A CRACK INDUCED BY A THIN RIGID INCLUSION PARTLY DEBONDED FROM THE MATRIX

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Summary

The interaction of a thin rigid inclusion with a finite crack is studied. Two plane problems of elasticity are considered. The first one concerns the case when the upper side of the inclusion of length $2b$ is completely debonded from the matrix, and the crack formed symmetrically penetrates into the medium; its length is $2a > 2b$. In the second model, the upper side of the inclusion is partly separated from the matrix, and $a < b$. It is shown that both problems are governed by singular integral equations that share the same kernel but have different right-hand sides, and their solution satisfy different additional conditions. Derivation of a closed-form solution to these integral equations is one of the main results of the article. The solution is found by reducing the integral equation to a vector Riemann–Hilbert problem with the Chebotarev–Khrapkov matrix coefficient. A feature of the method proposed is that the vector Riemann–Hilbert problem is set on a finite segment, while the original Khrapkov method of matrix factorization is developed for a closed contour. In the case, when the crack and inclusion lengths are the same, the solution is derived by passing to the limit $b/a \to 1$. It is demonstrated that the limiting case $a = b$ is unstable, and when $a < b$ and the crack tips approach the inclusion ends, the crack tends to accelerate in order to penetrate into the matrix. It is shown that the stresses and the tangential derivative of the displacement have the square root singularity at the crack and the inclusion tips. The stresses and the displacement derivative are monotonic at the external singular points, $\pm a$ and $\pm b$ for Models 1 and 2, and oscillate in small neighborhoods of the internal singular points, $|x/a - b/a| < \varepsilon$ and $|x/b - a/b| < \varepsilon$, for the first and second problems, respectively, and $0 < \varepsilon \leq 10^{-6}$. The potential energy release rate and the Griffith crack growth criterion are established for both models.

1. Introduction

Different aspects of interaction of cracks with inclusions have been studied in detail by many investigators. The fundamental two-dimensional elastic model problem for a finite set of slits lying on the real axis when the traction components and the displacements are prescribed in the upper and lower surfaces, respectively, was solved in (**1**) by the method of singular integral equations and in (**2**) by the method of complex potentials. Some generalizations of this setting and the method of the vector Riemann–Hilbert problem were proposed and the interaction of a semi-infinite inclusion

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and a finite cut was analyzed in (3). The model (1) for an inclusion whose upper side is completely separated from the matrix was generalized in (4) for the case when an inclusion is in the interface between two half-planes with different elastic constants. The method of complex potentials and the scalar Riemann–Hilbert problem on a hyperelliptic surface was advanced (5) for the mixed boundary value problem governing a system of collinear cuts when the points of change of boundary conditions are not necessarily the endpoints of the cuts. This technique was applied (6) for the solution of the plane problem for n inclusions whose upper sides are completely separated from the matrix, and slip lines emerge from the crack tips. The case of a rigid inclusion $-a < x < a, y = 0$ with a crack $-a < x < a$ on the inclusion upper side open in the interval $-b < x < b$ and contacting with the inclusion when $b < |x| < a$, $\sigma_{12} = 0$, $u_2 = 0$, was treated in (7). The governing system of four singular integral equations was converted into a single singular integral equation, reduced to a vector Riemann–Hilbert problem with the entries of the matrix coefficient being meromorphic and almost periodic functions. The Riemann–Hilbert problem was solved analytically in terms of some series with the exponential rate of convergence. It was shown that the solution was monotonic in neighborhoods of singular points. A closed-form solution to the plane problem on a rigid inclusion $-b < x < b$ located on the lower side of a crack $-a < x < a, y = 0$ $(b < a)$ when the tangential traction component vanishes on the segment $-b < x < b, y = 0^-$) (frictionless contact of the inclusion and the matrix) was found in (8). To model debonding of a rigid rod interacting with an elastic half-space, three mixed boundary value problems of plane elasticity were analyzed in (9). The first problem concerns the interior of an elastic quarter-plane, while the other two were set for the exterior of a rigid quarter-plane. The boundary conditions change their type in one side of the wedge and do not in the other side. The Mellin transform was employed (9) to reduce the problems to vector Wiener–Hopf functional equations in an infinite line (the Bromwich contour), which were solved by the Khrapkov factorization method (10).

