is the Gudermann function $\operatorname{gd} x = \cos^{-1}\operatorname{sech} x$. As $k_0 r \to \infty$, the electric and magnetic field can be represented as

$$E_z \sim E_z^i + E_{z+}^r + E_{z-}^r + E_{z+}^R + E_{z-}^R + E_z^s + E_z^d,$$
$$H_z \sim H_z^i + H_{z+}^r + H_{z-}^r + H_{z+}^R + H_{z-}^R + H_z^s + H_z^d, \quad k_0 r \to \infty, \tag{6.25}$$

E_z^s, H_z^s are the surface waves, and E_z^d, H_z^d are the diffracted waves. Write down the poles of the functions $S_1(s)$ and $S_2(s)$ in the interval $(-\pi, \pi)$ which generate the incident and reflected waves. These poles are

$$s = -\theta + \theta_0 \in (-\pi/2, \pi/2), \quad s + \theta \in (0, \pi/2),$$
$$s = \pi - \theta - \theta_0 \in (0, \pi), \quad s + \theta \in (\pi/2, \pi),$$
$$s = -\theta - \theta_0 \in (-\pi, 0), \quad s + \theta \in (-\pi/2, 0),$$
$$s = \pi - \theta + \theta_0 \in (\pi/2, \pi) \text{ if } \theta_0 < \theta < \pi/2, \quad s + \theta \in (\pi, 3\pi/2),$$
$$s = -\pi - \theta + \theta_0 \in (-\pi, -\pi/2) \text{ if } 0 < \theta < \theta_0, \quad s + \theta \in (-\pi, -\pi/2). \tag{6.26}$$

The incident and reflected electrical and magnetic waves can be recovered from (6.1) and (6.24)

$$E_z^i = i_1 e^{-ik_0\rho\cos(\theta-\theta_0)}, \quad ZH_z^i = i_2 e^{-ik_0\rho\cos(\theta-\theta_0)},$$
$$E_{z+}^r = (-i_1 + \Lambda_{1+}) e^{ik_0\rho\cos(\theta+\theta_0)}, \quad ZH_{z+}^r = (-i_2 + \Lambda_{2+}) e^{ik_0\rho\cos(\theta+\theta_0)},$$
$$E_{z-}^r = (-i_1 + \Lambda_{1-}) e^{-ik_0\rho\cos(\theta+\theta_0)}, \quad ZH_{z-}^r = (-i_2 + \Lambda_{2-}) e^{-ik_0\rho\cos(\theta+\theta_0)},$$
$$E_{z+}^R = (i_1 + M_{1-} - \Lambda_{1+}) e^{ik_0\rho\cos(\theta-\theta_0)} \omega(\theta; 0, \theta_0),$$
$$ZH_{z+}^R = (i_2 + M_{2-} - \Lambda_{2+}) e^{ik_0\rho\cos(\theta-\theta_0)} \omega(\theta; 0, \theta_0),$$
$$E_{z-}^R = (i_1 + M_{1+} - \Lambda_{1-}) e^{ik_0\rho\cos(\theta-\theta_0)} \omega(\theta; \theta_0, \pi/2),$$
$$ZH_{z-}^R = (i_2 + M_{2+} - \Lambda_{2-}) e^{ik_0\rho\cos(\theta-\theta_0)} \omega(\theta; \theta_0, \pi/2). \tag{6.27}$$

The diffracted waves are determined in a standard manner in the form

$$E_z^d = \frac{e^{ik_0\rho}}{\sqrt{k_0\rho}} D_1(\theta), \quad ZH_z^d = \frac{e^{ik_0\rho}}{\sqrt{k_0\rho}} D_2(\theta), \tag{6.28}$$

where $D_1(\theta)$ and $D_2(\theta)$ are the diffraction coefficients

$$D_j(\theta) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} [S_j(\theta - \pi) - S_j(\theta + \pi)]. \quad j = 1, 2. \tag{6.29}$$

As for the surface waves, they can be recovered in the same way as the incident and reflected waves as soon as the location of the surface wave poles is determined.

