SINGULAR INTEGRAL EQUATIONS WITH TWO FIXED SINGULARITIES AND APPLICATIONS TO FRACTURED COMPOSITES

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[Received 3 February 2015. Revise 22 September 2015. Accepted 26 September 2015]

Summary

A symmetric characteristic singular integral equation with two fixed singularities at the endpoints in the class of functions bounded at the ends is analysed. It reduces to a vector Hilbert problem for a half-disc and then to a vector Riemann–Hilbert problem on a real axis with a piecewise constant matrix coefficient that has two points of discontinuity. A condition of solvability and a closed-form solution to the integral equation are derived. For the Chebyshev polynomials of the first kind in the right hand-side, the solution of the integral equation is expressed in terms of two non-orthogonal polynomials with associated weights. Based on this new generalised spectral relation for the singular operator with two fixed singularities an approximate solution to the complete singular integral equation is derived by recasting it as an infinite system of linear algebraic equations of the second kind. The method is illustrated by solving two problems of fracture mechanics, the antiplane and plane strain problems for a finite crack in a composite plane. The plane is formed by a strip and two half-planes; the elastic constants of the strip are different from those of the half-planes. The crack is orthogonal to the interfaces, and it is located in the strip with the ends lying in the interfaces. Numerical results are reported and discussed.

1. Introduction

The method of orthogonal polynomials, a particular realisation of the general Bubnov–Galerkin method, has been successfully applied for singular integral equations since publication of the work by Klubin (**1**) who employed the spectral properties of the logarithmic kernel $\ln|x-\xi|$ and the Weber–Schafheitlin integral $W(x,\xi)=\int_0^\infty J_0(tx)J_0(t\xi)dt$, $J_0(y)$ is the Bessel function, to obtain series representations of the solutions to the corresponding singular integral equations. In particular, by making use of the spectral relation

$$\int_0^1 W(x,\xi)\frac{\xi P_{2n}(\sqrt{1-\xi^2})d\xi}{\sqrt{1-\xi^2}}=\frac{\pi}{2}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 P_{2n}(\sqrt{1-x^2}),\quad 0<x<1,\quad n=0,1,\dots, \tag{1.1}$$

where $P_m(x)$ are the Legendre polynomials, Klubin derived an efficient solution to the contact problem on a circular plate lying on an elastic foundation. The method of orthogonal polynomials was further developed and employed by many researchers including (**2**), (**3**), (**4**). This scheme applied to a singular integral equation $\int_a^b M(x,\xi)\varphi(\xi)d\xi=f(x)$, $a<x<b$, requires to determine

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Q. Jl Mech. Appl. Math, Vol. 68. No. 4 © The Author, 2015. Published by Oxford University Press; all rights reserved. For Permissions, please email: journals.permissions@oup.com
Advance Access publication 11 November 2015. doi:10.1093/qjmam/hbv016

the singularities of the solution at the endpoints in the class prescribed and represent the kernel as $M(x, \xi) = \Pi(x, \xi) + K(x, \xi)$. The function $\Pi(x, \xi)$ is the dominant singular kernel, while the second part is normally bounded or may have a weaker singularity as $x = \xi$. The method can be successfully applied if $\Pi(x, \xi)$ is a polynomial kernel (**3**), that is a function satisfying the spectral relations

$$\int_a^b \begin{pmatrix} \Pi(x, \xi) \\ \Pi(\xi, x) \end{pmatrix} p_\pm(\xi)\pi_n^\pm(\xi)d\xi = \sigma_n g_\pm(x)\pi_n^\mp(x), \quad a < x < b, \quad n = 0, 1, \dots, \tag{1.2}$$

where $\sigma_n \neq 0$ and $\pi_n^\pm(x)$ are orthonormal polynomials with the weights $w_\pm(x) = p_\pm(x)g_\mp(x)$ in the segment (a, b). Employing the system of functions $p_+(x)\pi_n^+(x)$ $(n = 1, 2, \dots)$ as basis functions, expanding the unknown function as

$$\varphi(x) = p_+(x) \sum_{m=0}^{\infty} a_m \pi_m^+(x) \tag{1.3}$$

and inserting the series into the integral equation enable us to reduce the equation to

$$\sigma_n a_n + \sum_{m=0}^{\infty} d_{nm} a_m = f_n, \quad n = 0, 1, \dots, \tag{1.4}$$

an infinite system of linear algebraic equations of the second kind. Under certain conditions, normally satisfied in applications, it is possible to prove the convergence of an approximate solution to the exact one. An approximate solution to the infinite system (1.4) is derived by the reduction method.

Many problems of thin plate theory (**5**) and contact and fracture mechanics (**6**) are governed by singular singular integral equations of the form

$$\int_0^1 [S(x, \xi) + K(x, \xi)]\varphi(\xi)d\xi = f(x), \quad 0 < x < 1, \tag{1.5}$$

where $S(x, \xi)$ is the sum of the Cauchy kernel and the kernel whose only singularities are at the endpoints $x = \xi = 0$ and $x = \xi = 1$. Motivated by the antiplane problem for a crack in a strip placed between two half-planes of different shear moduli when the crack is orthogonal to the interfaces, Moiseyev and Popov (**6**) analysed the case when

$$S(x, \xi) = \frac{1}{\xi - x} + \frac{\beta_0}{\xi + x} + \frac{\beta_1}{\xi + x - 2} + R_0(x, \xi), \tag{1.6}$$

β_0 and β_1 are real, $|\beta_j| < 1$, and $R_0(x, \xi)$ is a specifically chosen regular kernel. They reduced the problem to the vector Riemann–Hilbert problem with a piecewise constant matrix coefficient with three points of discontinuity and solved it in terms of some quadratures and the hypergeometric functions. They did not find a spectral relation for the singular operator with the fixed singularities. The approximate scheme outlined was not tested and found to be hard to implement.

The main goal of this article is to derive an exact solution to (1.5) in a simple form when $K(x, \xi) \equiv 0$, $2S(x, \xi) = \cot \frac{1}{2}\pi(\xi - x) + \beta \cot \frac{1}{2}\pi(\xi + x)$, $\beta_0 = \beta_1 = \beta$, $\beta \in (-\infty, +\infty)$, and develop a sufficient numerical scheme for the complete singular integral equation with an arbitrary kernel $K(x, \xi)$ having at most weak singularities in the line $x = \xi$ and weak fixed singularities at the endpoints. Specifically, we aim to derive a spectral relation of the form

$$\int_0^1 \left[\cot\frac{\pi(\xi-x)}{2}+\beta\cot\frac{\pi(\xi+x)}{2}\right]\phi_n(\xi)d\xi=\sigma_n g(x)\pi_n(x),\quad 0<x<1,\quad n=0,1,\ldots,\tag{1.7}$$

and use it as the core of an approximate scheme for (1.5). Here, $\phi_n(x)=p^{(1)}(x)\pi_n^{(1)}(x)+p^{(2)}(x)\pi_n^{(2)}(x)$, $\pi_n^{(j)}(x)$ $(j=1,2)$ are some degree-n trigonometric polynomials not necessary orthogonal, $p^{(j)}(x)$ are their weights, and $\pi_n(x)$ are degree-n orthogonal trigonometric polynomials, $\sigma_n\neq 0$, $g(x)>0$.

The article is organised as follows. In Section 2, we analyse the characteristic singular integral (2.1) by employing the method of the vector Hilbert problem for a half-disc proposed in (**6**). The characteristic equation (2.1) considered in the present article in the class of functions bounded at the endpoints leads to a vector Riemann–Hilbert problem whose matrix coefficient has two points of discontinuity. This enables us to write down the solvability condition and the solution to the integral equation explicitly. Motivated by possible applications that may arise in future applications we consider all possible cases for the major parameter β of the kernel, $\beta\in(-\infty,\infty)$, not only when $|\beta|<1$. In the case $\beta=0$, the kernel becomes the Hilbert kernel, and we show that our general formulas for the solvability condition and the solution reduce to the ones consistent with the known results (**7**, p. 426).

In Section 3, we consider the complete singular integral equation with two fixed singularities in the class of functions bounded at the ends when the right-hand side is defined up to an arbitrary constant. First we derive the relation (1.7) with the right-hand side chosen to be the Chebyshev trigonometric polynomial. We further employ these new spectral relations to derive an efficient approximate solution to the complete singular integral equation with two fixed singularities by converting the integral equation into an infinite system of linear algebraic equations of the second kind.

In Section 4, the method is tested by solving the antiplane problem on a finite crack orthogonal to the interfaces between a strip and two half-planes when the shear moduli of the half-planes are the same, while the shear modulus of the strip is different, and the crack lies in the strip. Section 5 generalises the method for the singular integral equation with two fixed singularities (5.9) that governs the corresponding plane strain problem. In this case, we analyse the singularities of the solution at the endpoints and construct a new singular integral operator associated with (5.9) and whose spectral properties are studied in Section 3. A quick numerical test applied to (5.18) confirms the efficiency of the method for plane problems as well.

In Appendix A, we adjust to our case the proof (**6**) of the equivalence of the singular integral equation and the vector Hilbert problem for two analytic functions in the half-disc. In Appendix B, we show that the cases $\beta=\pm 1$ reduce to an equation whose exact solution can be obtained by the Hilbert inversion formula (**7**, p. 244); (**8**, p. 69). Appendix C computes certain auxiliary integrals needed for the derivation of the spectral relation. Appendix D employs the method of classical orthogonal polynomials for an approximate solution of the complete singular integral equation with the Cauchy kernel in the class of functions vanishing at the endpoints.

2. Characteristic singular integral equation with fixed singularities

In this section, we aim to construct the exact solution to the singular integral equation

$$\frac{1}{2}\int_0^1 \left[\cot\frac{\pi(\xi-x)}{2}+\beta\cot\frac{\pi(\xi+x)}{2}\right]\phi(\xi)d\xi=f(x),\quad 0<x<1,\tag{2.1}$$

in the class of functions bounded at the points $x = 0$ and $x = 1$ and Hölder-continuous in the interval $(0, 1)$, $\phi(x) \in H(0, 1)$. Here, β is a real parameter, and $f(x) \in H[0, 1]$. Denote the kernel of the equation as

$$S(x, \xi) = \frac{1}{2} \cot \frac{\pi(\xi - x)}{2} + \frac{\beta}{2} \cot \frac{\pi(\xi + x)}{2}. \tag{2.2}$$

The first term of the kernel has a singularity in the line $\xi = x$, $x \in [0, 1]$, while the second term has fixed singularities at the endpoints $\xi = x = 0$ and $\xi = x = 1$. The kernel admits the representation

$$S(x, \xi) = \frac{1}{\pi} \left(\frac{1}{\xi - x} + \frac{\beta}{\xi + x} + \frac{\beta}{\xi + x - 2} \right) + R_0(x, \xi), \tag{2.3}$$

where $R_0(x, \xi)$ is a regular kernel. We call (2.1) the characteristic equation to distinguish it from the complete singular integral equation to be analysed in Section 3.

2.1 *Vector Hilbert and Riemann–Hilbert problems associated with the integral equation*

To construct an exact solution to the characteristic integral equation (2.1), first we transform it into an equation on the upper arc L of the unit circle centered at the origin with the starting and terminal points 1 and -1, respectively. Let $t = e^{i\pi x}$, $\tau = e^{i\pi\xi}$, $\phi(x) = u(e^{i\pi x})$ and $f(x) = v(e^{i\pi x})$. Then (2.1) becomes

$$\frac{1}{2\pi} \int_L \left[\frac{\tau + t}{\tau - t} - \frac{\beta(1 + \tau t)}{1 - \tau t} \right] \frac{u(\tau) d\tau}{\tau} = v(t), \quad t \in L. \tag{2.4}$$

According to the theorem to be stated below, this integral equation is equivalent to a certain vector Hilbert boundary value problem for a half-disc.

THEOREM 2.1. *Let $\varphi_1(z)$ and $\varphi_2(z)$ be two functions analytic in the upper half-disc $D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$, Hölder-continuous up to the boundary $\partial D = L \cup (-1, 1)$, bounded at the points $z = \pm 1$ and satisfying the boundary conditions*

$$\operatorname{Re}[\varphi_1(t) - \varphi_2(t)] = 0, \quad \operatorname{Im}[\beta_0 \varphi_1(t) + \varphi_2(t)] = 0, \quad -1 < t < 1, \quad \beta_0 = \frac{1 + \beta}{1 - \beta},$$

$$\operatorname{Re} \varphi_1(t) = 0, \quad \operatorname{Re} \varphi_2(t) = u(t), \quad t \in L, \tag{2.5}$$

and the additional condition

$$\lim_{z \to 0} [\varphi_1(z) - \varphi_2(z)] = 0. \tag{2.6}$$

Then the function $u(t)$, $t \in L$, solves the singular integral equation (2.4) with $v(t) = -\operatorname{Im} \varphi_2(t)$ in the class of functions $H(L)$ bounded at the points $t = \pm 1$.

Conversely, let $u(t)$ be a solution to the integral equation (2.4) in the class of functions Hölder-continuous everywhere in the contour L and bounded at the ending points. Denote

$$\gamma(z) = \frac{\beta}{2\pi i} [\ln z - \ln(-z)], \quad -\pi < \arg z < \pi. \tag{2.7}$$

Then the functions

$$\varphi_1(z) = \frac{1}{2\pi i} \int_L \left\{ [\gamma(z) - \gamma(\tau)] \frac{\tau + z}{\tau - z} + [1 - \gamma(z) - \gamma(\tau)] \frac{1 + \tau z}{1 - \tau z} \right\} \frac{u(\tau) d\tau}{\tau},$$

$$\varphi_2(z) = \frac{1}{2\pi i}\int_L \left\{[1+\gamma(z)-\gamma(\tau)]\frac{\tau+z}{\tau-z} - [\gamma(z)+\gamma(\tau)]\frac{1+\tau z}{1-\tau z}\right\}\frac{u(\tau)d\tau}{\tau} \tag{2.8}$$

form the solution of the Hilbert boundary value problem

$$\mathrm{Re}[\varphi_1(t)-\varphi_2(t)] = 0, \quad \mathrm{Im}[\beta_0\varphi_1(t)+\varphi_2(t)] = 0, \quad -1 < t < 1,$$

$$\mathrm{Re}\,\varphi_1(t) = 0, \quad \mathrm{Im}\,\varphi_2(t) = -v(t), \quad t \in L, \tag{2.9}$$

in the class of functions bounded at the points $z = \pm 1$, satisfying the condition (2.6), and $\mathrm{Re}\,\varphi_2(t) = u(t)$*, $t \in L$.*

This theorem follows from the results derived for a more general case in (**6**). A proof adjusted for the case under consideration is presented in Appendix A.

We now proceed to the construction of the solution to the Hilbert boundary value problem (2.5), (2.6). First we conformally map the domain D onto the lower half-plane $\mathrm{Im}\,s(z) < 0$,

$$s = \left(\frac{z-1}{z+1}\right)^2, \quad z = \frac{1+s^{1/2}}{1-s^{1/2}}, \quad z \in D, \quad \pi < \arg s < 2\pi. \tag{2.10}$$

In this way, the contour L is mapped onto the negative semi-axis $-\infty < s < 0$, while the segment $(-1, 1)$ is mapped onto the positive semi-axis $0 < s < +\infty$. The points $z = 1$ and $z = -1$ fall onto the points $s = 0$ and $s = \infty$, respectively. Next we define the vector

$$\mathbf{\Phi}(s) = \begin{pmatrix}\varphi_1(z(s))\\ \varphi_2(z(s))\end{pmatrix}, \quad \mathrm{Im}\,s < 0, \tag{2.11}$$

analytic in the lower s-half-plane and extend its definition into the upper half-plane by the symmetry law

$$\mathbf{\Phi}(s) = \mathrm{diag}\{-1, 1\}\overline{\mathbf{\Phi}(\bar{s})}, \quad \mathrm{Im}\,s > 0. \tag{2.12}$$

On denoting

$$\mathbf{\Phi}^-(\sigma) = \mathbf{\Phi}(\sigma - i0), \quad \mathbf{\Phi}^+(\sigma) = \mathrm{diag}\{-1, 1\}\overline{\mathbf{\Phi}(\sigma - i0)}, \quad -\infty < \sigma < +\infty, \tag{2.13}$$

we write the Hilbert problem (2.5) in the form of the vector Riemann–Hilbert problem with a piece-wise constant matrix coefficient

$$\mathbf{\Phi}^+(\sigma) = G(\sigma)\mathbf{\Phi}^-(\sigma) + \mathbf{g}(\sigma), \quad -\infty < \sigma < +\infty, \tag{2.14}$$

where

$$G(\sigma) = \begin{cases} I, & -\infty < \sigma < 0,\\ G_0, & 0 < \sigma < +\infty,\end{cases} \quad G_0 = \begin{pmatrix}-\beta & \beta-1\\ \beta+1 & -\beta\end{pmatrix}, \quad I = \mathrm{diag}\{1, 1\},$$

$$\mathbf{g}(\sigma) = \begin{cases} 2iv(z(\sigma))\mathbf{J}, & -\infty < \sigma < 0,\\ \mathbf{0}, & 0 < \sigma < +\infty,\end{cases} \quad \mathbf{J} = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{2.15}$$