Some other relevant works include (11, 12). In the former paper, the problem of the symmetric indentation of a penny-shaped crack by a smoothly embedded rigid circular thin disc was treated by the method of triple integral equations. The second work examines models of a crack inside, outside, penetrating or lying along the interface of an anisotropic elliptical inclusion and presents numerical solutions of the governing singular integral equations.

The main goal of this article was to develop a method of the vector Riemann–Hilbert on a segment for the singular integral equation

$$\frac{1}{\pi}\int_{-k}^{k}\left(1+\sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\omega(\tau)d\tau}{\tau-t}-i(1-\nu)\omega(t)=g(t), \quad -k<t<k, \quad 0<k<1, \tag{1.1}$$

whose solution is subjected to the condition

$$\int_{-k}^{k}\omega(t)dt=Q. \tag{1.2}$$

This equation governs the following two model problems. Problem 1 concerns a rigid inclusion $-b < x < b$, $y = 0$, whose lower side is bonded to the matrix, while the upper side is completely separated from the elastic medium, and the crack formed on the upper surface penetrates into the medium and occupies the segment $-a < x < a$, $y = 0$ $(a > b)$. The latter is on a rigid inclusion $-b < x < b$, whose upper part is debonded on the segment $-a < x < a$, and $a < b$. In particular, on passing to the limit $b/a \to 1$ we aim to examine the transition of the solution to the first problem

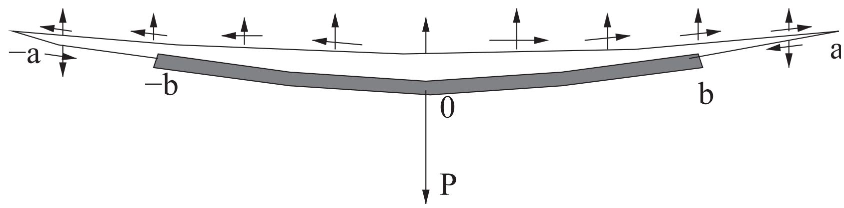

Fig. 1 The geometry of Model 1

with the square root singularities at the crack tips to the solution for the case $a = b$ that oscillates and has a stronger singularity of order $-3/4$ at the endpoints.

In section 2, we formulate the first problem as a discontinuous boundary value problem for the biharmonic operator. It is reduced to a singular integral equation in section 3. Then, in section 4, we transform the integral equation to a vector Riemann–Hilbert problem in a segment with the Chebotarev–Khrapkov matrix coefficient (**10**, **13**) and construct the solution of the Riemann–Hilbert problem. Based on this solution, we derive a closed-form solution of the singular integral equation in section 5. In the next section, we analyze the singularities of the solution, discuss the numerical results obtained and find the solution in the limiting case $b/a \to 0$. In section 7, we show that Problem 2 is governed by a singular integral equation whose kernel is the same as the one for the first problem. What is different is the right-hand side, the additional condition and the meaning of the parameters and the unknown function. We obtain an exact solution to the integral equation, implement the passage to the limit $a \to b$ and show that in a particular case the limiting solution coincides with the solution available in the literature (**2**). In section 8, for both models, we derive the energy release rate and the Griffith crack growth criterion.

2. Model 1: formulation

The problem to be addressed is a two-dimensional one for a thin rigid inclusion whose profile is described by a function $y = h(x)$, $-b < x < b$. The function $h(x)$ is even, $h(\pm b) = 0$, and the curvature of $h(x)$, ϵ, is small, $\epsilon << 1$. The medium is assumed to be elastic, uniform, infinite and to be loaded in a way that would generate, in the absence of the inclusion, a stress field $\sigma^\circ_{ij}(x, y)$ symmetric with respect to both axes, x and y. The Poisson ratio and the Young modulus of the medium are ν and E, respectively, and the conditions of plane stress are considered. It is assumed that along the upper side of the inclusion (Fig. 1) there is a crack that spreads not only over the whole upper surface of the inclusion, but also penetrates into the matrix. We aim to analyze the stress concentration in the vicinity of the tips of the inclusion and the crack.