Now, the reflected coefficients in (6.27) have to be consistent with those given by (3.14). This immediately determines the unknown constants C_1 and C_2

$$C_1 = r_1^+ + i_1, \quad C_2 = r_2^+ + i_2. \tag{6.30}$$

As for the other six relations

$$\Lambda_{j-} = r_j + i_j, \quad M_{j-} = r_j^+ + R_j^+, \quad M_{j+} = r_j^- + R_j^-, \quad j = 1, 2, \tag{6.31}$$

they are satisfied identically. On verifying these equalities we may test the validity of the computations. Notice that the coefficients Λ_{j-}, M_{j-} and M_{j+} depend upon the entries of the matrix $G(\eta_0)$ and do not depend on the actual solution to the vector RHP.

7. Normal incidence

Consider the scalar case when $\alpha = \beta = \pi/2$. The matrix coefficient $G(\eta)$ of both vector RHPs, 1 and 2, is a diagonal matrix,

$$G(\eta) = \mathrm{diag}\{(\gamma_1^- + \eta)(\gamma_1^- - \eta)^{-1}, (\gamma_4^- + \eta)(\gamma_4^- - \eta)^{-1}\},$$
$$\hat{G}(\eta) = \mathrm{diag}\{(\gamma_1^+ + \eta)(\gamma_1^+ - \eta)^{-1}, (\gamma_4^+ + \eta)(\gamma_4^+ - \eta)^{-1}\}, \tag{7.1}$$

and the boundary conditions (3.6) and (3.9) become

$$\Phi_j^+(\eta) = \frac{\gamma_{j-} + \eta}{\gamma_{j-} - \eta}\Phi_j^-(\eta), \quad \eta \in L, \quad j = 1, 2, \tag{7.2}$$

and

$$\hat{\Phi}_j^+(\eta) = \frac{\gamma_{j+} + \eta}{\gamma_{j+} - \eta}\hat{\Phi}_j^-(\eta), \quad \eta \in L, \quad j = 1, 2, \tag{7.3}$$

respectively. Here we adopted the notations $\gamma_{1\pm} = \gamma_1^\pm$ and $\gamma_{2\pm} = \gamma_4^\pm$. The coefficients of the problems are rational functions, and the solution can easily be derived. The polynomials $\delta_0(\eta)$ and $\hat{\delta}(\eta)$ have the form $\delta_0(\eta) = -(\eta - \gamma_{1-})(\eta - \gamma_{2-})$ and $\hat{\delta}_0(\eta) = -(\eta - \gamma_{1+})(\eta - \gamma_{2+})$. As before, we consider the cases (i), (ii) and (iii). In the first case, both zeros of the polynomials $\delta_0(\eta)$ and $\hat{\delta}_0(\eta)$ lie in the lower half-plane, $\mathrm{Im}\,\gamma_{j-} < 0$, $\mathrm{Im}\,\gamma_{j+} < 0$, $j = 1, 2$, and the solution to each RHP in (7.2) has two arbitrary constants, D_{0j} and D_{1j},

$$\Phi_j^\pm(\eta) = \frac{D_{0j} + D_{1j}\eta^2}{(\eta^2 - \eta_0^2)(\gamma_{j-} \mp \eta)}, \quad \eta \in \mathbb{C}^\pm, \quad j = 1, 2. \tag{7.4}$$

On employing (3.3) we can derive a formula for $\hat{\phi}_j(i\zeta, 0)$

$$\hat{\phi}_j(i\zeta, 0) = -\frac{(\gamma_{j+} + i\zeta)(D_{0j} + D_{1j}\eta^2)}{(\eta^2 - \eta_0^2)(\gamma_{j-}^2 - \eta^2)}. \tag{7.5}$$

We next denote $i\zeta = \hat{\eta}$ and have

$$\hat{\phi}_j(\pm\hat{\eta}, 0) = \frac{(\gamma_{j+} \pm \hat{\eta})[D_{0j} + D_{1j}(k_0^2 - \hat{\eta}^2)]}{(\hat{\eta}^2 - \hat{\eta}_0^2)(\gamma_{j-}^2 - k_0^2 + \hat{\eta}^2)}, \quad \hat{\eta} \in \mathbb{C}^\pm, \quad j = 1, 2. \tag{7.6}$$