2.2 *Case* $|\beta| < 1$

In this case the matrix G_0 has complex-conjugate eigenvalues, $\lambda_1 = -\beta + i\sqrt{1-\beta^2}$ and $\lambda_2 = -\beta - i\sqrt{1-\beta^2}$, and admits the splitting $G_0 = T\Lambda_0 T^{-1}$, where

$$T = \begin{pmatrix} 1 & 1 \\ -i\sqrt{\beta_0} & i\sqrt{\beta_0} \end{pmatrix}, \quad \Lambda_0 = \operatorname{diag}\{\lambda_1, \lambda_2\}, \tag{2.16}$$

and β_0 is the parameter defined in (2.5). It is natural now to introduce the following vectors and matrices:

$$\mathbf{\Phi}_0(s) = T^{-1}\mathbf{\Phi}(s), \quad \Lambda(\sigma) = \begin{cases} I, & -\infty < \sigma < 0, \\ \Lambda_0, & 0 < \sigma < +\infty, \end{cases} \quad \mathbf{g}_0(\sigma) = T^{-1}\mathbf{g}(\sigma). \tag{2.17}$$

The vector Riemann–Hilbert problem has been decoupled, and the one-sided limits $\mathbf{\Phi}_0^{\pm}(\sigma)$ of the new vector $\mathbf{\Phi}_0(s)$ satisfy the boundary condition

$$\mathbf{\Phi}_0^{+}(\sigma) = \Lambda(\sigma)\mathbf{\Phi}_0^{-}(\sigma) + \mathbf{g}_0(\sigma), \quad -\infty < \sigma < +\infty. \tag{2.18}$$

The diagonal entries of the matrix $\Lambda(\sigma)$ can be factorised by means of the Cauchy integral

$$\lambda_j = \frac{\chi_j^{+}(\sigma)}{\chi_j^{-}(\sigma)}, \quad \sigma \in (0, +\infty), \quad j = 1, 2, \tag{2.19}$$

where $\chi_j^{\pm}(\sigma)$ are the one-sided limits as $s \to \sigma \pm i0$ of the function

$$\chi_j(s) = \exp\left\{\frac{1}{2\pi i}\int_0^{\infty}\left(\frac{1}{\sigma - s} - \frac{1}{\sigma - s_0}\right)\log\lambda_j d\sigma\right\} = C_j^0 s^{-(2\pi i)^{-1}\log\lambda_j}, \tag{2.20}$$

and

$$-2\pi \leq \arg\lambda_j \leq 0. \tag{2.21}$$

The choice of the range for the argument of the eigenvalues is due to the class of solutions. Here, $s_0 \in \mathbb{C} \setminus (0, +\infty)$ is an arbitrary fixed point, and C_j^0 ($j = 1, 2$) are constants. Denote

$$\rho_j = -\frac{\log\lambda_j}{2\pi i}, \quad \delta = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta} \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad j = 1, 2. \tag{2.22}$$

Since $|\lambda_j| = 1, j = 1, 2$, and because of the range for $\arg\lambda_j$ defined by (2.21), we find

$$\rho_1 = \begin{cases} \frac{1}{2} + \frac{\delta}{2}, & \beta > 0, \\ 1 + \frac{\delta}{2}, & \beta < 0, \end{cases} \quad \rho_2 = \begin{cases} \frac{1}{2} - \frac{\delta}{2}, & \beta > 0, \\ -\frac{\delta}{2}, & \beta < 0. \end{cases} \tag{2.23}$$

We consequently derive the factorisation of the diagonal matrix $\Lambda(\sigma) = X^{+}(\sigma)[X^{-}(\sigma)]^{-1}$, $\sigma \in (-\infty, +\infty)$, where $X^{\pm}(\sigma)$ are the one-sided limits as $s \to \sigma \pm i0$ of the matrix $X(s)$

$$X(s) = \operatorname{diag}\{s^{\rho_1}, s^{\rho_2}\}. \tag{2.24}$$

The branches of the functions s^{ρ_1} and s^{ρ_2} are fixed by cutting the s-plane along the positive semi-axis and selecting $\arg s \in [0, 2\pi]$. Next, by the reasoning usual in the theory of the Riemann–Hilbert problem we obtain

$$\mathbf{\Phi}_0(s) = X(s)\mathbf{V}(s), \quad \mathbf{V}(s) = \frac{1}{2\pi i\sqrt{\beta_0}} \int_{-\infty}^{0} \frac{\nu(z(\sigma))}{\sigma - s} \begin{pmatrix} -\sigma^{-\rho_1} \\ \sigma^{-\rho_2} \end{pmatrix} d\sigma. \tag{2.25}$$

On returning to the z-plane by means of (2.10), we transform the vector-function $\mathbf{V}(s)$ as

$$\mathbf{V}(s) = \frac{(z+1)^2}{2\pi i\sqrt{\beta_0}} \int_L \frac{\nu(t)}{(t-z)(1-tz)} \begin{pmatrix} -\omega_1(t) \\ \omega_2(t) \end{pmatrix} dt. \tag{2.26}$$

Here,

$$\omega_j(z) = \left(\frac{z-1}{z+1}\right)^{1-2\rho_j}, \quad j = 1, 2, \tag{2.27}$$

and the branch of $\omega_j(z)$ in the z-plane cut along the line joining the points $z = 1$ and $z = -1$ and passing through the infinite point is chosen such that $\omega_j(0) = -e^{-2\pi i\rho_j}$, $j = 1, 2$. On employing formulas (2.17) and (2.11) it is now easy to derive from here representations for the functions $\varphi_1(z)$ and $\varphi_2(z)$, the solution of the Hilbert problem (2.5),

$$\varphi_1(z) = \frac{z^2-1}{2\pi i\sqrt{\beta_0}} \int_L \left[-\frac{\omega_1(t)}{\omega_1(z)} + \frac{\omega_2(t)}{\omega_2(z)}\right] \frac{\nu(t)dt}{(t-z)(1-tz)},$$

$$\varphi_2(z) = \frac{z^2-1}{2\pi} \int_L \left[\frac{\omega_1(t)}{\omega_1(z)} + \frac{\omega_2(t)}{\omega_2(z)}\right] \frac{\nu(t)dt}{(t-z)(1-tz)}. \tag{2.28}$$

These functions have to satisfy the necessary and sufficient condition (2.6) for the equivalence of the Hilbert problem (2.5) and the integral equation (2.4). It reads

$$\int_L \left[\left(\frac{1}{\sqrt{\beta_0}} + i\right)\frac{\omega_1(t)}{\omega_1(0)} - \left(\frac{1}{\sqrt{\beta_0}} - i\right)\frac{\omega_2(t)}{\omega_2(0)}\right]\frac{\nu(t)dt}{t} = 0. \tag{2.29}$$

If this condition is satisfied, then the solution to (2.4) is expressed through the solution to the Hilbert problem by the formula $u(t) = \operatorname{Re}\varphi_2(t)$, $t \in L$.

In order to recover the solution to the original characteristic equation (2.1) and verify the boundary conditions on the contour L, we make the reverse substitution $t = e^{i\pi x}$, $\tau = e^{i\pi\xi}$ and utilise the Sokhotski–Plemelj formulas. Since

$$\frac{(t^2-1)d\tau}{(\tau-t)(1-\tau t)} = \frac{\pi \sin\pi x d\xi}{\cos\pi\xi - \cos\pi x}, \tag{2.30}$$

we obtain

$$\varphi_1(t) = \frac{\sin \pi x}{2i\sqrt{\beta_0}} \int_0^1 \frac{v(e^{i\pi\xi})[a^{2\rho_1-1}(x,\xi) - a^{-2\rho_1+1}(x,\xi)]d\xi}{\cos \pi\xi - \cos \pi x},$$

$$\varphi_2(t) = -iv(t) + \frac{\sin \pi x}{2} \int_0^1 \frac{v(e^{i\pi\xi})[a^{2\rho_1-1}(x,\xi) + a^{-2\rho_1+1}(x,\xi)]d\xi}{\cos \pi\xi - \cos \pi x}. \quad t \in L, \quad 0 < x < 1, \tag{2.31}$$

where

$$a(x,\xi) = \tan \frac{\pi\xi}{2} \cot \frac{\pi x}{2}. \tag{2.32}$$

These formulas imply

$$\operatorname{Re} \varphi_1(t) = 0, \quad \operatorname{Im} \varphi_2(t) = -v(t), \quad t \in L,$$

$$\operatorname{Re} \varphi_2(t) = \frac{\sin \pi x}{2} \int_0^1 \frac{v(e^{i\pi\xi})[a^{2\rho_1-1}(x,\xi) + a^{-2\rho_1+1}(x,\xi)]d\xi}{\cos \pi\xi - \cos \pi x}, \quad t \in L, \quad 0 < x < 1. \tag{2.33}$$

Thus, the boundary conditions on the contour L in (2.5) are fulfilled, and since $v(e^{i\pi x}) = f(x)$, $u(e^{i\pi x}) = \phi(x)$ and $\operatorname{Re} \varphi_2(t) = u(t)$, $t \in L$, we deduce the following formula for the solution to the characteristic integral equation (2.1):

$$\phi(x) = \frac{\sin \pi x}{2} \int_0^1 \frac{[a^{2\rho_1-1}(x,\xi) + a^{-2\rho_1+1}(x,\xi)]f(\xi)d\xi}{\cos \pi\xi - \cos \pi x}, \quad 0 < x < 1, \tag{2.34}$$

with the function $a(x,\xi)$ given by (2.32). We wish now to transform the solvability condition (2.29). On making the substitution $t = e^{i\pi x}$ and employing the formulas

$$\frac{\omega_j(t)}{\omega_j(0)} = -ie^{\pi i\rho_j} \tan^{1-2\rho_j} \frac{\pi x}{2},$$

$$\cos \frac{\pi\delta}{2} = \sqrt{\frac{1+|\beta|}{2}}, \quad \sin \frac{\pi\delta}{2} = \sqrt{\frac{1-|\beta|}{2}} \operatorname{sgn} \beta, \tag{2.35}$$

we deduce

$$\int_0^1 \left(\tan^{2\rho_1-1} \frac{\pi x}{2} + \cot^{2\rho_1-1} \frac{\pi x}{2}\right) f(x)dx = 0. \tag{2.36}$$

We have thus shown that, in the case $-1 < \beta < 1$, the integral equation (2.1) is solvable in the class of bounded at the ends functions if and only if the function $f(x)$ meets the condition (2.36). If this condition is satisfied, then the solution is unique and given by (2.34). Note that the solution and the solvability condition derived admit an alternative representation. On making the substitutions $\eta = \cos \pi\xi$, $\zeta = \cos \pi x$, we recast the solution and the condition

(2.36) as

$$\phi\left(\frac{\cos^{-1}\zeta}{\pi}\right) = \frac{1}{2\pi}\int_{-1}^{1}\left[\left(\frac{1-\eta}{1+\eta}\frac{1+\zeta}{1-\zeta}\right)^{\rho_1-\frac{1}{2}} + \left(\frac{1+\eta}{1-\eta}\frac{1-\zeta}{1+\zeta}\right)^{\rho_1-\frac{1}{2}}\right] \times\sqrt{\frac{1-\zeta^2}{1-\eta^2}}\frac{f\left(\pi^{-1}\cos^{-1}\eta\right)d\eta}{\eta-\zeta},$$

$$\int_{-1}^{1}\left[\left(\frac{1+\zeta}{1-\zeta}\right)^{\rho_1-\frac{1}{2}} + \left(\frac{1-\zeta}{1+\zeta}\right)^{\rho_1-\frac{1}{2}}\right]f\left(\frac{\cos^{-1}\zeta}{\pi}\right)\frac{d\zeta}{\sqrt{1-\zeta^2}} = 0, \tag{2.37}$$

respectively.

Having defined the exact formula for the solution let us now show that the function $\phi(x)$ is not just bounded as $x \to 0$ and $x \to 1$, but vanishes at these points. On utilising the following formulas for the Cauchy integral when $h(\eta) \in H[-1, 1]$ and $0 < \operatorname{Re}\alpha < 1$ (**8**, p. 73):

$$\int_{-1}^{1}\frac{h(\eta)d\eta}{(\eta+1)^{\alpha}(\eta-\zeta)} \sim \pi h(-1)\cot\pi\alpha(\zeta+1)^{-\alpha} + A_1(\zeta), \quad \zeta \to -1^{+},$$

$$\int_{-1}^{1}\frac{h(\eta)d\eta}{(1-\eta)^{\alpha}(\eta-\zeta)} \sim -\pi h(1)\cot\pi\alpha(1-\zeta)^{-\alpha} + A_2(\zeta), \quad \zeta \to 1^{-}, \tag{2.38}$$

we notice that in the representation (2.37) of the function $\phi(x)$ the bounded terms are cancelled, and in the vicinity of the point $\zeta = -1$, the function $\phi(x)$ behaves as

$$\phi(x) = C_1^{-}(\zeta+1)^{\rho_1} + C_2^{-}(\zeta+1)^{1-\rho_1} + o((\zeta+1)^{\rho_0}), \quad \zeta \to -1^{+}, \tag{2.39}$$

Here, $A_1(\zeta)$ and $A_2(\zeta)$ are functions analytic in a neighbourhood of the points $\zeta = -1$ and $\zeta = 1$, respectively, C_1^{-}, and C_2^{-} are non-zero constants, and

$$\rho_0 = \min\{\rho_1, 1-\rho_1\} = 1-\rho_1 = \begin{cases} \frac{1}{2}-\frac{\delta}{2}, & \beta > 0, \\ -\frac{\delta}{2}, & \beta < 0. \end{cases} \tag{2.40}$$

A similar argument is applied to the case when $\zeta \to 1-0$, and we have

$$\phi(x) \sim C^{-}(\zeta+1)^{\rho_0}, \quad \zeta \to -1^{+}, \qquad \phi(x) \sim C^{+}(-\zeta+1)^{\rho_0}, \quad \zeta \to 1^{-}. \tag{2.41}$$

with C^{-} and C^{+} being non-zero constants. Now, since

$$1+\zeta \sim \frac{\pi^2}{2}(x-1)^2, \quad \zeta \to -1^{+}, \quad x \to 1^{-},$$

$$1-\zeta \sim \frac{\pi^2}{2}x^2, \quad \zeta \to 1^{-}, \quad x \to 0^{+}, \tag{2.42}$$

we deduce that the function $\phi(x)$ vanishes at the points $x = 0$ and $x = 1$ and

$$\phi(x) \sim C_0^-(1-x)^{2-2\rho_1}, \quad x \to 1^-, \qquad \phi(x) \sim C_0^+ x^{2-2\rho_1}, \quad x \to 0^+, \tag{2.43}$$

where

$$2 - 2\rho_1 = \begin{cases} 1-\delta \in \left(\frac{1}{2}, 1\right), & \beta > 0, \\ -\delta \in \left(0, \frac{1}{2}\right), & \beta < 0, \end{cases} \qquad \delta = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta}. \tag{2.44}$$

2.3 *Case* $\beta = 0$

In this particular case, the term with two fixed singularities in the kernel of (2.1) vanishes, and the equation becomes

$$\frac{1}{2}\int_0^1 \cot\frac{\pi(\xi - x)}{2}\phi(\xi)d\xi = f(x), \quad 0 < x < 1. \tag{2.45}$$

For the parameters introduced we have the following values:

$$\beta_0 = 1, \quad \lambda_1 = i, \quad \lambda_2 = -i, \quad \rho_1 = \frac{3}{4}, \quad \rho_2 = \frac{1}{4}, \tag{2.46}$$

and $\delta = \pm\frac{1}{2}$ if $\beta = 0^\pm$. The solvability condition (2.36) reduces to the form

$$\int_0^1 \frac{f(x)}{\sqrt{\sin\pi x}}\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right)dx = 0. \tag{2.47}$$

If this condition is satisfied, then according to Section 2.2 the solution to Equation (2.45) reads

$$\phi(x) = \frac{1}{2}\int_0^1 \sqrt{\frac{\sin\pi x}{\sin\pi\xi}}\frac{f(\xi)d\xi}{\sin\frac{\pi}{2}(x-\xi)}. \tag{2.48}$$

From here we immediately deduce that the solution vanishes at both ending points,

$$\phi(x) \sim D_0 x^{1/2}, \quad x \to 0^+, \qquad \phi(x) \sim D_1(1-x)^{1/2}, \quad x \to 1^-, \tag{2.49}$$

where D_0 and D_1 are non-zero constants.