Owing to small deviations of the function $h(x)$ from the x-axis, it is conventional in linear elasticity to write the boundary conditions on the line $y = 0$, not the actual curve $y = h(x)$. They read

$$\sigma_{12} + i\sigma_{22} = -\sigma^\circ_{12} - i\sigma^\circ_{22}, \quad |x| < a, \quad y = 0^+,$$

$$\sigma_{12} + i\sigma_{22} = -\sigma^\circ_{12} - i\sigma^\circ_{22}, \quad b < |x| < a, \quad y = 0^-,$$

$$\frac{\partial u_1}{\partial x} + i\frac{\partial u_2}{\partial x} = ih'(x) - w^\circ(x), \quad |x| < b, \quad y = 0^-. \tag{2.1}$$

Here, (u_1, u_2) is the displacement vector, $w^\circ(x) = \frac{\partial}{\partial x}(u_1^\circ + iu_2^\circ)(x, 0)$, and (u_1°, u_2°) is the displacement vector associated with the stress field σ_{ij}°.

It will be convenient to introduce the functions

$$\phi_j(x) = E\left[\frac{\partial u_j}{\partial x}\right](x), \quad \psi_j(x) = [\sigma_{j2}](x), \quad j = 1, 2, \tag{2.2}$$

where

$$[f](x) = f(x, 0^+) - f(x, 0^-). \tag{2.3}$$

Since the displacements are discontinuous across the whole crack surface and the traction components are discontinuous across the inclusion surface and continuous across the segments $-a < x < -b$ and $b < x < a$, we have $\operatorname{supp} \phi_j \subset [-a, a]$ and $\operatorname{supp} \psi_j \subset [-b, b]$, $j = 1, 2$.

Let $U(x, y)$ be the Airy function of the problem. Then

$$\sigma_{12} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial x^2},$$

$$\frac{\partial u_1}{\partial x} = \frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial x^2}\right), \quad \frac{\partial u_2}{\partial y} = \frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right),$$

$$\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = -\frac{2(1+\nu)}{E}\frac{\partial^2 U}{\partial x \partial y}, \tag{2.4}$$

and the model problem is equivalent to the following discontinuous boundary value problem for the biharmonic operator:

$$\Delta^2 U(x, y) = 0, \quad (x, y) \in \mathbb{R}^2 \setminus \{-a < x < a, y = 0\},$$

$$[U](x) = \tilde{\psi}_2(x), \quad \left[\frac{\partial U}{\partial y}\right](x) = -\hat{\psi}_1(x),$$

$$\left[\frac{\partial^2 U}{\partial y^2}\right](x) = \phi_1(x) + \nu\psi_2(x), \quad \left[\frac{\partial^3 U}{\partial y^3}\right](x) = -\phi_2'(x) + (\nu + 2)\psi_1'(x), \tag{2.5}$$

subject to the boundary conditions (2.1), the inclusion equilibrium conditions

$$\int_{-b}^{b} \psi_1(x)dx = 0, \quad \int_{-b}^{b} \psi_2(x)dx = \Sigma + P, \tag{2.6}$$

and the crack closedness conditions

$$\int_{-a}^{a} \phi_1(x)dx = 0, \quad \int_{-a}^{a} \phi_2(x)dx = 0. \tag{2.7}$$

Here, $\hat{\psi}_1'(x) = \psi_1(x)$ and $\tilde{\psi}_2''(x) = \psi_2(x)$,

$$P = -\int_{-b}^{b} \sigma_{22}(x, 0^-)dx, \quad \Sigma = -\int_{-b}^{b} \sigma_{22}^\circ(x, 0^+)dx, \quad j = 1, 2, \tag{2.8}$$

P is the magnitude of the total normal force applied to the inclusion central point and directed downwards.

3. Singular integral equation

Aiming to reduce the discontinuous boundary value problem to an integral equation we apply first the Fourier transform across the discontinuity line

$$\hat{U}_\alpha(x) = \int_{-\infty}^{\infty} U(x, y)e^{i\alpha y}dy \tag{3.1}$$

and deduce the following ordinary differential equation for the Fourier transform $\hat{U}_\alpha(x)$:

$$\left(\frac{d^4}{dx^4} - 2\alpha^2\frac{d^2}{dx^2} + \alpha^4\right)\hat{U}_\alpha(x)$$

$$= -i\alpha\phi_1(x) - \phi_2'(x) + \nu\psi_1'(x) + \alpha^2\hat{\psi}_1(x) - i\alpha(2+\nu)\psi_2(x) + i\alpha^3\tilde{\psi}_2(x), \quad -\infty < x < \infty. \tag{3.2}$$