On the other hand, the decoupled RHP 2 (7.3) yields an alternative formula for the functions $\hat{\phi}_j(\pm\eta, 0) = \hat{\Phi}_j^{\pm}(\eta)$,

$$\hat{\Phi}_j^{\pm}(\eta) = \frac{\hat{D}_{0j} + \hat{D}_{1j}\eta^2}{(\eta^2 - \hat{\eta}_0^2)(\gamma_{j+} \mp \eta)}, \quad \eta \in \mathbb{C}^{\pm}, \quad j = 1, 2, \tag{7.7}$$

where $\hat{D}_{0j}$ and $\hat{D}_{1j}$ are arbitrary constants. The compatibility condition

$$\hat{\phi}_j(\pm\eta, 0)|_{RHP\,1} = \hat{\Phi}_j^{\pm}(\eta)|_{RHP\,2} \tag{7.8}$$

reads

$$(\gamma_{j+}^2 - \eta^2)[D_{0j} + D_{1j}(k_0^2 - \eta^2)] = (\hat{D}_{0j} + \hat{D}_{1j}\eta^2)(\gamma_{j-}^2 - k_0^2 + \eta^2), \quad \eta \in \mathbb{C}. \tag{7.9}$$

This gives six conditions for the eight constants. They are fulfilled if and only if

$$D_{1j} = \hat{D}_{1j} = -\frac{D_{0j}}{\gamma_{j-}^2}, \quad \hat{D}_{0j} = \frac{\gamma_{j+}^2 D_{0j}}{\gamma_{j-}^2}. \tag{7.10}$$

In this case the solution of both RHPs 1 and 2 has one arbitrary constant, $D_j = D_{0j}/\gamma_{j-}^2$,

$$\Phi^{\pm}(\eta) = \frac{D_j(\gamma_{j-} \pm \eta)}{\eta^2 - \eta_0^2}, \quad \hat{\Phi}^{\pm}(\eta) = \frac{D_j(\gamma_{j+} \pm \eta)}{\eta^2 - \hat{\eta}_0^2}, \quad \eta \in \mathbb{C}^{\pm}, \quad j = 1, 2. \tag{7.11}$$

Consider next the second case when $\operatorname{Im}\gamma_{j-} < 0$ and $\operatorname{Im}\gamma_{j+} > 0, j = 1, 2$. As before, the functions $\hat{\phi}_j(\pm i\zeta, 0) = \hat{\Phi}_j^{\pm}(\tilde{\eta})$ derived from the solution of the RHP 1, have the form (7.5). The same functions $\hat{\Phi}^{\pm}(\eta)$ comprise the solution to the RHP 2 and coincide with (7.11). The formulas (7.6) and (7.11) are compatible if and only if

$$D_{0j} + D_{1j}(k_0^2 - \eta^2) = \hat{D}_j(\gamma_{j-}^2 - k_0 + \eta^2). \tag{7.12}$$

This implies four equations for the six constants. They give $D_{0j} = \gamma_{j-}^2 D_j$ and $D_{1j} = -D_j$. Then the functions $\Phi^{\pm}(\eta)$ are the same as the ones in (7.11).

In the last case, $\operatorname{Im}\gamma_{j-} > 0$ and $\operatorname{Im}\gamma_{j+} > 0, j = 1, 2$, by solving the RHPs 1 and 2 we find that the compatibility condition (7.8) gives $D_j = \hat{D}_j$, and the functions $\Phi_j^{\pm}(\eta)$ and $\hat{\Phi}_j^{\pm}(\eta)$ coincide with the functions given in (7.11).

Our intention next is to fix the constants D_j. In all the three cases, regardless of whether $\operatorname{Im}\gamma_{j\pm}$ is positive or negative, the functions $\Phi_j^{\pm}(\eta)$ have the same form (7.11). Formulas (6.14) and (7.11) imply

$$C_j = -\frac{iD_j(\gamma_{j-} + \eta_0)}{2\eta_0}, \tag{7.13}$$

and from (6.30) we fix the constants D_j

$$D_j = \frac{4i\eta_0\hat{\eta}_0 i_j}{(\eta_0 + \gamma_{j-})(\hat{\eta}_0 + \gamma_{j+})}, \quad j = 1, 2. \tag{7.14}$$