We show now that the results found are consistent with the ones recovered from the classical theory (**7**). First, by making the substitutions $t = e^{i\pi x}$, $\tau = e^{i\pi\xi}$ and denoting $\tilde{\phi}(t) = \phi(x), \tilde{f}(t) = f(x)$, we rewrite (2.45) as

$$\frac{1}{\pi}\int_L \frac{\tilde{\phi}(\tau)d\tau}{\tau - t} = \tilde{f}(t) + \tilde{\phi}_0, \quad t \in L, \tag{2.50}$$

where

$$\tilde{\phi}_0 = \frac{1}{2\pi}\int_L \frac{\tilde{\phi}(t)dt}{t}, \tag{2.51}$$

and, as before, $L = \{z \in \mathbb{C} : |z| = 1, \operatorname{Im} z > 0\}$ with the starting point $z = 1$. In the class of functions bounded at the endpoints of the contour L the solution to (2.50) does not exist unless the condition

$$\int_L \frac{\tilde{f}(t) + \tilde{\phi}_0}{\sqrt{t^2 - 1}} dt = 0 \tag{2.52}$$

is fulfilled. Then the solution is unique and has the form (**7**)

$$\tilde{\phi}(t) = -\frac{1}{\pi}\sqrt{t^2 - 1} \int_L \frac{\tilde{f}(\tau) + \tilde{\phi}_0}{\sqrt{\tau^2 - 1}} \frac{d\tau}{\tau - t}. \tag{2.53}$$

Here, the branch of the square root in the plane cut along the contour L is fixed by the condition $\sqrt{z^2 - 1} \sim z$, $z \to \infty$. Note that the lower side of the cut $|z| = 1 - 0$, $\arg z \in [0, \pi]$, is identified as the contour L itself. Employing the relation

$$\int_L \frac{d\tau}{\sqrt{\tau^2 - 1}(\tau - t)} = 0, \quad t \in L, \tag{2.54}$$

and coming back to the original variables and functions we transform formula (2.53) into the form (2.48). Analyse now the solvability condition (2.52). Owing to (2.51) and (2.53) we thereby represent the constant $\tilde{\phi}_0$ as the double integral

$$\tilde{\phi}_0 = -\frac{1}{2\pi^2} \int_L \frac{\sqrt{t^2 - 1}dt}{t} \int_L \frac{\tilde{f}(\tau)d\tau}{\sqrt{\tau^2 - 1}(\tau - t)}. \tag{2.55}$$

To convert the double integral into a single one, we introduce the complex potential

$$\Omega(z) = \frac{1}{2\pi i} \int_L \frac{\sqrt{\tau^2 - 1}d\tau}{\tau(\tau - z)}. \quad z \neq 0. \tag{2.56}$$

Applying the theory of residues and taking into account that for the branch fixed, $\sqrt{z^2 - 1}|_{z=0} = -i$, we discover

$$\Omega(z) = -\frac{1}{2} + \frac{i}{2z} + \frac{\sqrt{z^2 - 1}}{2z}. \tag{2.57}$$

To determine the principal value $\Omega(t)$ of the integral (2.56), we apply the Sokhotski–Plemelj formulas and obtain

$$\frac{1}{2\pi i} \int_L \frac{\sqrt{\tau^2 - 1}d\tau}{\tau(\tau - t)} = -\frac{1}{2} + \frac{i}{2t}. \tag{2.58}$$

Therefore

$$\tilde{\phi}_0 = -\frac{1}{2\pi} \int_L \frac{\tilde{f}(\tau)}{\sqrt{\tau^2 - 1}} \left(i + \frac{1}{\tau}\right) d\tau. \tag{2.59}$$

Next, by substituting this expression in (2.52) and making use of the integral

$$\int_L \frac{d\tau}{\sqrt{\tau^2 - 1}} = -\pi i \tag{2.60}$$

we discover that $\tilde{\phi}_0 = 0$ that is the solvability condition becomes

$$\int_L \frac{\tilde{f}(\tau)}{\sqrt{\tau^2 - 1}} \left(i + \frac{1}{\tau}\right) d\tau = 0. \tag{2.61}$$

We finally come back to the variable x and the function $f(x)$ and reduce the condition (2.61) to the desired form (2.47).

2.4 *Case* $\beta > 1$

Now we assume that $\beta > 1$. In this case the eigenvalues of the matrix G_0, $\lambda_j = -\beta - (-1)^j \sqrt{\beta^2 - 1}$, $j = 1, 2$, are real and negative. The entries of the matrix of transformation T are also real,

$$T = \begin{pmatrix} 1 & 1 \\ \sqrt{-\beta_0} & -\sqrt{-\beta_0} \end{pmatrix}, \quad \beta_0 = \frac{1+\beta}{1-\beta} < 0. \tag{2.62}$$

The matrix of factorisation $X(s)$ has the same form as (2.24). However the parameters ρ_1 and ρ_2 are not real anymore,

$$\rho_1 = \frac{1}{2} - i\varepsilon, \quad \rho_2 = \frac{1}{2} + i\varepsilon, \tag{2.63}$$

where

$$\varepsilon = \frac{1}{2\pi} \log(\beta + \sqrt{\beta^2 - 1}) > 0. \tag{2.64}$$

Following the scheme of Section 2.2 we derive the solution of the Hilbert problem (2.5) in the form

$$\varphi_1(z) = \frac{z^2 - 1}{2\pi\sqrt{-\beta_0}} \int_L \left[\frac{\omega_1(t)}{\omega_1(z)} - \frac{\omega_2(t)}{\omega_2(z)}\right] \frac{v(t)dt}{(t-z)(1-tz)},$$

$$\varphi_2(z) = \frac{z^2 - 1}{2\pi} \int_L \left[\frac{\omega_1(t)}{\omega_1(z)} + \frac{\omega_2(t)}{\omega_2(z)}\right] \frac{v(t)dt}{(t-z)(1-tz)}, \tag{2.65}$$

where

$$\omega_j(t) = ie^{-\pi i \rho_j} \tan^{1-2\rho_j} \frac{1}{2}\pi x, \quad j = 1, 2, \quad 1 - 2\rho_1 = 2i\varepsilon, \quad 1 - 2\rho_2 = -2i\varepsilon. \tag{2.66}$$

The condition (2.6) that guarantees that the Hilbert problem (2.5) is equivalent to the integral equation (2.4) reads

$$\int_0^1 \left(\tan^{2i\varepsilon} \frac{\pi x}{2} + \cot^{2i\varepsilon} \frac{\pi x}{2}\right) v(e^{i\pi x})dx = 0. \tag{2.67}$$

Here we used the relations

$$e^{\pi\varepsilon} = \frac{1}{\sqrt{\beta - \sqrt{\beta^2 - 1}}}, \quad \left(\frac{1}{\sqrt{-\beta_0}} \mp 1\right) e^{\pm\pi\varepsilon} = \mp\sqrt{\frac{2}{\beta + 1}}. \tag{2.68}$$

We assume further that the condition (2.67) is satisfied. Our task now is to find the solution to the integral equation (2.1). The Sokhotski–Plemelj formulas applied to (2.65) yield

$$\varphi_1(e^{i\pi x}) = \frac{i\sin\pi x}{\sqrt{-\beta_0}}\int_0^1 \frac{v(e^{i\pi\xi})}{\cos\pi\xi - \cos\pi x}\sin\left(2\varepsilon\log a(x,\xi)\right)d\xi,$$

$$\varphi_2(e^{i\pi x}) = -iv(e^{i\pi x}) + \sin\pi x\int_0^1 \frac{v(e^{i\pi\xi})}{\cos\pi\xi - \cos\pi x}\cos\left(2\varepsilon\log a(x,\xi)\right)d\xi, \tag{2.69}$$

where $a(x, \xi)$ is given by (2.32). These formulas enable us to verify the boundary conditions of the Hilbert problem (2.5) on the contour L

$$\operatorname{Re}\varphi_1(t) = 0, \quad \operatorname{Im}\varphi_2(t) = -v(t), \quad t \in L, \tag{2.70}$$

and also to derive the solution to the integral equation (2.4), $u(t) = \operatorname{Re}\varphi_2(t)$. On putting $f(\xi) = v(e^{i\pi\xi})$ and $\phi(x) = u(e^{i\pi x})$ we have from (2.69)

$$\phi(x) = \int_0^1 \frac{\sin\pi x f(\xi)}{\cos\pi\xi - \cos\pi x}\cos\left(2\varepsilon\log a(x,\xi)\right)d\xi, \quad 0 < x < 1. \tag{2.71}$$

We assert that the integral equation (2.1) has at most one solution in the class of functions bounded at the ends. If the condition

$$\int_0^1 \cos\left(2\varepsilon\log\tan\frac{\pi x}{2}\right)f(x)dx = 0 \tag{2.72}$$

is satisfied, then the solution exists and is given by (2.71). Alternatively, applying the map $\zeta = \cos\pi x$, $\eta = \cos\pi\xi$ we can write this condition and the solution as

$$\int_{-1}^1 \left[\left(\frac{1+\zeta}{1-\zeta}\right)^{i\varepsilon} + \left(\frac{1+\zeta}{1-\zeta}\right)^{-i\varepsilon}\right] f\left(\frac{\cos^{-1}\zeta}{\pi}\right)\frac{d\zeta}{\sqrt{1-\zeta^2}} = 0,$$

$$\phi\left(\frac{\cos^{-1}\zeta}{\pi}\right) = \frac{1}{2\pi}\int_{-1}^1 \left[\left(\frac{1-\eta}{1+\eta}\frac{1+\zeta}{1-\zeta}\right)^{i\varepsilon} + \left(\frac{1-\eta}{1+\eta}\frac{1+\zeta}{1-\zeta}\right)^{-i\varepsilon}\right]\sqrt{\frac{1-\zeta^2}{1-\eta^2}}\frac{f\left(\pi^{-1}\cos^{-1}\eta\right)d\eta}{\eta-\zeta}, \tag{2.73}$$

respectively. To discover the asymptotics of the solution at the endpoints, we employ formulas (2.38) as we did in the case $-1 < \beta < 1$. At the point $\zeta = -1$, we have

$$\phi(x) \sim C_0(1+\zeta)^{1/2+i\varepsilon} + \overline{C_0}(1+\zeta)^{1/2-i\varepsilon}, \quad \zeta \to -1^+, \tag{2.74}$$

and then, due to (2.42), derive

$$\phi(x) \sim (1-x)[C_{10}\cos(2\varepsilon\log(1-x)) + C_{11}\sin(2\varepsilon\log(1-x))], \quad x \to 1^-, \tag{2.75}$$

Similarly, at the second endpoint $x = 0$,

$$\phi(x) \sim x[C_{00}\cos(2\varepsilon\log x) + C_{01}\sin(2\varepsilon\log x)], \quad x \to 0^+. \tag{2.76}$$

Here, C_{jm} $(j, m = 0, 1)$ are real non-zero constants.

2.5 *Case* $\beta < -1$

In this case, the eigenvalues λ_1 and λ_2 of the matrix G_0 are positive, $\lambda_j = -\beta - (-1)^j\sqrt{\beta^2 - 1}$, $j = 1, 2$, and the transformation matrix T has the form

$$T = \begin{pmatrix} 1 & 1 \\ -\sqrt{-\beta_0} & \sqrt{-\beta_0} \end{pmatrix}, \quad \beta_0 = \frac{1+\beta}{1-\beta} < 0. \tag{2.77}$$

There are two possibilities for the choice of the parameters ρ_1 and ρ_2, (i) $\rho_1 = i\varepsilon$, $\rho_2 = -i\varepsilon$, and (ii) $\rho_1 = 1 + i\varepsilon$, $\rho_2 = 1 - i\varepsilon$. Here,

$$\varepsilon = \frac{1}{2\pi}\log(-\beta + \sqrt{\beta^2 - 1}) > 0. \tag{2.78}$$

Note that the pairs $\rho_1 = i\varepsilon$, $\rho_2 = 1 - i\varepsilon$ and $\rho_1 = 1 + i\varepsilon$, $\rho_2 = -i\varepsilon$ lead to the functions $\varphi_1(z)$ and $\varphi_2(z)$ which do not satisfy the boundary conditions (2.70).

For the pair $\rho_1 = i\varepsilon$, $\rho_2 = -i\varepsilon$, similarly to Sections 2.2 and 2.4, we derive the solution to the Hilbert problem (2.5) in the form

$$\varphi_1(z) = -\frac{z^2 - 1}{2\pi\sqrt{-\beta_0}}\int_L \left[\frac{\omega_1(t)}{\omega_1(z)} - \frac{\omega_2(t)}{\omega_2(z)}\right]\frac{v(t)dt}{(t-z)(1-tz)},$$

$$\varphi_2(z) = \frac{z^2 - 1}{2\pi}\int_L \left[\frac{\omega_1(t)}{\omega_1(z)} + \frac{\omega_2(t)}{\omega_2(z)}\right]\frac{v(t)dt}{(t-z)(1-tz)}, \tag{2.79}$$

where

$$\omega_1(t) = ie^{\pi\varepsilon}\tan^{1-2i\varepsilon}\frac{\pi x}{2}, \quad \omega_2(t) = ie^{-\pi\varepsilon}\tan^{1+2i\varepsilon}\frac{\pi x}{2}. \tag{2.80}$$

As before, by applying the Sokhotski-Plemelj formulas to the singular integrals (2.79) we determine

$$\varphi_1(e^{i\pi x}) = \frac{i\sin\pi x}{\sqrt{-\beta_0}}\int_0^1 \frac{v(e^{i\pi\xi})a(x,\xi)}{\cos\pi\xi - \cos\pi x}\sin(2\varepsilon\log a(x,\xi))\,d\xi,$$

$$\varphi_2(e^{i\pi x}) = -iv(e^{i\pi x}) + \sin\pi x\int_0^1 \frac{v(e^{i\pi\xi})a(x,\xi)}{\cos\pi\xi - \cos\pi x}\cos(2\varepsilon\log a(x,\xi))\,d\xi, \tag{2.81}$$

with $a(x, \xi)$ being the function given by (2.32). The functions (2.81) satisfy the boundary conditions on L in (2.5), $\operatorname{Re}\varphi_1(t) = 0$, $\operatorname{Im}\varphi_2(t) = -v(t)$, and yield the solution to the integral equation (2.1), $\phi(x) = \operatorname{Re}\varphi_2(e^{i\pi x})$ with $v(e^{i\pi\xi}) = f(\xi)$,

$$\phi(x) = \int_0^1 \frac{\sin\pi x a(x,\xi)\cos(2\varepsilon\log a(x,\xi))}{\cos\pi\xi - \cos\pi x}f(\xi)d\xi, \quad 0 < x < 1. \tag{2.82}$$

For this function to be the solution to the integral equation (2.1) in the class of functions chosen, it is necessary and sufficient that the function $f(x)$ meets the condition (2.6) or, equivalently,

$$\int_0^1 \tan\frac{\pi x}{2}\cos\left(2\varepsilon\log\tan\frac{\pi x}{2}\right)f(x)dx = 0. \tag{2.83}$$

This condition can be also represented in the form

$$\int_{-1}^{1}\left[\left(\frac{1+\zeta}{1-\zeta}\right)^{i\varepsilon}+\left(\frac{1+\zeta}{1-\zeta}\right)^{-i\varepsilon}\right]f\left(\frac{\cos^{-1}\zeta}{\pi}\right)\frac{d\zeta}{1+\zeta}=0, \tag{2.84}$$

and the corresponding solution to the integral equation becomes

$$\phi\left(\frac{\cos^{-1}\zeta}{\pi}\right)=\frac{1}{2\pi}\int_{-1}^{1}\left[\left(\frac{1-\eta}{1+\eta}\frac{1+\zeta}{1-\zeta}\right)^{i\varepsilon}+\left(\frac{1-\eta}{1+\eta}\frac{1+\zeta}{1-\zeta}\right)^{-i\varepsilon}\right]\frac{f\left(\pi^{-1}\cos^{-1}\eta\right)}{\eta-\zeta}\frac{(1+\zeta)d\eta}{1+\eta}. \tag{2.85}$$

Its analysis shows that the solution vanishes at the point $\zeta=-1$, while at the second end, $\zeta=1$, it is bounded and oscillates. Indeed, in view of the relations (2.38) applied to (2.85) we have

$$\phi\left(\frac{\cos^{-1}\zeta}{\pi}\right)\sim D_1(1+\zeta)^{1+i\varepsilon}+\overline{D_1}(1+\zeta)^{1-i\varepsilon},\quad \zeta\to-1^+,$$

$$\phi\left(\frac{\cos^{-1}\zeta}{\pi}\right)\sim D_0(1-\zeta)^{i\varepsilon}+\overline{D_0}(1-\zeta)^{-i\varepsilon},\quad \zeta\to 1^-. \tag{2.86}$$

for some non-zero complex constants D_0 and D_1. On returning to the original variables we deduce

$$\phi(x)\sim D_{00}\cos(2\varepsilon\log x)+D_{01}\sin(2\varepsilon\log x),\quad x\to 0^+,$$

$$\phi(x)\sim(1-x)^2[D_{10}\cos(2\varepsilon\log(1-x))+D_{11}\sin(2\varepsilon\log(1-x))],\quad x\to 1^-, \tag{2.87}$$

where D_{jm} $(j,m=0,1)$ are real non-zero constants.

For the second choice of the parameters ρ_1 and ρ_2, $\rho_1=1+i\varepsilon$ and $\rho_2=1-i\varepsilon$, in the case $\beta<-1$, we again derive the solution to the Hilbert problem (2.5), (2.6), take the real part of the second potential $\varphi_2(z)$, put $v(e^{i\pi x})=f(x)$ and obtain the solvability condition of (2.1)

$$\int_0^1\cot\frac{\pi x}{2}\cos\left(2\varepsilon\log\tan\frac{\pi x}{2}\right)f(x)dx=0 \tag{2.88}$$

and the solution to the integral equation

$$\phi(x)=\int_0^1\frac{\sin\pi x\cos\left(2\varepsilon\log a(x,\xi)\right)}{(\cos\pi\xi-\cos\pi x)a(x,\xi)}f(\xi)d\xi,\quad 0<x<1. \tag{2.89}$$

On making the substitutions $\eta=\cos\pi\xi$, $\zeta=\cos\pi x$, we have another form of the solvability condition and the solution to the integral equation. It coincides with formulas (2.84) and (2.85) if we replace there $d\zeta/(1+\zeta)$ and $(1+\zeta)d\eta/(1+\eta)$ by $d\zeta/(1-\zeta)$ and $(1-\zeta)d\eta/(1-\eta)$, respectively. The solution (2.89) associated with the parameters $\rho_1=1+i\varepsilon$ and $\rho_2=1-i\varepsilon$ and the condition (2.88) vanishes at the point $x=0$ and is bounded and oscillates at the point $x=1$,

$$\phi(x)\sim x^2[E_{00}\cos(2\varepsilon\log x)+E_{01}\sin(2\varepsilon\log x)],\quad x\to 0^+,$$

$$\phi(x)\sim E_{10}\cos(2\varepsilon\log(1-x))+E_{11}\sin(2\varepsilon\log(1-x)),\quad x\to 1^-, \tag{2.90}$$

where E_{jm} $(j,m=0,1)$ are real non-zero constants.