We next utilize the fundamental function of the differential operator in (3.2):

$$\frac{1+|\alpha||x-\xi|}{4|\alpha|^3}e^{-|\alpha||x-\xi|} \tag{3.3}$$

and integration by parts to derive

$$\hat{U}_\alpha(x) = \frac{1}{4\alpha^2}\int_{-a}^{a}\{-i\,\mathrm{sgn}\,\alpha(1+|\alpha||x-\xi|)\phi_1(\xi) + |\alpha|(x-\xi)\phi_2(\xi)$$

$$-[2\,\mathrm{sgn}(x-\xi) + (1+\nu)|\alpha|(x-\xi)]\psi_1(\xi) + i\,\mathrm{sgn}\,\alpha[1-\nu-(1+\nu)|\alpha||x-\xi|]\psi_2(\xi)\}e^{-|\alpha||x-\xi|}d\xi. \tag{3.4}$$

On inverting the Fourier transform, using formulas (2.4) and evaluating the corresponding integrals, we obtain the integral representations of the stresses

$$\sigma_{12}(x,y) = \int_{-a}^{a}\{-M_1(x-\xi,y)\phi_1(\xi) + [N_2(x-\xi,y) - M_2(x-\xi,y)]\phi_2(\xi)\}d\xi$$

$$+\int_{-b}^{b}\{[(1-\nu)N_2(x-\xi,y) + (1+\nu)M_2(x-\xi,y)]\psi_1(\xi) + [N_1(x-\xi,y)$$

$$-(1+\nu)M_1(x-\xi,y)]\psi_2(\xi)\}d\xi,$$

$$\sigma_{22}(x,y) = \int_{-a}^{a}\{[N_2(x-\xi,y) - M_2(x-\xi,y)]\phi_1(\xi) - [N_1(x-\xi,y) - M_1(x-\xi,y)]\phi_2(\xi)\}d\xi$$

$$+\int_{-b}^{b}\{[\nu N_1(x-\xi,y) - (1+\nu)M_1(x-\xi,y)]\psi_1(\xi) + [(3+\nu)N_2(x-\xi,y)$$

$$-(1+\nu)M_2(x-\xi,y)]\psi_2(\xi)\}d\xi, \tag{3.5}$$

and the displacement derivatives

$$E\frac{\partial u_1}{\partial x}(x,y)=\int_{-a}^{a}\{[(1-\nu)N_2(x-\xi,y)+(1+\nu)M_2(x-\xi,y)]\phi_1(\xi)+[\nu N_1(x-\xi,y)$$

$$-(1+\nu)M_1(x-\xi,y)]\phi_2(\xi)\}d\xi+\int_{-b}^{b}\{[(1-\nu^2)N_1(x-\xi,y)+(1+\nu)^2M_1(x-\xi,y)]\psi_1(\xi)$$

$$-(1+\nu)^2[N_2(x-\xi,y)-M_2(x-\xi,y)]\psi_2(\xi)\}d\xi,$$

$$E\frac{\partial u_2}{\partial x}(x,y)=\int_{-a}^{a}\{[N_1(x-\xi,y)-(1+\nu)M_1(x-\xi,y)]\phi_1(\xi)+[(3+\nu)N_2(x-\xi,y)$$

$$-(1+\nu)M_2(x-\xi,y)]\phi_2(\xi)\}d\xi+\int_{-b}^{b}\{(1+\nu)^2[-N_2(x-\xi,y)+M_2(x-\xi,y)]\psi_1(\xi)$$

$$+(1+\nu)[2N_1(x-\xi,y)-(1+\nu)M_1(x-\xi,y)]\psi_2(\xi)\}d\xi \tag{3.6}$$

in terms of the jumps introduced in (2.2). Here,

$$N_1(t,y)=\frac{t}{2\pi(t^2+y^2)},\quad N_2(t,y)=\frac{y}{4\pi(t^2+y^2)},$$

$$M_1(t,y)=\frac{t(t^2-y^2)}{4\pi(t^2+y^2)^2},\quad M_2(t,y)=\frac{t^2y}{2\pi(t^2+y^2)^2}. \tag{3.7}$$