On putting in (6.20) $\beta = \pi/2$ we obtain

$$\mu_{12} = \mu_{21} = 0, \quad \mu_{jj} = \frac{\gamma_{j-} - \eta_0}{\gamma_{j-} + \eta_0}, \quad \Lambda_{j-} = \frac{2\eta_0 i_j}{\gamma_{j-} + \eta_0},$$
$$M_{j-} = \frac{2\eta_0(\hat{\eta}_0 - \gamma_{j+})i_j}{(\eta_0 + \gamma_{j-})(\hat{\eta}_0 + \gamma_{j+})}, \quad M_{j+} = \frac{2\hat{\eta}_0(\eta_0 - \gamma_{j-})i_j}{(\eta_0 + \gamma_{j-})(\hat{\eta}_0 + \gamma_{j+})}. \tag{7.15}$$

Employing (3.14) and (3.15) we reduce the formulas for the reflection coefficients

$$r_j^+ = \frac{\hat{\eta}_0 - \gamma_{j+}}{\hat{\eta}_0 + \gamma_{j+}} i_j, \quad r_j^- = \frac{\eta_0 - \gamma_{j-}}{\eta_0 + \gamma_{j-}} i_j, \quad R_j^+ = R_j^- = \frac{(\hat{\eta}_0 - \gamma_{j+})(\eta_0 - \gamma_{j-})}{(\hat{\eta}_0 + \gamma_{j+})(\eta_0 + \gamma_{j-})} i_j. \tag{7.16}$$

Simple derivations show that the six relations (6.31) are identities.

8. Conclusions

We have found a closed-form solution of the classical problem on scattering of an electromagnetic plane wave obliquely incident from an impedance right-angled concave wedge. Two Helmholtz equations coupled by the boundary conditions have been reduced to two vector RHPs subject to the symmetry condition for the unknown vectors, $\Phi^+(\eta) = \Phi^-(-\eta)$ and $\hat{\Phi}^+(\eta) = \hat{\Phi}^-(-\eta)$. The unknown vectors $\Phi^+(\eta)$ and $\hat{\Phi}^+(\eta)$ are the Laplace transform of the electric and magnetic components on the vertical and horizontal faces of the wedge, respectively. It has been found convenient not to specify the residues of the solution to the RHPs at the geometric optics poles at this stage. The matrix factorization problem has been solved by the technique of the RHP on a genus-3 Riemann surface in the case of the boundary conditions (1.1). In the general case (1.2) of the boundary conditions the matrix can be factorized as well. However, this requires tedious computations of the analogues of formulas (3.26), (4.1) and (4.2). Due to the symmetry of the problem we have managed to avoid the genus-3 Riemann θ-function and solved the Jacobi inversion problem in terms of elliptic functions. The Wiener-Hopf matrix-factors have been found by quadratures and, eventually, an exact solution to the two vector RHPs associated with the diffraction problem of interest has been constructed. Either vector RHP is unconditionally solvable and has $\kappa + 3$ (RHP 1) and $\hat{\kappa} + 3$ (RHP 2) free constants, where $2\kappa = \text{ind}\det G(\eta)$, $2\hat{\kappa} = \text{ind}\det \hat{G}(\eta)$, and G and $\hat{G}$ are the matrix coefficients of the vector RHP 1 and 2. Depending on the location of the zeros of quadratic polynomials associated with each RHP the integers κ and $\hat{\kappa}$ can be 1, 0 and -1. We have shown that the solutions to the RHPs are not independent and have to satisfy certain compatibility conditions. When satisfied these conditions reduce the number n of free constants to 2 and therefore make n independent of the indices κ and $\hat{\kappa}$. The electric and magnetic field components E_z and H_z have been derived in terms of Sommerfeld's integrals with the spectra expressed through the solution of the vector RHP 1. By analytic continuation of the spectra to the left and to the right we have determined the poles responsible for the reflection waves E_{z+}^r and H_{z+}^r and fixed the two free constants left. We have also considered the case of normal incidence, derived a closed-form solution to the RHPs 1 and 2, verified the compatibility conditions and recovered the incident and reflected waves.

Although we have not implemented numerical computations, the core of the procedure, the factorization method on a Riemann surface, has been tested several times ((**21**) and (**26**) in the genus-1 case and (**4**) in the genus-3 case). Certainly, the technique is robust, and the determination of the diffraction coefficients in (6.29) for example requires computing just several integrals given in (6.12), (5.2) and (4.42).

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