2.6 *Solution to the characteristic integral equation*

We now summarise the results.

THEOREM 2.2. *Let* $f(x) \in H[0, 1]$ *and* β *be a real parameter. Denote*

$$\varepsilon = \begin{cases} \delta, & \beta \in (0, 1), \\ 1 + \delta, & \beta \in (-1, 0), \\ (2\pi)^{-1} \log(|\beta| + \sqrt{\beta^2 - 1}), & |\beta| > 1, \end{cases} \qquad \delta = \frac{1}{\pi} \tan^{-1} \frac{\sqrt{1 - \beta^2}}{\beta},$$

$$a(x, \xi) = \tan \frac{\pi \xi}{2} \cot \frac{\pi x}{2}. \tag{2.91}$$

In the class of functions Hölder-continuous everywhere in the interval $(0, 1)$ *and bounded at the endpoints, generally, the singular integral equation*

$$\mathcal{S}[\phi](x) \equiv \int_0^1 S(x, \xi)\phi(\xi)d\xi = f(x), \quad 0 < x < 1, \tag{2.92}$$

with the kernel $S(x, \xi) = \frac{1}{2} \cot \frac{\pi}{2}(\xi - x) + \frac{\beta}{2} \cot \frac{\pi}{2}(\xi + x)$ *does not have a solution. If* $|\beta| < 1$ *or* $\beta > 1$*, the integral equation becomes solvable if and only if the function* $f(x)$ *meets the condition*

$$\int_0^1 V(x) f(x) dx = 0, \tag{2.93}$$

where

$$V(x) = \begin{cases} (\sin \pi x)^{-1/2} \cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right), & \beta = 0, \\ \tan^{\varepsilon} \frac{\pi x}{2} + \cot^{\varepsilon} \frac{\pi x}{2}, & |\beta| < 1, \ \beta \neq 0, \\ \cos\left(2\varepsilon \log \tan \frac{\pi x}{2}\right), & \beta > 1. \end{cases} \tag{2.94}$$

If the condition of solvability is fulfilled, then the inverse operator S^{-1} *exists, and the solution is unique and given by* $\phi(x) = \mathcal{S}^{-1}[f](x)$*, where*

$$\mathcal{S}^{-1}[f](x) = \frac{1}{2} \int_0^1 \sqrt{\frac{\sin \pi x}{\sin \pi \xi}} \frac{f(\xi) d\xi}{\sin \frac{1}{2}\pi(x - \xi)}, \quad \beta = 0,$$

$$\mathcal{S}^{-1}[f](x) = \frac{1}{2} \int_0^1 \left[a^{\varepsilon}(x, \xi) + a^{-\varepsilon}(x, \xi)\right] \frac{\sin \pi x f(\xi) d\xi}{\cos \pi \xi - \cos \pi x}, \quad |\beta| < 1, \quad \beta \neq 0,$$

$$\mathcal{S}^{-1}[f](x) = \int_0^1 \cos\left(2\varepsilon \log a(x, \xi)\right) \frac{\sin \pi x f(\xi) d\xi}{\cos \pi \xi - \cos \pi x}, \quad \beta > 1. \tag{2.95}$$

The solution vanishes at the points $x = 0$ *and* $x = 1$*, and its asymptotics is described by (2.49) if* $\beta = 0$*, (2.43) in the case* $|\beta| < 1$ *and (2.75), (2.76) if* $\beta > 1$*.*

In the case $\beta < -1$*, in the class of functions chosen, the integral equation is solvable if and only if the function* $f(x)$ *satisfies one of the following two conditions:*

$$\int_0^1 \left(\tan \frac{\pi x}{2}\right)^{\pm 1} \cos\left(2\varepsilon \log \tan \frac{\pi x}{2}\right) f(x) dx = 0. \tag{2.96}$$

Then the solution is unique and given by $\phi(x) = \mathcal{S}^{-1}[f](x)$*, where*

$$\mathcal{S}^{-1}[f](x) = \int_0^1 \frac{\sin \pi x [a(x,\xi)]^{\pm 1} \cos(2\varepsilon \log a(x,\xi))}{\cos \pi\xi - \cos \pi x} f(\xi) d\xi, \quad 0 < x < 1. \tag{2.97}$$

It is bounded and oscillates at the point $p^{\pm}$ *and vanishes at the point* $p^{\mp}$*, where* $p^+ = 0$ *and* $p^- = 1$*. The asymptotics of the solution is described by (2.87) and (2.90).*

The two cases left, $\beta = 1$ and $\beta = -1$, can be easily treated by reducing the integral equation to an equation solvable by the Hilbert inversion formula (see Appendix B).

3. Complete singular integral equation

In this section, we will develop an algorithm for the solution of the complete singular integral equation

$$\int_0^1 [S(x,\xi) + K(x,\xi)]\phi(\xi) d\xi = -F(x) + C, \quad 0 < x < 1, \tag{3.1}$$

where $S(x,\xi)$ is the singular kernel introduced in (2.2), $K(x,\xi)$ is a regular kernel, $F(x) \in H[0,1]$, and C is an unknown constant. We seek the solution to this equation, $\phi(x)$, in the class of Hölder functions bounded at the endpoints.

3.1 *Regularisation of the complete equation*

First we regularize (3.1) and reduce it to a Fredholm integral equation. Since the inverse operator $\mathcal{S}^{-1}$ has been constructed, it is reasonable to apply the Carleman–Vekua regularisation procedure (**7**). This brings us to the equation

$$\phi(x) + \mathcal{S}^{-1}\mathcal{K}[\phi](x) = \mathcal{S}^{-1}[-F + C](x), \quad 0 < x < 1, \tag{3.2}$$

provided the constant C is selected to be

$$C = \left(\int_0^1 V(x) dx\right)^{-1} \int_0^1 V(x)\,[F(x) + \mathcal{K}[\phi](x)]\,dx. \tag{3.3}$$

Here, $\mathcal{K}$ is the Fredholm operator with the kernel $K(x,\xi)$, and $V(x)$ is the function defined by (2.94). Motivated by applications to fracture mechanics we restrict ourselves to considering the case $|\beta| < 1$ and exclude the trivial case $\beta = 0$. Note that a similar procedure can be worked out when $|\beta| > 1$. So, here and further we suppose $0 < |\beta| < 1$. It turns out (see Appendix C) that the first integral in (3.3) can be evaluated explicitly and the solvability condition becomes

$$C = \frac{\sin \pi\rho_1}{2} \int_0^1 \left(\tan^{2\rho_1 - 1}\frac{\pi x}{2} + \cot^{2\rho_1 - 1}\frac{\pi x}{2}\right)[F(x) + \mathcal{K}[\phi](x)]\,dx. \tag{3.4}$$

If this condition holds, then the integral equation (3.1) is equivalent to the following Fredholm integral equation:

$$\phi(x) + \int_0^1 \tilde{K}(x,\xi)\phi(\xi) d\xi = -\tilde{F}(x), \quad 0 < x < 1, \tag{3.5}$$

where $\tilde{K}(x, \xi)$ and $\tilde{F}(x)$ are bounded in the sets $0 \le x, \xi \le 1$ and $0 \le x \le 1$, respectively, and have the form

$$\tilde{K}(x, \xi) = \frac{\sin \pi x}{2} \int_0^1 \frac{[a^{2\rho_1 - 1}(x, \tau) + a^{-2\rho_1 + 1}(x, \tau)]K(\tau, \xi)d\tau}{\cos \pi\tau - \cos \pi x},$$

$$\tilde{F}(x) = \frac{\sin \pi x}{2} \int_0^1 \frac{[a^{2\rho_1 - 1}(x, \tau) + a^{-2\rho_1 + 1}(x, \tau)]F(\tau)d\tau}{\cos \pi\tau - \cos \pi x}. \quad (3.6)$$

Here, we used the relation $\mathcal{S}^{-1}[1] = 0$ to be proved in Section 3.2. As in the case of the complete singular integral equation with the Cauchy kernel, for numerical purposes, it is preferable to deal with the singular equation (3.1) directly and bypass its regularisation. In the next sections, we generalise the method of orthogonal polynomials efficient for equations with the Cauchy kernel to the case of (3.1).

3.2 *Spectral relation for the operator $\mathcal{S}$*

The heart of the numerical method to be proposed is the derivation of the solutions $\phi_j(x), j = 0, 1, \ldots,$ of the characteristic equation $\mathcal{S}[\phi_j](x) = f_j(x)$ with the right-hand side chosen to be $f_j(x) = C_j - \cos[(j+1)\pi x]$,

$$\frac{1}{2}\int_0^1 \left[\cot\frac{\pi(\xi - x)}{2} + \beta \cot\frac{\pi(\xi + x)}{2}\right]\phi_j(\xi)d\xi = C_j - \cos[(j+1)\pi x], \quad 0 < x < 1, \quad j = 0, 1, \ldots, \quad (3.7)$$

where C_j are constants to be fixed. The solutions are sought in the class of functions vanishing at the endpoints and meeting the condition (2.36) that is $\mathcal{S} : L^2_w(0, 1) \cap H^\circ(0, 1) \to L^2(0, 1) \cap H[0, 1]$, where $L^2_w(0, 1)$ is the weight Hilbert space, $w(x) = \sin^{2(\rho_1 - 1)} \pi x$, and H° is the class of Hölder functions meeting the condition (2.36). The functions $\phi_j(x)$ admit the following explicit expressions in terms of some trigonometric polynomials:

$$\phi_j(x) = \cos^{2\rho_1}\frac{\pi x}{2}\sin^{2(1-\rho_1)}\frac{\pi x}{2} q_j^{(\rho_1)}(x) + \cos^{2(1-\rho_1)}\frac{\pi x}{2}\sin^{2\rho_1}\frac{\pi x}{2} q_j^{(1-\rho_1)}(x), \quad (3.8)$$

where

$$q_j^{(\alpha)}(x) = \sum_{\nu=0}^{j} c_{j\nu}^{(\alpha)} \sin^{2\nu}\frac{\pi x}{2},$$

$$c_{j\nu}^{(\alpha)} = \frac{1}{2\sin\pi\alpha}\sum_{m=\nu+1}^{j+1}\frac{(-j-1)_m(j+1)_m(\alpha)_{m-1-\nu}}{(1/2)_m m!(m-1-\nu)!}, \quad (3.9)$$

and $(\cdot)_m$ is the factorial symbol. On making the substitution $\zeta = \cos \pi x$ it is possible to rewrite the spectral relation (3.7) in the form

$$\frac{1}{2\pi}\int_{-1}^1 \left[\frac{\sqrt{(1+\eta)(1+\zeta)} + \sqrt{(1-\eta)(1-\zeta)}}{\sqrt{(1-\eta)(1+\zeta)} - \sqrt{(1+\eta)(1-\zeta)}} + \beta\frac{\sqrt{(1+\eta)(1+\zeta)} - \sqrt{(1-\eta)(1-\zeta)}}{\sqrt{(1-\eta)(1+\zeta)} + \sqrt{(1+\eta)(1-\zeta)}}\right]\frac{\psi_j(\eta)d\eta}{\sqrt{1-\eta^2}}$$

$$= C_j - T_{j+1}(\zeta), \quad -1 < \zeta < 1, \quad (3.10)$$

where the functions $\psi_j(\zeta) = \phi_j(\frac{1}{\pi}\cos^{-1}\zeta)$ are given by

$$\psi_j(\zeta) = \frac{1}{2}\left[(1+\zeta)^{\rho_1}(1-\zeta)^{1-\rho_1}p_j^{(\rho_1)}(\zeta) + (1+\zeta)^{1-\rho_1}(1-\zeta)^{\rho_1}p_j^{(1-\rho_1)}(\zeta)\right],$$

$$p_j^{(\alpha)}(\zeta) = \sum_{\nu=0}^{j} c_{j\nu}\left(\frac{1-\zeta}{2}\right)^{\nu}. \tag{3.11}$$

To prove the correctness of these formulas, first we must satisfy the solvability condition (2.93). It fixes the constants C_j

$$C_j = \frac{M_{j+1}}{M_0}, \tag{3.12}$$

where

$$M_j = \int_0^1 \left(\tan^{2\rho_1-1}\frac{\pi x}{2} + \cot^{2\rho_1-1}\frac{\pi x}{2}\right) T_j(\cos\pi x)dx, \quad j = 0, 1, \ldots. \tag{3.13}$$

These integrals evaluated explicitly in Appendix C have the form

$$M_0 = 2\csc\pi\rho_1, \quad M_{2m-1} = 0,$$

$$M_{2m} = \frac{2}{\sin\pi\rho_1}\sum_{j=0}^{2m}\frac{(-2m)_j(2m)_j(\rho_1)_j}{(1/2)_j(j!)^2}, \quad m = 1, 2, \ldots. \tag{3.14}$$

After we have computed the constants C_j we wish to simplify the expression (2.34) ($|\beta| < 1$) for the solution of the integral equation (3.7) and therefore verify the representation (3.8). On making the substitutions $\zeta = \cos\pi x$ and $\eta = \cos\pi\xi$ and using the first formula in (2.37) we discover

$$\phi_j(x) = \frac{1}{2}\left[(1+\zeta)^{\rho_1}(1-\zeta)^{1-\rho_1}I_j^{(\rho_1)}(\zeta) + (1+\zeta)^{1-\rho_1}(1-\zeta)^{\rho_1}I_j^{(1-\rho_1)}(\zeta)\right], \tag{3.15}$$

where

$$I_j^{(\alpha)}(\zeta) = C_j J_0^{(\alpha)}(\zeta) - J_{j+1}^{(\alpha)}(\zeta),$$

$$J_j^{(\alpha)}(\zeta) = \frac{1}{\pi}\int_{-1}^{1}(1-\eta)^{\alpha-1}(1+\eta)^{-\alpha}\frac{T_j(\eta)d\eta}{\eta-\zeta}, \quad 0 < \alpha < 1, \quad j = 0, 1, \ldots. \tag{3.16}$$

The method we apply to compute the latter integral is built upon the convolution theorem and the theory of residues. Recast the integral $J_j^{(\alpha)}(\zeta)$ as

$$J_j^{(\alpha)}(\zeta) = \int_0^{\infty} h_1(\tau)h_2\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}, \quad -1 < \zeta < 1, \quad 0 < t < 1, \tag{3.17}$$

where $t = (1-\zeta)/2$,

$$h_1(\tau) = \begin{cases} \tau^{\alpha-1}(1-\tau)^{-\alpha}T_j(1-2\tau), & 0 < \tau < 1, \\ 0, & \tau > 1, \end{cases} \quad h_2(\tau) = -\frac{1}{2\pi(1-\tau)}, \tag{3.18}$$

and apply the convolution theorem. It reads

$$J_j^{(\alpha)}(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} H_1(s) H_2(s) t^{-s} ds. \tag{3.19}$$

Here, H_1 and H_2 are the Mellin transforms of the functions h_1 and h_2, respectively, and $c \in (1-\alpha, 1)$ $(0 < \alpha < 1)$. Formula (C.5) enables us to find a series representation of $H_1(s)$,

$$H_1(s) = \frac{\Gamma(s+\alpha-1)\Gamma(1-\alpha)}{\Gamma(s)} {}_3F_2(-j, j, s+\alpha-1; 1/2, s; 1), \quad \operatorname{Re} s > 1-\alpha, \tag{3.20}$$

and since

$$H_2(s) = -\frac{1}{2} \cot \pi s, \quad 0 < \operatorname{Re} s < 1, \tag{3.21}$$

the integral $J_j^{(\alpha)}(\zeta)$ becomes

$$J_j^{(\alpha)}(\zeta) = -\frac{\Gamma(1-\alpha)}{2\pi} \sum_{m=0}^{j} \frac{(-1)^m (-j)_m (j)_m}{(1/2)_m m!} L_m(t), \tag{3.22}$$

where

$$L_m(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \cos \pi s \Gamma(1-s-m) \Gamma(s+\alpha-1+m) t^{-s} ds, \quad 0 < t < 1. \tag{3.23}$$

Next, by employing the theory of residues we compute the integral $L_m(t)$ and deduce the series representation of the function $J_j^{(\alpha)}(\zeta)$

$$J_j^{(\alpha)}(\zeta) = -\frac{1}{2 \sin \pi \alpha} \sum_{m=0}^{j} \frac{(-j)_m (j)_m t^{m-1}}{(1/2)_m m!} \left[\sum_{k=0}^{m-1} \frac{(\alpha)_k t^{-k}}{k!} - \cos \pi \alpha \left(\frac{t}{1-t} \right)^{\alpha} \right]. \tag{3.24}$$

Note that the sum $\sum_{k=0}^{m-1} (\alpha)_k t^{-k} (k!)^{-1}$ is equal to zero when $m = 0$. Now we use the connection (C.4) between the Gauss hypergeometric function and the Chebyshev polynomials of the first kind to obtain

$$J_j^{(\alpha)}(\zeta) = \cot \pi \alpha (1-\zeta)^{\alpha-1} (1+\zeta)^{-\alpha} T_j(\zeta) - q_{j-1}^{(\alpha)}(\zeta), \quad -1 < \zeta < 1, \quad j = 0, 1, \ldots, \tag{3.25}$$

where $q_{-1}^{(\alpha)}(\zeta) \equiv 0$. To finalise our computations, we substitute this expression into the first formula in (3.16) to have $I_j^{(\alpha)}(\zeta)$. According to (3.15) the function $\phi_j(x)$ is the sum of the functions

$$\frac{1}{2}(1+\zeta)^{\rho_1}(1-\zeta)^{1-\rho_1} I_j^{(\rho_1)}(\zeta) = B(\zeta) + \frac{1}{2}(1+\zeta)^{\rho_1}(1-\zeta)^{1-\rho_1} q_j^{(\rho_1)}(\zeta),$$

$$\frac{1}{2}(1+\zeta)^{1-\rho_1}(1-\zeta)^{\rho_1} I_j^{(1-\rho_1)}(\zeta) = -B(\zeta) + \frac{1}{2}(1+\zeta)^{1-\rho_1}(1-\zeta)^{\rho_1} q_j^{(1-\rho_1)}(\zeta), \tag{3.26}$$

where

$$B(\zeta) = \frac{1}{2} \cot \pi \rho_1 [C_j - T_{j+1}(\zeta)]. \tag{3.27}$$

After some obvious simplifications this eventually yields the spectral relation for the operator $\mathcal{S}$

$$\mathcal{S}[\phi_j](x) = N_j - \cos(j+1)\pi x, \quad 0 < x < 1, \quad j = 0, 1, \dots, \tag{3.28}$$

where

$$N_{2m+1} = 0, \quad N_{2m} = \sum_{j=0}^{2m} \frac{(-2m)_j (2m)_j (\rho_1)_j}{(1/2)_j (j!)^2}, \quad m = 0, 1, \dots, \tag{3.29}$$

and $\phi_j(x)$ are expressed through the degree-$2j$ polynomials on $\sin \frac{\pi x}{2}$,

$$\phi_j(x) = \cos^{2\rho_1} \frac{\pi x}{2} \sin^{2(1-\rho_1)} \frac{\pi x}{2} q_j^{(\rho_1)}(x) + \cos^{2(1-\rho_1)} \frac{\pi x}{2} \sin^{2\rho_1} \frac{\pi x}{2} q_j^{(1-\rho_1)}(x), \tag{3.30}$$

where

$$q_j^{(\alpha)}(x) = \frac{1}{2 \sin \pi \alpha} \sum_{m=1}^{j+1} \frac{(-j-1)_m (j+1)_m}{(1/2)_m m!} \sum_{k=0}^{m-1} \frac{(\alpha)_k}{k!} \sin^{2(m-1-k)} \frac{\pi x}{2}. \tag{3.31}$$

Evidently, this expression coincides with (3.9).