To satisfy the boundary conditions (2.1), we need the boundary values $\sigma_{j2}(x,0^\pm)$ and $\frac{\partial}{\partial x}u_j(x,0^\pm)$, $j=1,2$. In view of the properties of the Dirac delta-function

$$\lim_{y\to 0^\pm}\frac{y}{\pi(t^2+y^2)}=\pm\delta(t),\quad \lim_{y\to 0^\pm}\frac{yt^2}{\pi(t^2+y^2)^2}=\pm\frac{1}{2}\delta(t), \tag{3.8}$$

the integral representations (3.5) and (3.6) yield

$$\sigma_{12}(x,0^\pm)=-\frac{1}{4\pi}\int_{-a}^{a}\frac{\phi_1(\xi)d\xi}{x-\xi}\pm\frac{1}{2}\psi_1(x)+\frac{1-\nu}{4\pi}\int_{-b}^{b}\frac{\psi_2(\xi)d\xi}{x-\xi},$$

$$\sigma_{22}(x,0^\pm)=-\frac{1}{4\pi}\int_{-a}^{a}\frac{\phi_2(\xi)d\xi}{x-\xi}-\frac{1-\nu}{4\pi}\int_{-b}^{b}\frac{\psi_1(\xi)d\xi}{x-\xi}\pm\frac{1}{2}\psi_2(x),$$

$$E\frac{\partial u_1}{\partial x}(x,0^\pm)=\pm\frac{1}{2}\phi_1(x)-\frac{1-\nu}{4\pi}\int_{-a}^{a}\frac{\phi_2(\xi)d\xi}{x-\xi}+\frac{\nu_1}{4\pi}\int_{-b}^{b}\frac{\psi_1(\xi)d\xi}{x-\xi},$$

$$E\frac{\partial u_2}{\partial x}(x,0^\pm)=\frac{1-\nu}{4\pi}\int_{-a}^{a}\frac{\phi_1(\xi)d\xi}{x-\xi}\pm\frac{1}{2}\phi_2(x)+\frac{\nu_1}{4\pi}\int_{-b}^{b}\frac{\psi_2(\xi)d\xi}{x-\xi}, \tag{3.9}$$

where

$$\nu_1=(3-\nu)(1+\nu). \tag{3.10}$$

It is convenient to introduce the following complex-valued functions:

$$\phi(x)=\phi_1(x)+i\phi_2(x),\quad \psi(x)=\psi_1(x)+i\psi_2(x),$$

$$\sigma_\pm(x)=\sigma_{12}(x,0^\pm)+i\sigma_{22}(x,0^\pm),\quad w_\pm(x)=\frac{\partial u_1}{\partial x}(x,0^\pm)+i\frac{\partial u_2}{\partial x}(x,0^\pm). \tag{3.11}$$

Equations (3.9), after rearrangement, become

$$\sigma_\pm(x)=\pm\frac{1}{2}\psi(x)+\frac{i(1-\nu)}{4\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi-x}+\frac{1}{4\pi}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi-x},$$

$$Ew_\pm(x)=\pm\frac{1}{2}\phi(x)-\frac{i(1-\nu)}{4\pi}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi-x}-\frac{\nu_1}{4\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi-x}. \tag{3.12}$$

We now proceed with the boundary conditions on the crack sides and the lower side of the inclusion. From (3.12), we have

$$2\sigma_+(x)=\frac{1}{2\pi}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi-x}+\psi(x)+\frac{i(1-\nu)}{2\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi-x},\quad -a<x<a,$$

$$2Ew_-(x)=-\phi(x)-i(1-\nu)[2\sigma_+(x)-\psi(x)]-\frac{2}{\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi-x},\quad -b<x<b. \tag{3.13}$$

On inverting the singular operator in the space of functions having integrable singularities at the endpoints $\pm a$ (**14**) we express the function $\phi(x)$ from the first equation in (3.13)

$$\phi(x)=-\frac{2}{\pi\sqrt{a^2-x^2}}\left\{\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{\xi-x}[2\sigma_+(\xi)-\psi(\xi)]d\xi+C\right\}+i(1-\nu)\Theta(x), \tag{3.14}$$

where C is an arbitrary constant and

$$\Theta(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{\xi-x}\frac{1}{\pi}\int_{-a}^{a}\frac{\psi(\eta)d\eta}{\eta-\xi}. \tag{3.15}$$