Note that on putting $j = -1$ in (3.15) and (3.26) we obtain the relation

$$\frac{\sin \pi x}{2} \int_0^1 \frac{[a^{2\rho_1 - 1}(x, \tau) + a^{-2\rho_1 + 1}(x, \tau)] d\tau}{\cos \pi \tau - \cos \pi x} = 0, \quad 0 < x < 1, \tag{3.32}$$

that is

$$\mathcal{S}^{-1}[1] = 0, \quad 0 < x < 1. \tag{3.33}$$

In Figure 1, we plot the functions $\phi_j(x)$, $j = 0, 1, \dots, 4$. They vanish at the endpoints $x = 0$ and $x = 1$. This numerical result is consistent with formulas (2.43). Also, the functions $\phi_j(x)$ have some properties which make them resemble orthogonal polynomials. First, the functions $\phi_j(x)$ have exactly j real roots on the interval $(0, 1)$. Secondly, although $\{\phi_j(x)\}, j = 0, 1, \dots,$ is not an orthogonal system, the functions $\{\phi_{2m+1}(x)\}$ are orthogonal to $\{\phi_{2k}(x)\}$ in the associated weight Hilbert space $L_w^2(0, 1)$, $w(x) = \sin^{2(\rho_1 - 1)} \pi x$,

$$\int_0^1 \phi_{2m+1}(x) \phi_{2k}(x) w(x) dx = 0, \quad m, k = 0, 1, \dots. \tag{3.34}$$

Finally, the functions $\phi_j(x)$ $(j = 0, 1, \dots)$ can be employed for constructing a series-form solution of the characteristic equation in the same fashion as the Chebyshev polynomials are used for the integral equation with the Cauchy kernel in the segment.

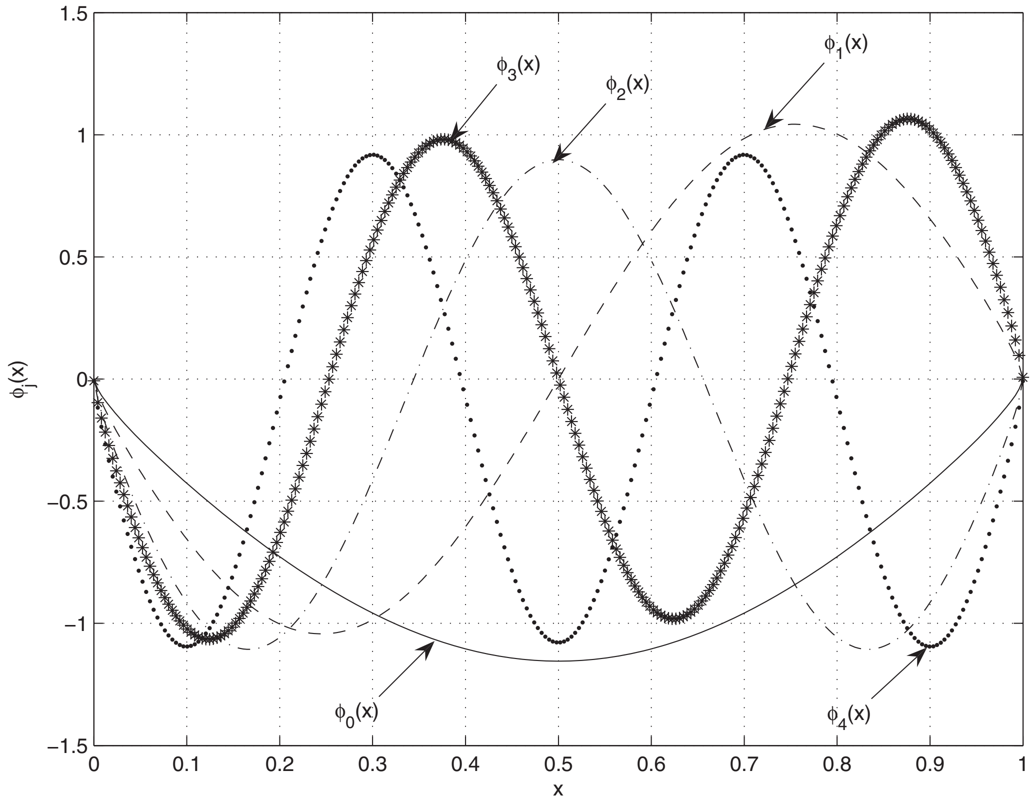

Fig. 1 Functions $\phi_j(x)$, $j = 0, 1, \dots, 4$, $\beta = 0.5$

3.3 *Series-form solution of the characteristic equation*

We consider the characteristic equation

$$\mathcal{S}[\phi](x) = C - F(x), \quad 0 < x < 1, \tag{3.35}$$

with the constant C given by

$$C = \frac{\sin \pi\rho_1}{2} \int_0^1 \left(\tan^{2\rho_1 - 1} \frac{\pi x}{2} + \cot^{2\rho_1 - 1} \frac{\pi x}{2}\right) F(x)dx \tag{3.36}$$

and the function $F(x)$ represented by its cosine Fourier series

$$F(x) = f_0 + 2\sum_{j=1}^{\infty} f_j \cos \pi j x, \quad 0 < x < 1,$$

$$f_j = \int_0^1 F(x) \cos \pi j x dx. \tag{3.37}$$

Table 1 The values of the function $\phi(x)$ as $\beta = 0.5$, $x = 0.5$ and $x = 0.25$ for some values of m_0

	$m_0 = 5$	$m_0 = 10$	$m_0 = 15$	$m_0 = 20$	$m_0 = 22$	$m_0 = 25$
$x_0 = 0.5$	0.445026	0.439400	0.440180	0.441492	0.441439	0.434170
$x = 0.25$	0.371760	0.372910	0.370147	0.371442	0.371252	0.363680

According to Theorem 2.2 and since $\mathcal{S}^{-1}[1] = 0$, we derive $\phi(x) = -\mathcal{S}^{-1}[F](x)$, $0 < x < 1$. On replacing $F(x)$ by its cosine series and changing the order of integration and summation we have

$$\phi(x) = -2\sum_{j=1}^{\infty} f_j \mathcal{S}^{-1}[\cos \pi j\xi](x). \tag{3.38}$$

We recall that $\mathcal{S}^{-1}[\cos \pi j\xi](x) = -\phi_{j-1}(x)$, $j = 1, 2, \ldots$. This brings us to

$$\phi(x) = 2\sum_{j=0}^{\infty} f_{j+1}\phi_j(x). \tag{3.39}$$

We consider two examples, $F(x) = x$ and $F(x) = x^2$. The cosine Fourier series of these functions have the form

$$x = \frac{1}{2} + \frac{2}{\pi^2}\sum_{j=1}^{\infty} \frac{(-1)^j - 1}{j^2} \cos \pi jx, \quad 0 < x < 1,$$

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \cos \pi jx, \quad 0 < x < 1. \tag{3.40}$$

We substitute these series into the relation (3.36), use formula (3.14) and determine that if $F(x) = x$, then $C = \frac{1}{2}$ for all ρ_1 ($|\beta| < 1$). In the case $F(x) = x^2$,

$$C = \frac{1}{3} + \frac{1}{\pi^2}\sum_{m=1}^{\infty} \Pi_m, \quad \Pi_m = \frac{1}{m^2}\sum_{j=0}^{2m} \frac{(-2m)_j(2m)_j(\rho_1)_j}{(1/2)_j(j!)^2}. \tag{3.41}$$

Our computations show that for $\beta = \frac{1}{2}$, $C(M) = \frac{1}{3} + \pi^{-2}\sum_{m=1}^{M} \Pi_m \approx \frac{2}{3}$ for large M: $C(100) = 0.665659$, $C(1000) = 0.666565$, $C(2000) = 0.666616$, $C(3000) = 0.666633$. As for the solution itself, it turns out that the error of approximation decreases as m_0 in $\phi(x) \approx 2\sum_{j=0}^{m_0} F_{j+1}\phi_j(x)$ approaches 20, and the algoritm becomes unstable for $m_0 \geq 25$ (Table 1). This is a typical feature even in the case of the classical method of orthogonal polynomials caused by their oscilation that increases as m_0 grows. In Figure 2, we plot the function $\phi(x)$ when $m_0 = 20$ for the cases $F(x) = x$ and $F(x) = x^2$.

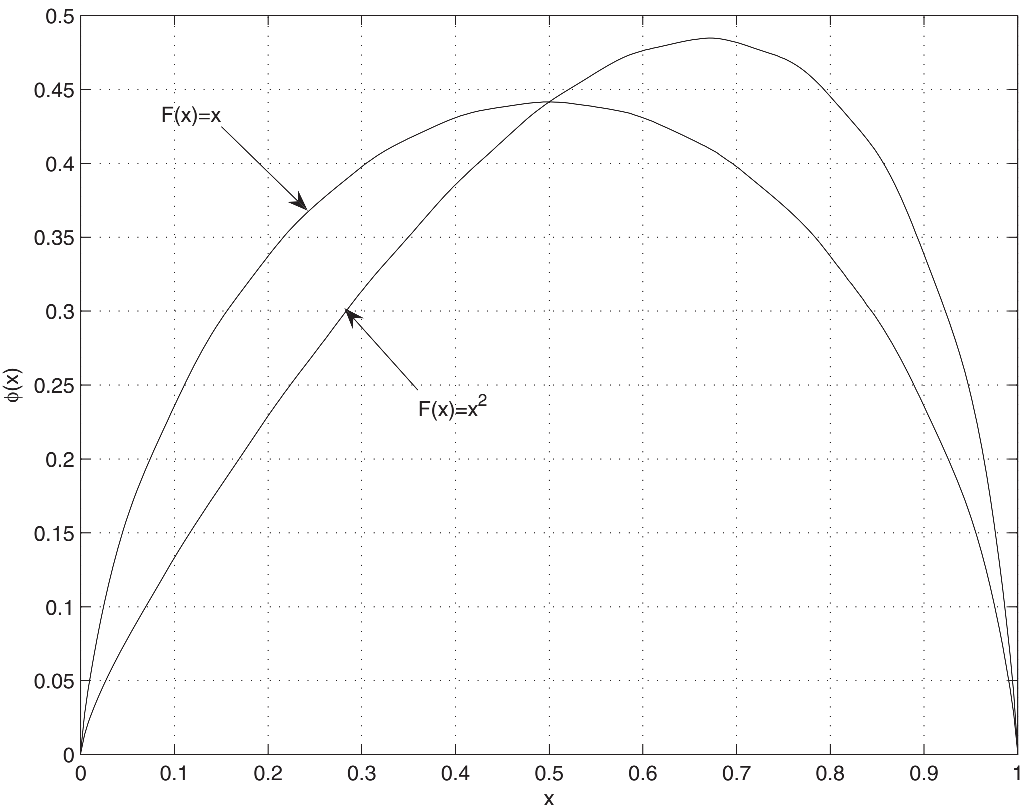

Fig. 2 The solution of the characteristic equation $\phi(x)$ for $F(x) = x$ and $F(x) = x^2$ when $\beta = 0.5$

3.4 *Solution of the complete integral equation*

We continue considering the case $|\beta| < 1$. In the class of functions bounded at the ends the complete singular integral equation (3.1) is solvable if and only if the following condition is met:

$$\int_0^1 \left(\tan^{2\rho_1 - 1} \frac{\pi x}{2} + \cot^{2\rho_1 - 1} \frac{\pi x}{2}\right) \left[C - F(x) - \int_0^1 K(x, \xi)\phi(\xi)d\xi\right] dx = 0. \tag{3.42}$$

Suppose this condition holds. We build up the solution to the integral equation (3.1) in the series form

$$\phi(x) = \sum_{j=0}^{\infty} b_j \phi_j(x), \tag{3.43}$$

where the coefficients b_j ($j = 0, 1, \ldots$) are to be determined. On substituting this series into (3.1), utilising the expansion (3.43) and the orthogonality relation

$$\int_0^1 \cos j\pi x \cos n\pi x dx = \begin{cases} 1, & n = j = 0, \\ \frac{1}{2}, & n = j \neq 0, \\ 0, & n \neq j, \end{cases} \tag{3.44}$$

we derive the following infinite system of algebraic equations:

$$\sum_{j=0}^{\infty} \left[N_{j+1}\delta_{n,0} - \frac{1}{2}\delta_{n,j+1} + k_{nj} \right] b_j = C\delta_{n,0} - f_n, \quad n = 0, 1, \ldots . \tag{3.45}$$

Here, $\delta_{n,j}$ is the Kronecker delta, and

$$k_{nj} = \int_0^1 \int_0^1 K(x, \xi)\phi_j(\xi) \cos n\pi x d\xi dx,$$

$$f_n = \int_0^1 F(x) \cos n\pi x dx. \tag{3.46}$$

It is convenient to split this system into two parts

$$n = 0: \quad \sum_{j=0}^{\infty} \left(N_{j+1} + k_{0j}\right) b_j = -f_0 + C, \tag{3.47}$$

and

$$n = 1, 2, \ldots: \quad -\frac{1}{2}b_{n-1} + \sum_{j=0}^{\infty} k_{nj} b_j = -f_n. \tag{3.48}$$

It turns out that (3.47) is equivalent to the solvability relation (3.42). To prove this, we substitute the Fourier expansions

$$\int_0^1 K(x, \xi)\phi_j(\xi) d\xi = k_{0j} + 2\sum_{l=1}^{\infty} k_{nj} \cos \pi l x,$$

$$F(x) = f_0 + 2\sum_{j=0}^{\infty} f_j \cos \pi j x, \tag{3.49}$$

into (3.42), recall formula (3.13) and rewrite the solvability condition (3.42) as

$$M_0 \left(C - f_0 - \sum_{j=0}^{\infty} k_{0j} b_j \right) - 2\sum_{n=1}^{\infty} M_n \left(f_n + \sum_{j=0}^{\infty} k_{nj} b_j \right) = 0. \tag{3.50}$$

Now, the coefficients b_n solve the system of (3.47) and (3.48). This immediately brings us to the identity

$$\sum_{j=0}^{\infty} b_j M_{j+1} - \sum_{n=1}^{\infty} b_{n-1} M_n = 0. \tag{3.51}$$

and therefore, in the class of functions bounded at the ends, the singular integral equation (3.1) is equivalent to the system of linear algebraic equations (3.47) and (3.48); its solution automatically satisfies the solvability condition (3.42). In Appendix D, we use the classical method of orthogonal polynomials to analyse the complete singular integral equation with the Cauchy kernel in the class of bounded at the endpoints functions and show that the corresponding solvability condition is also equivalent to the first ($n = 0$) equation of the associated infinite system of algebraic equations.

4. Antiplane problem for a crack in a composite plane

The problem under consideration is one of antiplane strain on a crack $0 < x < 1$, $y = 0^\pm$. The elastic medium is formed by two half-planes $x < 0$ and $x > 1$ and an infinite strip $0 < x < 1$. The shear moduli of the half-planes and the strip are G_1 and G_2, respectively. The faces of the crack are subjected to traction $\tau_{yz} = -f(x)$, $0 < x < 1$, $y = 0^\pm$. The problem is governed by the following boundary value problem for the Laplace operator in the plane:

$$\Delta w(x, y) = 0, \quad |x| < \infty, \quad |y| < \infty, \quad x \neq 0, \quad x \neq 1,$$

$$w(0^-, y) - w(0^+, y) = 0, \quad w(1^-, y) - w(1^+, y) = 0, \quad |y| < \infty,$$

$$G_1 w_x(0^-, y) = G_2 w_x(0^+, y), \quad G_2 w_x(1^-, y) = G_1 w_x(1^+, y), \quad |y| < \infty,$$

$$G_2 w_y(x, 0^\pm) = -f(x),\ 0 < x < 1. \tag{4.1}$$

We note that due to the symmetry of the problem, $w(x, 0^\pm) = 0$, $-\infty < x < 0$, $1 < x < \infty$, and the problem can be restated for say, the upper half-plane. On the crack faces, the displacement w is discontinuous, $w(x, 0^+) = -w(x, 0^-)$, $0 < x < 1$, and the displacement jump is to be determined. Denote $\phi(x) = G_2 w(x, 0^+)$, $0 < x < 1$. By the method of integral transformations the problem reduces to the following integral equation (**6**):

$$-\frac{d}{dx}\int_0^1 M(x, \xi)\phi(\xi)d\xi = f(x), \quad 0 < x < 1, \tag{4.2}$$

where

$$M(x, \xi) = \frac{1}{\pi}\left[\frac{1}{\xi - x} + \frac{\beta}{\xi + x} + \frac{\beta}{x + \xi - 2} + R(x, \xi)\right],$$

$$R(x, \xi) = \beta[D(x + \xi) - D(2 - x - \xi)] + \beta^2\left[D(2 - x + \xi) - D(2 + x - \xi) + \frac{2(x - \xi)}{4 - (x - \xi)^2}\right],$$

$$D(x) = \sum_{j=1}^{\infty}\frac{\beta^{2j}}{x + 2j}, \quad \beta = \frac{\lambda - 1}{\lambda + 1} \in (-1, 1), \quad \lambda = \frac{G_1}{G_2} \in (0, \infty). \tag{4.3}$$

By integrating equation (4.3) with respect to x we can rewrite the new equation in the form used in the previous section

$$\int_0^1 [S(x, \xi) + K(x, \xi)]\phi(\xi)d\xi = -F(x) + C, \quad 0 < x < 1, \tag{4.4}$$

where $F(x) = \int f(x)dx$ and

$$K(x,\xi) = \frac{1}{\pi}\left[\frac{1}{\xi - x} + \frac{\beta}{\xi + x} + \frac{\beta}{x + \xi - 2} + R(x,\xi)\right] - \frac{1}{2}\cot\frac{\pi(\xi - x)}{2} - \frac{\beta}{2}\cot\frac{\pi(\xi + x)}{2}. \tag{4.5}$$

is a regular kernel. This equation has been solved in the previous section by reducing it to the infinite system of linear algebraic equations of the second kind (3.48). The coefficients (3.46) of the infinite system can be represented in the form

$$f_n = \frac{1}{\pi}\int_{-1}^{1} F\left(\frac{\cos^{-1}\zeta}{\pi}\right)\frac{T_n(\zeta)d\zeta}{\sqrt{1-\zeta^2}},$$

$$k_{nj} = \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} K\left(\frac{\cos^{-1}\zeta}{\pi}, \frac{\cos^{-1}\eta}{\pi}\right)\phi_j\left(\frac{\cos^{-1}\eta}{\pi}\right)\frac{T_n(\zeta)d\zeta d\eta}{\sqrt{1-\zeta^2}\sqrt{1-\eta^2}}, \tag{4.6}$$

and computed by the Gauss quadrature formulas

$$f_n = \frac{1}{t_1}\sum_{m=1}^{t_1} F\left(\frac{2m-1}{2t_1}\right)\cos\frac{(2m-1)n\pi}{2t_1},$$

$$k_{nj} = \frac{1}{t_1 t_2}\sum_{m=1}^{t_1}\sum_{l=1}^{t_2} K\left(\frac{2m-1}{2t_1}, \frac{2l-1}{2t_2}\right)\phi_j\left(\frac{2l-1}{2t_2}\right)\cos\frac{(2m-1)n\pi}{2t_1}, \tag{4.7}$$

where t_1 and t_2 are the numbers of the Gauss nodes. An approximate solution

$$\phi^{(N)}(x) = \sum_{j=0}^{N-1} b_j^{(N)}\phi_j(x), \tag{4.8}$$

of the system (3.48) is found by the truncation method; the coefficients $b_j^{(N)}$ solve the system

$$-\frac{1}{2}b_{n-1}^{(N)} + \sum_{j=0}^{N-1} k_{nj}b_j^{(N)} = -f_n, \quad n = 1, 2, \dots, N. \tag{4.9}$$

To test the efficiency of the numerical scheme, we consider the case of a uniform load, $f(x) = P = \text{const}$ or, equivalently, the case $F(x) = Px$ (for computations, we select $P = 1$). Table 2 shows how the approximate solution $\phi^{(N)}(x)$ given by (4.8) depends on the numbers t_1 and t_2 of the Gauss nodes when $x = 0.5$, $\lambda = 0.5$, $F(x) = x$, and $N = 17$. In Table 3 we report the values $\phi^{(N)}(0.5)$ when $\lambda = 0.5$, $F(x) = x$, $t_1 = 200$, and $t_2 = 210$ for some values of the truncation parameter N. Again, as in the case of the characteristic equation, when N becomes greater than 20, then there is no improvement of the the accuracy of the numerical results, and the error of approximations grows when $N \geq 25$.

Figure 3 presents the computations for the displacement of the points on the upper crack face, the function $\phi(x)$, for some values of the parameter $\lambda = G_1/G_2$. It is seen that the crack opening is growing when the parameter λ is decreasing. Figure 4 shows that when $\beta = (\lambda - 1)(\lambda + 1)^{-1} \to -1$,

Table 2 Antiplane strain: the values of the function $\phi^{(N)}(x)$ at $x = 0.5$ for some numbers t_1 and t_2 of the Gauss nodes

	$t_1 = 10$	$t_1 = 50$		$t_1 = 100$	$t_1 = 300$
$t_2 = 11$	0.599598	0.0.600131	$t_2 = 110$	0.601067	0.601099
$t_2 = 51$	0.600073	0.600895	$t_2 = 310$	0.601091	0.601123

Table 3 Antiplane strain: the values of the function $\phi^{(N)}(x)$ at $x = 0.5$ for some values of N

	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 25$
$\phi^{(N)}(0.5)$	0.582829	0.601814	0.602258	0.601812	0.604167

that is when $\lambda \to 0$, the crack opening is growing to infinity. If $\beta \to 1$, then $\lambda \to \infty$, and the crack opening tends to a certain limit, a function $\phi_\infty(x)$.

5. Plane strain of a composite plane with a crack

In this section, we generalise the algorithm to the biharmonic case. As before, we consider a composite plane with a crack $0 < x < 1$, $y = 0^\pm$. The shear moduli and Poisson ratios of the half-planes $x < 0$ and $x > 1$ are the same, G_1 and ν_1, respectively, while the corresponding elastic constants for the strip $0 < x < 1$ are G_2 and ν_2. The faces of the crack are subjected to loading $\sigma_y = -f(x)$, $\tau_{xy} = 0$, $0 < x < 1$, $y = 0^\pm$. The normal displacement v is discontinuous across the crack faces, while the tangential displacement u is continuous. Denote

$$\phi(x) = v(x, 0^+) - v(x, 0^-), \quad \operatorname{supp} \phi(x) \subset [0, 1]. \tag{5.1}$$

Let $U(x, y)$ be the Airy function of the problem. Consider the conditions of plane strain. Then the function U solves the following discontinuous boundary value problem for the biharmonic operator:

$$\Delta^2 U(x, y) = 0, \quad |x| < \infty, \quad |y| < \infty, \quad x \neq 0, \quad x \neq 1,$$

$$\frac{\partial^j U}{\partial y^j}(x, 0^+) - \frac{\partial^j U}{\partial y^j}(x, 0^-) = 0,\ j = 0, 1, 2, \quad \frac{\partial^3 U}{\partial y^3}(x, 0^+) - \frac{\partial^3 U}{\partial y^3}(x, 0^-) = -\frac{2G_2}{1 - \nu_2}\phi''(x),\ |x| < \infty,$$

$$\frac{\partial^j U}{\partial x^j}(0^+, y) - \frac{\partial^j U}{\partial x^j}(0^-, y) = 0, \quad \frac{\partial^j U}{\partial x^j}(1^+, y) - \frac{\partial^j U}{\partial x^j}(1^-, y) = 0, \quad j = 0, 1, \quad |y| < \infty.$$

$$\frac{1 - \nu_1}{G_1}\frac{\partial^2 U}{\partial x^2}(0^-, y) - \frac{1 - \nu_2}{G_2}\frac{\partial^2 U}{\partial x^2}(0^+, y) = \left(\frac{\nu_1}{G_1} - \frac{\nu_2}{G_2}\right)\frac{\partial^2 U}{\partial y^2}(0, y), \quad |y| < \infty.$$

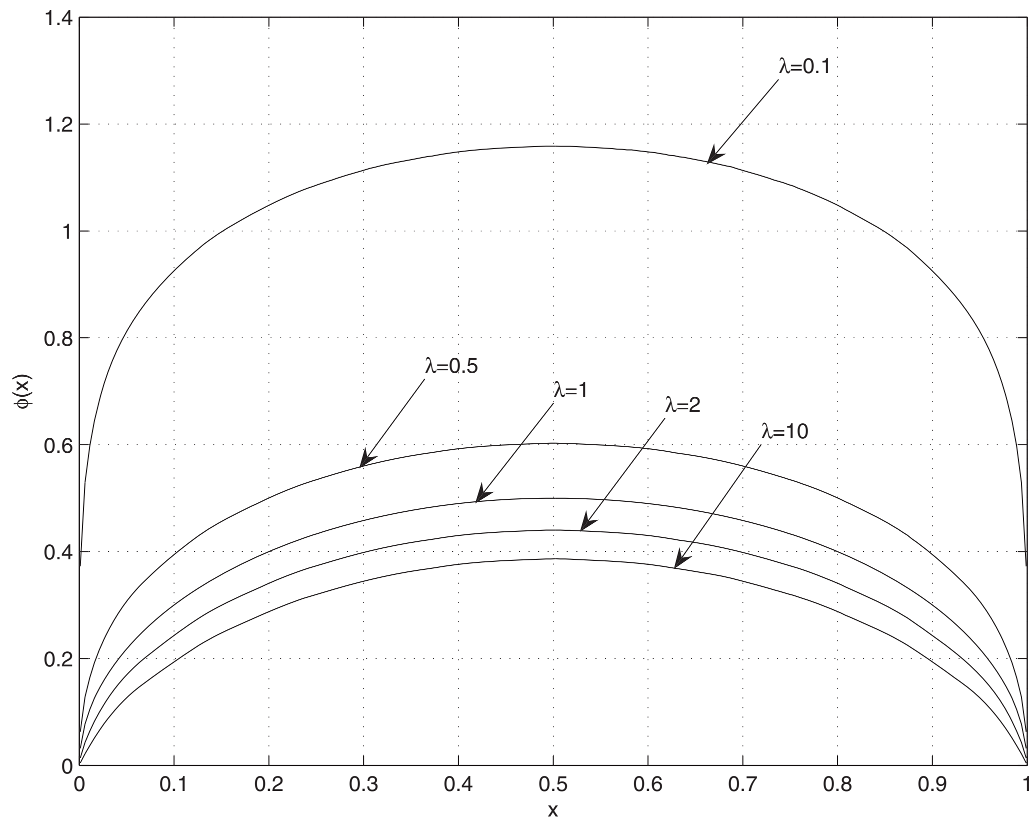

Fig. 3 Antiplane strain: the displacement function $\phi(x)$ for some values of the parameter $\lambda = G_1/G_2$

$$\frac{1-\nu_1}{G_1}\frac{\partial^3 U}{\partial x^3}(0^-, y) - \frac{1-\nu_2}{G_2}\frac{\partial^3 U}{\partial x^3}(0^+, y) = \left(\frac{\nu_1 - 2}{G_1} - \frac{\nu_2 - 2}{G_2}\right)\frac{\partial^3 U}{\partial x \partial y^2}(0, y), \quad |y| < \infty,$$

$$\frac{1-\nu_2}{G_2}\frac{\partial^2 U}{\partial x^2}(1^-, y) - \frac{1-\nu_1}{G_1}\frac{\partial^2 U}{\partial x^2}(1^+, y) = \left(\frac{\nu_2}{G_2} - \frac{\nu_1}{G_1}\right)\frac{\partial^2 U}{\partial y^2}(1, y), \quad |y| < \infty,$$

$$\frac{1-\nu_2}{G_2}\frac{\partial^3 U}{\partial x^3}(1^-, y) - \frac{1-\nu_1}{G_1}\frac{\partial^3 U}{\partial x^3}(1^+, y) = \left(\frac{\nu_2 - 2}{G_2} - \frac{\nu_1 - 2}{G_1}\right)\frac{\partial^3 U}{\partial x \partial y^2}(1, y), \quad |y| < \infty,$$

$$\frac{\partial^2 U}{\partial x^2}(x, 0^\pm) = -f(x), \quad 0 < x < 1. \tag{5.2}$$

It is natural to apply the Fourier transform with respect to y

$$U_\alpha(x) = \int_{-\infty}^{\infty} U(x, y) e^{i\alpha y} dy \tag{5.3}$$

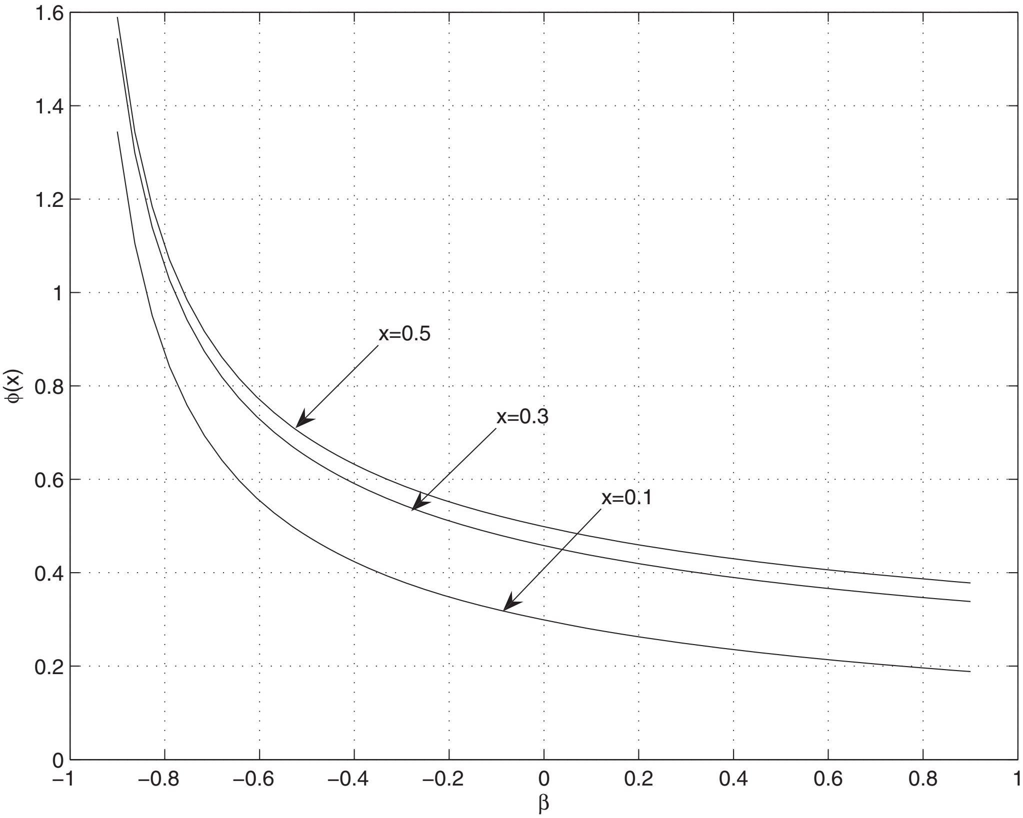

Fig. 4 Antiplane strain: the displacement function $\phi(x)$ as a function of $\beta = (G_1 - G_2)/(G_1 + G_2)$ for some values of x

and reduce the problem to the one-dimensional discontinuous boundary value problem

$$\left(\frac{d^4}{dx^4} - 2\alpha^2\frac{d^2}{dx^2} + \alpha^4\right)U_\alpha(x) = -\frac{2G_2}{1-\nu_2}\phi''(x), \quad |x| < \infty, \quad x \neq 0, \quad x \neq 1,$$

$$\frac{d^j}{dx^j}U_\alpha(0^-) = \frac{d^j}{dx^j}U_\alpha(0^+), \quad \frac{d^j}{dx^j}U_\alpha(1^-) = \frac{d^j}{dx^j}U_\alpha(1^+),$$

$$\frac{1-\nu_1}{G_1}\frac{d^{j+2}}{dx^{j+2}}U_\alpha(0^-) - \frac{1-\nu_2}{G_2}\frac{d^{j+2}}{dx^{j+2}}U_\alpha(0^+) + \alpha^2\lambda_j\frac{d^j}{dx^j}U_\alpha(0) = 0,$$

$$\frac{1-\nu_2}{G_2}\frac{d^{j+2}}{dx^{j+2}}U_\alpha(1^-) - \frac{1-\nu_1}{G_1}\frac{d^{j+2}}{dx^{j+2}}U_\alpha(1^+) - \alpha^2\lambda_j\frac{d^j}{dx^j}U^j_\alpha(1) = 0, \quad j = 0, 1, \tag{5.4}$$

where

$$\lambda_0 = \frac{\nu_1}{G_1} - \frac{\nu_2}{G_2}, \quad \lambda_1 = \frac{\nu_1 - 2}{G_1} - \frac{\nu_2 - 2}{G_2}. \tag{5.5}$$