It turns out that C has to be zero. Indeed, from the crack closedness conditions (2.7) and the definition (3.11) of the function $\phi(x)$ we have

$$\int_{-a}^{a}\phi(x)dx=0. \tag{3.16}$$

On the other hand,

$$\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}(x-\xi)}=0,\quad -a<\xi<a, \tag{3.17}$$

where the integral in the left-hand side is understood in the sense of the Cauchy principal value. Therefore, from the representation (3.14) of $\phi(x)$ it follows that $C=0$. Our next step is to simplify the relation (3.15) for the function $\Theta(x)$. On employing the Poincaré–Bertrand formula (**14**)

$$\frac{1}{\pi}\int_{-a}^{a}\frac{d\xi}{\xi-x}\frac{1}{\pi}\int_{-a}^{a}\frac{\varphi(\xi,\eta)d\eta}{\eta-\xi}=-\varphi(x,x)+\frac{1}{\pi}\int_{-a}^{a}\frac{d\eta}{\pi}\int_{-a}^{a}\frac{\varphi(\xi,\eta)d\xi}{(\xi-x)(\eta-\xi)}, \tag{3.18}$$

evaluating the integral

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}d\xi}{(\xi-x)(\eta-\xi)}=1, \quad -a<x<a, \quad -a<\eta<a, \tag{3.19}$$

and taking into account the equilibrium conditions (2.6) and the formula for the function $\psi(x)$ in (3.11), we arrive ultimately at the following simple relation:

$$\Theta(x)=-\psi(x)+\frac{iP_*}{\pi\sqrt{a^2-x^2}}, \quad -a<x<a, \tag{3.20}$$

where $P_*=P+\Sigma$. Now we replace the function $\phi(x)$ in the second equation (3.13) by its representation

$$\phi(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\left[2\int_{-b}^{b}\frac{\sqrt{a^2-\xi^2}}{\xi-x}\psi(\xi)d\xi+4\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{\xi-x}\sigma^\circ(\xi)d\xi-(1-\nu)P_*\right]$$
$$-i(1-\nu)\psi(x), \quad -a<x<a, \tag{3.21}$$

following from (3.14) and obtain a governing singular integral equation for the function $\psi(x)$. It reads

$$\frac{1}{\pi}\int_{-k}^{k}\left(1+\sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\psi(a\tau)d\tau}{\tau-t}-i(1-\nu)\psi(at)=g(t), \quad -k<t<k. \tag{3.22}$$

Here, $k=b/a<1$,

$$g(t)=-iEh'(at)+Ew^\circ(at)+i(1-\nu)\sigma^\circ(at)-\frac{2}{\pi\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}\sigma^\circ(a\tau)d\tau+\frac{(1-\nu)P_*}{2\pi a\sqrt{1-t^2}}, \tag{3.23}$$

$\sigma^\circ(x)=\sigma_{12}^\circ(x,0)+i\sigma_{22}^\circ(x,0)$, and due to (2.6) the function $\psi(a\tau)$ has to satisfy the additional condition

$$\int_{-k}^{k}\psi(at)dt=\frac{iP_*}{a}. \tag{3.24}$$

4. Vector Riemann–Hilbert problem and its solution

In this section, we aim to rewrite the singular integral equation (3.22) as a vector Riemann–Hilbert problem and derive its closed-form solution. Introduce two Cauchy integrals

$$\Phi_1(z)=\frac{1}{2\pi i}\int_{-k}^{k}\frac{\psi(a\tau)d\tau}{\tau-z}, \quad \Phi_2(z)=\frac{1}{2\pi i}\int_{-k}^{k}\frac{\sqrt{1-\tau^2}\psi(a\tau)d\tau}{\tau-z}, \tag{4.1}$$

analytic in the whole complex plane except at the segment $[-k, k]$. By the Sokhotski–Plemelj formulas their limiting values $\Phi_j^{\pm}(t) = \Phi_j(t \pm i0)$, $-k < t < k$, satisfy the relations

$$\Phi_1^+(t) + \Phi_1^-(t) = \frac{1}{\pi i}\int_{-k}^{k} \frac{\psi(a\tau)d\tau}{\tau - t}, \quad \Phi_2^+(t) + \Phi_2^-(t) = \frac{1}{\pi i}\int_{-k}^{k} \frac{\sqrt{1-\tau^2}\psi(a\tau)d\tau}{\tau - t}, \tag{4.2}$$

and

$$\Phi_1^+(t) - \Phi_1^-(t) = \psi(at), \quad \Phi_2^+(t) - \Phi_2^-(t) = \sqrt{1-t^2}\psi(at). \tag{4.3}$$