Associated with the differential operator in (5.4) is the fundamental function

$$\Phi_\alpha(x,\xi)=\frac{1+|\alpha||x-\xi|}{4|\alpha|^3}e^{-|\alpha||x-\xi|}. \tag{5.6}$$

After integration by parts two times the general solution of the problem (5.4) becomes

$$U_\alpha(x)=\frac{G_2}{2(1-\nu_2)|\alpha|}\int_0^1(1-|\alpha||x-\xi|)e^{-|\alpha||x-\xi|}\phi(\xi)d\xi$$

$$+\begin{cases}(c_{00}+c_{01}x)e^{|\alpha|x}, & x<0,\\ (c_{10}+c_{11}x)\cosh|\alpha|x+(c_{12}+c_{13}x)\sinh|\alpha|x, & 0<x<1,\\ (c_{20}+c_{21}x)e^{-|\alpha|x}, & x>1.\end{cases} \tag{5.7}$$

The eight arbitrary constants in this solution are fixed by the eight conditions at the points $x=0$ and $x=1$ in (5.4). By inversion of the Fourier transform and satisfying the last condition in (5.2) equivalent to

$$\frac{d}{dx}\frac{1}{2\pi}\int_{-\infty}^{\infty}U'_\alpha(x)e^{-i\alpha y}d\alpha|_{y=0}=-f(x),\quad 0<x<1, \tag{5.8}$$

we eventually arrive at the following singular integral equation with two fixed singularities at the ends:

$$\frac{1}{\pi}\int_0^1\left[\frac{1}{\xi-x}+\frac{b_1\xi^2+b_2\xi x+b_3x^2}{(\xi+x)^3}+\frac{b_1(\xi-1)^2+b_2(\xi-1)(x-1)+b_3(x-1)^2}{(\xi+x-2)^3}\right.$$

$$\left.+K_0(x,\xi)\right]\phi(\xi)d\xi=-F(x)+C,\quad 0<x<1. \tag{5.9}$$

Here,

$$b_1=\frac{1}{\delta_0}[(\nu_0+\mu_0-1)^2-4(1-\mu_0^2)],\quad b_2=\frac{4}{\delta_0}[\nu_0(\nu_0-2)-3(1-\mu_0^2)],$$

$$b_3=\frac{1}{\delta_0}[-4\nu_0(\nu_0-2)+3(\nu_0+\mu_0-1)^2],\quad \delta_0=(3+\mu_0-\nu_0)(1+3\mu_0+\nu_0),$$

$$\mu_0=\frac{G_1(1-\nu_2)}{G_2(1-\nu_1)},\quad \nu_0=\frac{\nu_1}{1-\nu_1}-\mu_0\frac{\nu_2}{1-\nu_2}, \tag{5.10}$$

C is an arbitrary constant due to integration of (5.8),

$$F(x)=\frac{4(1-\nu_2)}{G_2}\int f(x)dx, \tag{5.11}$$

and the function $K_0(x,\xi)$ is a regular kernel whose representation is quite complicated and omitted.

Since the structure of the singular kernel in (5.9) is different from the one in the governing equation (4.4) for the antiplane problem, we cannot directly apply the algorithm of Section 3. To adjust the scheme to the plane case, we build up a new singular operator that generates a solution with the same singularities at the endpoints. Assume that $\phi(x)\sim Ax^\gamma$, $x\to 0^+$, A is a non-zero constant. Because the function $\phi(x)$ is the displacement jump of the normal displacement across the crack,

$\operatorname{Re}\gamma$ is positive. To deal with the largest class of functions possible, we assume $\operatorname{Re}\gamma \in (0, 1)$. Then we employ the formulas

$$\frac{1}{\pi}\int_0^1 \frac{\xi^\gamma d\xi}{\xi - x} = -\cot\pi\gamma x^\gamma + \omega_0(x), \quad x \to 0^+,$$

$$\frac{1}{\pi}\int_0^1 \frac{\xi^{\gamma+2} d\xi}{(\xi + x)^3} = -\frac{(\gamma+1)(\gamma+2)x^\gamma}{2\sin\pi\gamma} + \omega_1(x), \quad x \to 0^+, \tag{5.12}$$

where $\omega_j(x)$ are continuously differentiable functions in the segment $0 \le x < \varepsilon$ for some positive ε, and $\omega_j(0) \ne 0, j = 0, 1$. Analysis of the singular integrals in (5.9) as $x \to 0^+$ leads to the following transcendental equation for the parameter γ:

$$\Lambda(\gamma) \equiv \delta_0 \cos\pi\gamma - 2[\mu_0^2 - 3 - 2\mu_0(\nu_0 - 1) + \nu_0(\nu_0 - 2)]\gamma^2 - 4(1 - \mu_0^2) + (\nu_0 + \mu_0 - 1)^2 = 0. \tag{5.13}$$

It turns out that in the strip $0 < \operatorname{Re}\gamma < 1$ the function $\Lambda(\gamma)$ has one and only one zero, γ_0, and it is real. Its dependence on the parameter $\lambda = G_1/G_2$ for some values of the Poisson ratio ν_1 when $\nu_2 = 0.3$ is shown in Figure 5. It is seen that when $\nu_1 = \nu_2 = 0.3$ and $\lambda = 1$, that is when the plane is homogeneous, the parameter γ_0 is equal to $1/2$. Also, if the shear modulus of the internal strip is greater than that of the surrounding matrix ($\lambda < 1$) and $\nu_1 = \nu_2$, then $\gamma_0 < 1/2$, and $\gamma_0 \to 0$ as $\lambda \to 0$. On the other hand, if $\lambda \to \infty$, then the parameter $\gamma_0 \to \gamma_\infty$, where γ_∞ is independent of ν_1 and ν_2 and $\gamma_\infty \approx 0.7111773$.

Introduce now a new singular integral equation

$$\frac{1}{\pi}\int_0^1 \left[\frac{1}{\xi - x} + \frac{\beta}{\xi + x} + \frac{\beta}{x + \xi - 2}\right]\psi(\xi)d\xi = -F(x) + C, \quad 0 < x < 1, \quad \beta = -\cos\pi\gamma_0, \tag{5.14}$$

associated with the complete singular equation (5.9). Analysis of the singular integrals in (5.14) shows that the derivatives of the functions $\phi(x)$ and $\psi(x)$ have identical asymptotic representations at the ends. We have

$$\phi'(x) \sim \psi'(x) \sim C_0 x^{\gamma-1}, \quad x \to 0^+,$$

$$\phi'(x) \sim \psi'(x) \sim C_1(1 - x)^{\gamma-1}, \quad x \to 1^-, \tag{5.15}$$

where $C_j \ne 0, j = 0, 1$. Therefore it is natural to rewrite the integral equation (5.9) in the form

$$\frac{1}{\pi}\int_0^1 \left[\frac{1}{\xi - x} + \frac{\beta}{\xi + x} + \frac{\beta}{x + \xi - 2} + R(x, \xi)\right]\phi(\xi)d\xi = -F(x) + C, \quad 0 < x < 1, \tag{5.16}$$

where

$$R(x, \xi) = \frac{b_1\xi^2 + b_2\xi x + b_3 x^2}{(\xi + x)^3} + \frac{b_1(\xi - 1)^2 + b_2(\xi - 1)(x - 1) + b_3(x - 1)^2}{(\xi + x - 2)^3}$$

$$+\frac{\cos\pi\gamma_0}{\xi + x} + \frac{\cos\pi\gamma_0}{x + \xi - 2} + K_0(x, \xi), \tag{5.17}$$

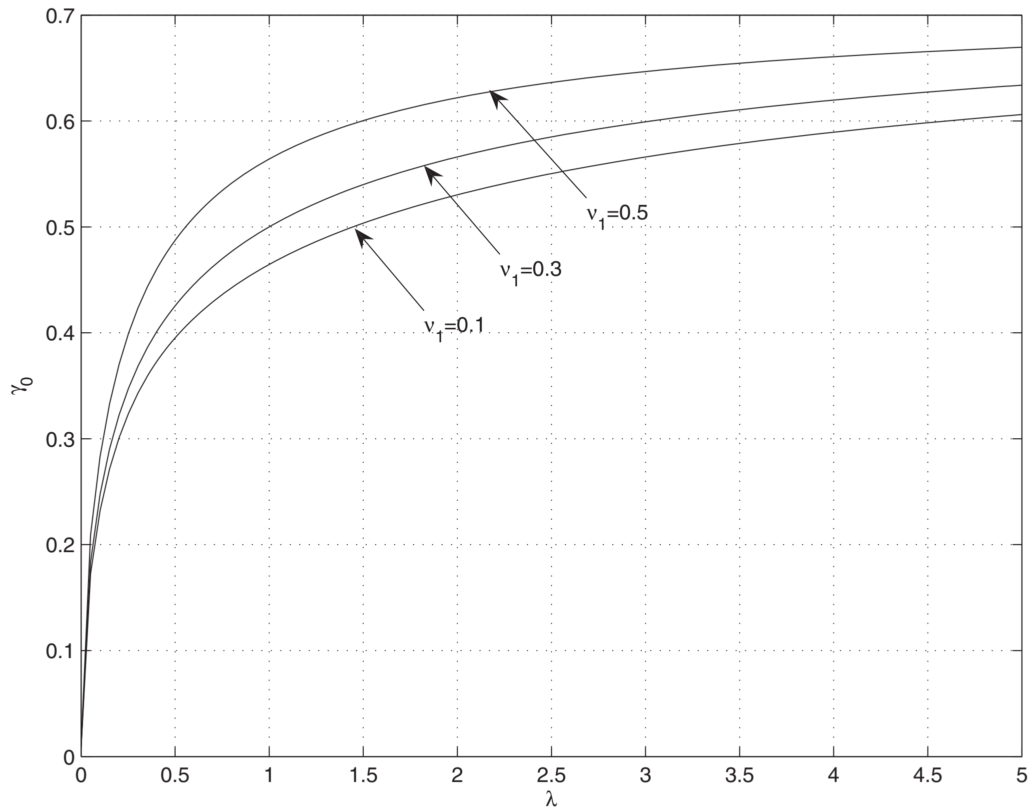

Fig. 5 The zero $\gamma_0 \in (0, 1)$ of (5.13) as a function of $\lambda = G_1/G_2$ for some values of ν_1 when $\nu_2 = 0.3$

which can be recast as (4.4) with the kernel (4.5), the function $R(x, \xi)$ given by (5.17) and the parameter $\beta = -\cos \pi \gamma_0$. An approximate solution of this equation is constructed in Section 3.

To verify the technique, we drop the regular part $K_0(x, \xi)$ and derive an approximate solution to the equation

$$\frac{1}{\pi}\int_0^1 \left[\frac{1}{\xi - x} + \frac{b_1\xi^2 + b_2\xi x + b_3x^2}{(\xi + x)^3} + \frac{b_1(\xi - 1)^2 + b_2(\xi - 1)(x - 1) + b_3(x - 1)^2}{(\xi + x - 2)^3}\right]\phi(\xi)d\xi =$$

$$-F(x) + C, \quad 0 < x < 1, \tag{5.18}$$

when $f(x) = P$ and $P = [4(1 - \nu_2)]^{-1}G_2$. In this case $F(x) = x$. The function $\phi(x)$ is plotted in Figure 6 for some values of the parameter $\lambda = G_1/G_2$ when $\nu_1 = \nu_2 = 0.3$. The algorithm provides a good accuracy for all values of λ. Our computations show (Table 4) that for small values of the parameter λ the same accuracy requires a larger dimension N of the truncated system (4.9).

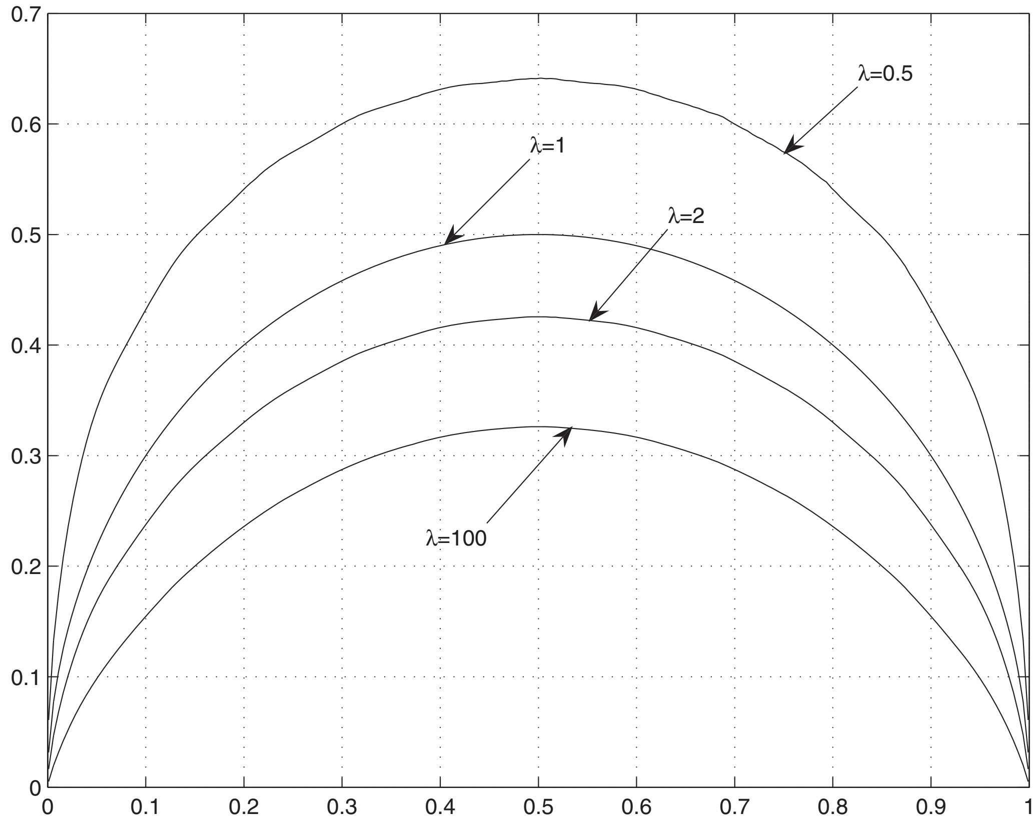

Fig. 6 Plane strain: the solution to the singular equation (5.18), the function $\phi_(x)$, for some values of the parameter $\lambda = G_1/G_2$ when $F(x) = x$, $\nu_1 = \nu_2 = 0.3$

Table 4 Plane strain: the values of the function $\Phi^{(N)}(x)$ at $x = 0.5$ for some values of N and λ when $\nu_1 = \nu_2 = 0.3$

	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 25$
$\lambda = 0.3$	0.744136	0.785573	0.787570	0.786676	0.782490
$\lambda = 0.5$	0.612515	0.640310	0.640704	0.639210	0.637523
$\lambda = 2$	0.414315	0.426399	0.425883	0.424791	0.424394
$\lambda = 100$	0.320104	0.326844	0.326405	0.325752	0.325526

6. Conclusions

We have analysed the singular integral equation $\mathcal{S}[\phi](x) = f(x)$, $0 < x < 1$, in the class of functions bounded at the ends. The singular operator $\mathcal{S}$ is defined by

$$\mathcal{S}[\phi](x) = \int_0^1 \left[\frac{1}{2}\cot\frac{\pi(\xi - x)}{2} + \frac{\beta}{2}\cot\frac{\pi(\xi + x)}{2}\right]\phi(\xi)d\xi, \tag{6.1}$$

and its kernel has fixed singularities at the points $x = \xi = 0$ and $x = \xi = 1$. By reducing it to a vector Riemann–Hilbert problem with a piece-wise constant matrix coefficient we have found that, in general, in the class of Hölder functions bounded at the ends, the solution does not exist. If a certain integral condition is satisfied, then the solution exists, it is unique and given by a quadrature. We have shown that in the case $|\beta| < 1$ the solution is monotonically decaying at the ends. If $\beta > 1$, the solution oscillates at the ends and vanishes. If $\beta < -1$, then there are two possibilities to derive a solution. Each one requires a certain condition of solvability and gives a solution that oscillates and does not vanish at one end and oscillates and vanishes at the second end.

We have managed to obtain a spectral relation $\mathcal{S}[\phi_j](x) = N_{j+1} - \cos[(j+1)\pi x]$, $0 < x < 1$, $j = 0, 1, \ldots$, for the singular operator $\mathcal{S}$. Here, the numbers N_j are given by (3.29), the functions $\phi_j(x)$ are

$$\phi_j(x) = \cos^{2\rho_1}\frac{\pi x}{2}\sin^{2(1-\rho_1)}\frac{\pi x}{2}q_j^{(\rho_1)}(x) + \cos^{2(1-\rho_1)}\frac{\pi x}{2}\sin^{2\rho_1}\frac{\pi x}{2}q_j^{(1-\rho_1)}(x), \tag{6.2}$$

with $q_j^{(\alpha)}(x)$ being the degree-j trigonometric polynomials

$$q_j^{(\alpha)}(x) = \sum_{\nu=0}^{j} c_{j\nu}^{(\alpha)}\left(\frac{1-\cos\pi jx}{2}\right)^{\nu}, \quad c_{j\nu}^{(\alpha)} = \frac{1}{2\sin\pi\alpha}\sum_{m=\nu+1}^{j+1}\frac{(-j-1)_m(j+1)_m(\alpha)_{m-1-\nu}}{(1/2)_m m!(m-1-\nu)!}. \tag{6.3}$$

This spectral relation has been used as the key step in the approximate scheme for the complete singular integral equation with two fixed singularities. On expanding the unknown function in terms of the functions $\phi_j(x)$, we have reduced the integral equation to an infinite system of linear algebraic equations of the second kind and solved it by the reduction method.

The method has been applied to the antiplane problem for a finite crack in a composite plane when the crack is orthogonal to the interfaces between a strip and two half-planes. The crack lies in the strip, and its tips fall in the interfaces. The shear moduli are the same for the half-planes, while the shear modulus of the strip is different. The problem is governed by a complete singular integral equation with two fixed singularities, and the generalised method of orthogonal polynomials proposed has been applied. The numerical algorithm has been successfully tested; it has a good accuracy and it is rapidly convergent. We have further modified the method to adjust it to the solution of the singular integral equation with two fixed singularities arising in biharmonic problems. We have derived the governing singular integral equation for the plane strain problem with the same geometry as in the antiplane case. We have shown that if the singularities of the solution at the endpoints are real, then it is possible to replace the singular operator associated with the plane problem by a simpler operator which satisfies the spectral relation used in the antiplane case. A numerical test for the dominant singular integral equation associated with the plane strain problem has been implemented. A good accuracy and fast convergence of the algorithm has been achieved.

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A. Proof of Theorem 2.1

To prove the first part of Theorem 2.1, we follow (**6**) and introduce a vector function

$$\boldsymbol{\Psi}(z) = \begin{pmatrix} \varphi_1(z) \\ \varphi_2(z) \end{pmatrix} \tag{A.1}$$

analytic in the semi-disc D and Hölder continuous up to the boundary $L \cup (-1, 1)$. To extend its definition into the whole plane, we continue analytically this vector first into the lower half-disc by the relation

$$\boldsymbol{\Psi}(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\Psi(\bar{z})}, \quad \operatorname{Im} z < 0, \quad |z| < 1, \tag{A.2}$$

and then into the exterior of the disc $|z| < 1$ by the law

$$\boldsymbol{\Psi}(z) = -\overline{\boldsymbol{\Psi}\left(\frac{1}{\bar{z}}\right)}, \quad |z| > 1. \tag{A.3}$$

Notice that then the vectors

$$\boldsymbol{\Psi}^{+}(t) = \lim_{z \to t \in L, z \in D} \boldsymbol{\Psi}(z), \quad \boldsymbol{\Psi}^{-}(t) = -\lim_{z \to t \in L, z \in D} \overline{\boldsymbol{\Psi}(\overline{z^{-1}})}, \tag{A.4}$$

admit analytic continuation from the contour L into the domains D and $\{|z| > 1, \operatorname{Im} z > 0\}$, respectively.

Likewise, the vectors

$$\mathbf{\Psi}^+(t) = \lim_{z\to t\in(\infty,+\infty), z\in\mathbb{C}^+} \Psi(z), \quad \mathbf{\Psi}^-(t) = \lim_{z\to t\in(\infty,+\infty), z\in\mathbb{C}^+} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\mathbf{\Psi}(\overline{z})} \tag{A.5}$$

admit analytic continuation from the real axis into the upper and lower half-planes, $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. We now invoke the boundary conditions (2.5) to derive a Riemann–Hilbert problem for the vector $\Psi(z)$ with a piecewise constant matrix coefficient. Its boundary condition reads

$$\mathbf{\Psi}^+(t) = A(t)\mathbf{\Psi}^-(t) + \mathbf{b}(t), \quad t \in L \cup (-\infty, +\infty), \tag{A.6}$$

where

$$A(t) = \begin{cases} A_0, & t \in (-\infty, +\infty), \\ I, & t \in L, \end{cases} \quad A_0 = \begin{pmatrix} 1-\beta & \beta \\ -\beta & 1+\beta \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{b}(t) = \begin{cases} \mathbf{0}, & t \in (-\infty, +\infty), \\ \mathbf{b}_0(t), & t \in L, \end{cases} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_0(t) = \begin{pmatrix} 0 \\ 2u(t) \end{pmatrix}. \tag{A.7}$$

The matrix $A(t)$ can be factorised as

$$A(t) = X^+(t)[X^-(t)]^{-1}, \quad t \in L \cup (-\infty, +\infty), \tag{A.8}$$

where

$$X(z) = \begin{pmatrix} 1-\gamma(z) & \gamma(z) \\ -\gamma(z) & 1+\gamma(z) \end{pmatrix}. \tag{A.9}$$

The function $\gamma(z)$ solves the problem

$$\gamma^+(t) - \gamma^-(t) = \beta, \quad t \in L \cup (-\infty, +\infty), \tag{A.10}$$

and has the form

$$\gamma(z) = \frac{\beta}{2\pi i}[\log z - \log(-z)], \quad -\pi \le \arg z \le \pi. \tag{A.11}$$

Because of the following properties of the function $\gamma(z)$:

$$\gamma^\pm(t) = \pm\frac{\beta}{2}, \quad t \in (-\infty, +\infty), \qquad \gamma^+(t) = \gamma^-(t), \quad t \in L,$$

$$\gamma(z) = -\overline{\gamma(\overline{z})}, \quad \operatorname{Im} z < 0, \qquad \gamma(z) = \overline{\gamma\left(\frac{1}{\overline{z}}\right)}, \quad |z| > 1, \tag{A.12}$$

the general solution of the vector Riemann–Hilbert problem (A.6) in the class of symmetric functions (A.2) and (A.3) has the form

$$\varphi_j(z) = \frac{1}{2\pi i}\int_L W_j(z,\tau)\frac{u(\tau)d\tau}{\tau} + ic[(-1)^j + 2\gamma(z)], \quad j = 1, 2, \tag{A.13}$$

where c is an arbitrary real constant and

$$W_1(z,\tau) = [\gamma(z) - \gamma(\tau)]\frac{\tau + z}{\tau - z} + [1 - \gamma(z) - \gamma(\tau)]\frac{1 + \tau z}{1 - \tau z},$$

$$W_2(z,\tau) = [1 + \gamma(z) - \gamma(\tau)]\frac{\tau + z}{\tau - z} - [\gamma(z) + \gamma(\tau)]\frac{1 + \tau z}{1 - \tau z}. \tag{A.14}$$

It is directly verified that

$$\overline{W_{3-j}(\bar{z},\tau)} = W_j(z,\tau), \quad \operatorname{Im} z < 0,$$

$$\overline{W_j\left(\frac{1}{\bar{z}},\tau\right)} = -W_j(z,\tau), \quad |z| > 1, \quad \tau \in L, \quad j = 1, 2, \tag{A.15}$$

and therefore the conditions (A.2) and (A.3) are fulfilled.

Next we determine $\operatorname{Im}\varphi_2(t)$, $t \in L$, from (A.13)

$$\operatorname{Im}\varphi_2(t) = c[1 + 2\gamma(t)] - \frac{1}{2\pi}\int_L \left[\frac{\tau + t}{\tau - t} - \frac{\beta(1 + \tau t)}{1 - \tau t}\right]\frac{u(\tau)d\tau}{\tau}, \quad t \in L. \tag{A.16}$$

Henceforth, the function $u(\tau) = \operatorname{Re}\varphi_2(\tau)$ solves the integral equation (2.4) if $c = 0$ and $\operatorname{Im}\varphi_2(t) = -v(t)$. Finally notice that since

$$\lim_{z\to 0}[\varphi_1(z) - \varphi_2(z)] = -2ic, \tag{A.17}$$

the constant c vanishes if the functions in (A.13) meet the condition (2.6).

The inverse statement of Theorem 2.1 is proved on the basis of the Sokhotski-Plemelj formulas and the representations (2.8).

B. Cases $\beta = \pm 1$

If $\beta = \pm 1$, then we extend the definition of the functions $\phi(x)$ and $f(x)$ to the interval $(-1, 0]$ by the relations

$$\phi(x) = -\beta\phi(-x), \quad f(x) = \beta f(-x), \quad -1 < x \le 0, \tag{B.1}$$

and convert (2.1) into the equation

$$\frac{1}{2}\int_{-1}^{1} \cot\frac{\pi(\xi - x)}{2}\phi(\xi)d\xi = f(x), \quad -1 < x < 1. \tag{B.2}$$

By making the substitutions $\sigma = \pi(\xi + 1)$ and $s = \pi(x + 1)$ and denoting $\phi(\xi) = \hat{\phi}(\sigma)$ and $f(x) = \hat{f}(s)$ we arrive at the equation

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\sigma - s}{2}\hat{\phi}(\sigma)d\sigma = \hat{f}(s), \quad 0 < s < 2\pi. \tag{B.3}$$

It is known (**7**), pp. 44, 244 that it is solvable if and only if

$$\int_0^{2\pi} \hat{f}(s)ds = 0, \tag{B.4}$$

and its solution is given by

$$\hat{\phi}(s) = -\frac{1}{2\pi}\int_0^{2\pi} \hat{f}(\sigma)\cot\frac{\sigma - s}{2}d\sigma + C, \quad C = \frac{1}{2\pi}\int_0^{2\pi}\hat{\phi}(\sigma)d\sigma. \tag{B.5}$$

Therefore, in the case $\beta = 1$, the condition

$$\int_0^1 f(\xi)d\xi = 0 \tag{B.6}$$

is necessary and sufficient for (2.1) to be solvable. If it is satisfied, the solution is unique and has the form

$$\phi(x) = -\frac{1}{2}\int_0^1 \left[\cot\frac{\pi(\xi - x)}{2} - \cot\frac{\pi(\xi + x)}{2}\right] f(\xi)d\xi, \quad 0 < x < 1. \tag{B.7}$$

In the case $\beta = -1$, the solution always exists, and its solution is defined up to an arbitrary constant

$$\phi(x) = -\frac{1}{2}\int_0^1 \left[\cot\frac{\pi(\xi - x)}{2} + \cot\frac{\pi(\xi + x)}{2}\right] f(\xi)d\xi + C, \quad 0 < x < 1. \tag{B.8}$$

C. Coefficients M_j

To evaluate the coefficients

$$M_j = \int_0^1 \left(\tan^{2\rho_1 - 1}\frac{\pi x}{2} + \cot^{2\rho_1 - 1}\frac{\pi x}{2}\right) T_j(\cos \pi x)dx, \quad j = 0, 1, \ldots, \tag{C.1}$$

we make the substitution $\zeta = \cos \pi x$ and rewrite this expression as

$$M_j = \frac{1 + (-1)^j}{\pi}\int_{-1}^1 (1 - \zeta)^{\rho_1 - 1}(1 + \zeta)^{-\rho_1} T_j(\zeta)d\zeta. \tag{C.2}$$

It follows immediately that $M_1 = M_3 = \ldots = 0$. To compute the integral in the even case, $j = 2m$, we consider the integral

$$\int_{-1}^1 (1 - \zeta)^{\alpha_1}(1 + \zeta)^{\alpha_2} T_j(\zeta)d\zeta, \tag{C.3}$$

make the substitution $\zeta = 2t - 1$ and express the Chebyshev polynomials of the first kind through the hypergeometric function

$$T_j(2t - 1) = F(-j, j; 1/2; 1 - t). \tag{C.4}$$

Then we change the order of integration and summation, evaluate the new integrals in terms of the Γ-functions and have

$$\int_{-1}^1 (1 - \zeta)^{\alpha_1}(1 + \zeta)^{\alpha_2} T_j(\zeta)d\zeta = \frac{2^{\alpha_1 + \alpha_2 + 1}\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_1 + \alpha_2 + 2)}$$

$$\times {}_3F_2\left(-j, j, \alpha_1 + 1; 1/2, \alpha_1 + \alpha_2 + 2; 1\right), \quad \mathrm{Re}\,\alpha_1 > -1, \quad \mathrm{Re}\,\alpha_2 > -1. \tag{C.5}$$

Alternatively, this result can be derived from formula 16.4 (3) in (**9**).

Note that the formulas 16.1 (2) in (**9**) and 7.347 (1) in (**10**) corresponding to (C.5) have the same error. In addition, formulas 16.1 (1), 16.1 (21) and 16.1 (22) in (**9**) and the corresponding to 16.1 (22) formula 7.347 (2) in (**10**) need also to be corrected. They should read, respectively,

$$\int_{-1}^{1}(1-\zeta)^{-1/2}(1+\zeta)^{\alpha}T_j(\zeta)d\zeta = \frac{2^{\alpha+1/2}\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\alpha+3/2)}{\Gamma(\alpha+3/2+j)\Gamma(\alpha+3/2-j)}.$$

$$\int_{-1}^{1}(1-\zeta)^{1/2}(1+\zeta)^{\alpha}U_j(\zeta)d\zeta = \frac{2^{\alpha+1/2}\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\alpha+1/2)(j+1)}{\Gamma(\alpha+5/2+j)\Gamma(\alpha+1/2-j)},$$

$$\int_{-1}^{1}(1-\zeta)^{\alpha_1}(1+\zeta)^{\alpha_2}U_j(\zeta)d\zeta = \frac{2^{\alpha_1+\alpha_2+1}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)(j+1)}{\Gamma(\alpha_1+\alpha_2+2)}$$

$$\times {}_3F_2\left(-j, j+2, \alpha_1+1; 3/2, \alpha_1+\alpha_2+2; 1\right), \quad \operatorname{Re}\alpha_1 > -1, \quad \operatorname{Re}\alpha_2 > -1. \tag{C.6}$$

Here, $U_j(\zeta)$ are the Chebyshev polynomials of the second kind. On employing now formulas (C.2) and (C.5) we obtain that M_{2m} is a finite sum given by

$$M_{2m} = \frac{2}{\sin\pi\rho_1}\sum_{j=0}^{2m}\frac{(-2m)_j(2m)_j(\rho_1)_j}{(1/2)_j(j!)^2}, \quad m = 0, 1, \ldots, \tag{C.7}$$

and, in particular, $M_0 = 2\csc\pi\rho_1$.

D. Complete singular integral equation with the Cauchy kernel

We aim to show that it is not surprising that the solvability condition of (3.1) coincides with the first equation (3.47) of the infinite system of algebraic equations (3.45). The same result can be derived for the classical equation with the Cauchy kernel

$$\frac{1}{\pi}\int_0^1\left[\frac{1}{\xi-x}+K(x,\xi)\right]\phi(\xi)d\xi = C - F(x), \quad 0 < x < 1, \tag{D.1}$$

in the class of Hölder functions bounded at the endpoints. It is known (**7**) that this equation is solvable if and only if the constant C is chosen to be

$$C = \frac{1}{\pi}\int_0^1\frac{F(x)dx}{\sqrt{x(1-x)}} + \frac{1}{\pi^2}\int_0^1\int_0^1\frac{K(x,\xi)\phi(\xi)d\xi dx}{\sqrt{x(1-x)}}. \tag{D.2}$$

Then the characteristic equation ($K(x,\xi)\equiv 0$), $\hat{S}[\phi](x) = C - F(x)$, $0 < x < 1$, in the class of functions chosen, admits the unique solution $\phi(x) = -\hat{\mathcal{S}}^{-1}[F](x)$, where the inverse operator is given by

$$\hat{\mathcal{S}}^{-1}[F](x) = -\frac{\sqrt{x(1-x)}}{\pi}\int_0^1\frac{F(\xi)d\xi}{\sqrt{\xi(1-\xi)}(\xi-x)}, \quad 0 < x < 1, \tag{D.3}$$

and $\hat{\mathcal{S}}^{-1}[1] = 0$, $0 < x < 1$ (recall that we derived the same result (3.33) for the operator $\mathcal{S}^{-1}$).

Now, on returning to the complete equation with the Cauchy kernel, we expand the unknown function $\phi(x)$ as

$$\phi(x) = \sqrt{x(1-x)} \sum_{j=0}^{\infty} b_j U_j(2x-1) \tag{D.4}$$

and employ the spectral relation for the Chebyshev polynomials of the second kind

$$\int_0^1 \frac{\sqrt{\xi(1-\xi)}}{\xi - x} U_j(2\xi - 1) d\xi = -\frac{\pi}{2} T_{j+1}(2x-1), \quad 0 < x < 1, \quad j = 0, 1, \ldots. \tag{D.5}$$

Then because of the orthogonality of the Chebyshev polynomials, we reduce the integral equation (D.1) to the following system of linear algebraic equations:

$$\sum_{j=0}^{\infty} k_{0j} b_j = C - f_0 \tag{D.6}$$

and

$$-\frac{b_n}{4} + \sum_{j=0}^{\infty} k_{nj} b_j = -f_n, \quad n = 1, 2, \ldots. \tag{D.7}$$

Here,

$$k_{nj} = \frac{1}{\pi^2} \int_0^1 \int_0^1 K(x, \xi) \sqrt{\xi(1-\xi)} U_j(2\xi - 1) \frac{T_n(2x-1) d\xi \, dx}{\sqrt{x(1-x)}},$$

$$f_n = \frac{1}{\pi} \int_0^1 \frac{F(x) T_n(2x-1) dx}{\sqrt{x(1-x)}}. \tag{D.8}$$

It becomes evident, upon substituting the series (D.4) into the solvability condition (D.2) and using the notations (D.8), that (D.2) and (D.6) are equivalent.