On substituting these relations into (3.22), we obtain that the limiting values of the functions (4.1) are connected by

$$\nu\Phi_1^+(t) + (2-\nu)\Phi_1^-(t) + \frac{\Phi_2^+(t) + \Phi_2^-(t)}{\sqrt{1-t^2}} = -ig(t),$$

$$\Phi_2^+(t) - \Phi_2^-(t) = \sqrt{1-t^2}[\Phi_1^+(t) - \Phi_1^-(t)], \quad -k < t < k. \tag{4.4}$$

This, after rearrangement, can be written as the following vector Riemann–Hilbert boundary value problem on a segment:

$$\mathbf{\Phi}^+(t) = G(t)\mathbf{\Phi}^-(t) - \frac{i}{\nu+1}\mathbf{T}(t)g(t), \quad -k < t < k, \tag{4.5}$$

where

$$G(t) = \frac{1}{\nu+1}\begin{pmatrix} \nu - 1 & -\frac{2}{\sqrt{1-t^2}} \\ -2\sqrt{1-t^2} & \nu - 1 \end{pmatrix},$$

$$\mathbf{\Phi}(z) = \begin{pmatrix} \Phi_1(z) \\ \Phi_2(z) \end{pmatrix}, \quad \mathbf{T}(t) = \begin{pmatrix} 1 \\ \sqrt{1-t^2} \end{pmatrix}. \tag{4.6}$$

The vector $\Phi(z)$ has a simple zero at the infinite point, and due to the condition (3.24) and the first relation (4.1) the function $\Phi_1(z)$ behaves at infinity as

$$\Phi_1(z) \sim -\frac{P_*}{2\pi a z}, \quad z \to \infty. \tag{4.7}$$

The matrix coefficient to be factorized has the Chebotarev–Khrapkov structure (**10**, **13**), and its eigenvalues are constants, $\lambda_1 = -(3-\nu)(1+\nu)^{-1}$, $\lambda_2 = 1$. We point out that the classical scheme (**10**) is designed and has been employed in the literature so far for the case of a closed contour. In the case of a segment, although the structure of the Wiener–Hopf factors is preserved, additional efforts need to be made to study the behavior of the solution at the endpoints in order to guarantee the solution derived is within the class of integrable functions. The matrix factorization problem

$$G(t) = X^+(t)[X^-(t)]^{-1} = [X^-(t)]^{-1}X^+(t), \quad -k < t < k, \tag{4.8}$$

has the solution

$$X(z)=\Lambda(z)\begin{pmatrix} c(z) & s_+(z)\\ s_-(z) & c(z)\end{pmatrix}. \tag{4.9}$$

Here, we denoted

$$c(z)=\cosh[\sqrt{1-z^2}\beta(z)],\quad s_\pm(z)=(1-z^2)^{\mp 1/2}\sinh[\sqrt{1-z^2}\beta(z)]. \tag{4.10}$$

The function $\sqrt{1-z^2}$ is the single branch of the two-valued function $\omega^2=1-z^2$ fixed by the condition $\sqrt{1-z^2}|_{z=0}=1$ in the plane cut along the straight line joining the branch points $z=-1$ and $z=1$ and passing through the infinite point. For $-k<t<k$, it is directly verified that

$$X^+(t)[X^-(t)]^{-1}=\frac{\Lambda^+(t)}{\Lambda^-(t)}$$

$$\times\begin{pmatrix} \cosh[\sqrt{1-t^2}(\beta^+(t)-\beta^-(t))] & \frac{1}{\sqrt{1-t^2}}\sinh[\sqrt{1-t^2}(\beta^+(t)-\beta^-(t))]\\ \sqrt{1-t^2}\sinh[\sqrt{1-t^2}(\beta^+(t)-\beta^-(t))] & \cosh[\sqrt{1-t^2}(\beta^+(t)-\beta^-(t))]\end{pmatrix}. \tag{4.11}$$

The matrix $X(z)$ is a solution of the factorization problem if the functions $\Lambda(z)$ and $\beta(z)$ solve the following scalar Riemann–Hilbert problems:

$$\frac{\Lambda^+(t)}{\Lambda^-(t)}=\sqrt{\lambda_1},\quad -k<t<k, \tag{4.12}$$

and

$$\beta^+(t)-\beta^-(t)=\frac{\ln\sqrt{\lambda_1}}{\sqrt{1-t^2}},\quad -k<t<k, \tag{4.13}$$

where

$$\sqrt{\lambda_1}=i\sqrt{\nu_0},\quad \ln\sqrt{\lambda_1}=\frac{1}{2}\ln\nu_0+\frac{i\pi}{2},\quad \nu_0=\frac{3-\nu}{1+\nu}>0. \tag{4.14}$$

The solution of the factorization problem (4.12) is defined up to a rational factor. For our purposes, we choose it in the form

$$\Lambda(z)=\frac{1}{z-k}\exp\left\{\frac{1}{2\pi i}\int_{-k}^{k}\frac{\ln\sqrt{\lambda_1}d\tau}{\tau-z}\right\}=(z-k)^{-3/4-i\gamma}(z+k)^{-1/4+i\gamma},\quad \gamma=\frac{\ln\nu_0}{4\pi}. \tag{4.15}$$

Here, $\Lambda(z)$ is the branch fixed by the condition $\Lambda(z)\sim z^{-1}$, $z\to\infty$, in the plane cut along the segment $[-k,k]$ passing through the point $z=0$. This branch is positive for $z=t$, $t\in(k,+\infty)$ and negative for $t\in(-\infty,-k)$. The solution of the problem (4.13) is given by the Cauchy integral

$$\beta(z)=\frac{\ln\sqrt{\lambda_1}}{2\pi i}\int_{-k}^{k}\frac{d\tau}{\sqrt{1-\tau^2}(\tau-z)}. \tag{4.16}$$

It has logarithmic singularities at the endpoints, and the functions $c(z)$ and $s_\pm(z)$ have power singularities

$$c(z) \sim C_\pm (z \mp k)^{-1/4+i\gamma}, \quad s_+(z) \sim \mp \frac{C_\pm}{\sqrt{1-k^2}} (z \mp k)^{-1/4+i\gamma},$$

$$s_-(z) \sim \mp C_\pm \sqrt{1-k^2} (z \mp k)^{-1/4+i\gamma}, \quad z \to \pm k, \quad C_\pm = \text{const} \neq 0. \tag{4.17}$$

In view of the power singularities of the function $\Lambda(z)$, the matrix $X(z)$ and its inverse

$$[X(z)]^{-1} = (z-k)^{3/4+i\gamma} (z+k)^{1/4-i\gamma} \begin{pmatrix} c(z) & -s_+(z) \\ -s_-(z) & c(z) \end{pmatrix} \tag{4.18}$$

in neighborhoods of the endpoints behave as

$$X(z) \sim (z-k)^{-1} Y'_+(z), \quad [X(z)]^{-1} \sim (z-k)^{1/2} Y''_+(z), \quad z \to k,$$

$$X(z) \sim (z+k)^{-1/2} Y'_-(z), \quad [X(z)]^{-1} \sim Y''_-(z), \quad z \to -k, \tag{4.19}$$

where $Y'_\pm(z)$ and $Y''_\pm(z)$ are 2×2 matrices whose elements are bounded as $z \to \pm k$.

We next study the behavior of the matrices $X(z)$ and $[X(z)]^{-1}$ at the infinite point. From (4.16), we derive

$$\beta(z) \sim \frac{\beta_0}{z}, \quad z \to \infty, \quad \beta_0 = \left(2i\gamma - \frac{1}{2}\right) \sin^{-1} k. \tag{4.20}$$

The branch of the function $\sqrt{1-z^2}$ chosen before is discontinuous at the infinite point since the cut passes through the point $z = \infty$. We have

$$\sqrt{1-z^2} \sim -iz \,\text{sgn}\, \text{Im}\, z, \quad z \to \infty. \tag{4.21}$$

However, for the functions $c(z)$, $s_+(z)$ and $s_-(z)$, the infinite point is a regular point, a simple zero and a simple pole, respectively. Combining this with (4.20), (4.15), (4.9), (4.10) and (4.18) we find the asymptotics of the matrix $X(z)$ and its inverse as

$$X(z) \sim \frac{1}{z} \begin{pmatrix} \cos\beta_0 & z^{-1}\sin\beta_0 \\ -z\sin\beta_0 & \cos\beta_0 \end{pmatrix}, \quad [X(z)]^{-1} \sim z \begin{pmatrix} \cos\beta_0 & -z^{-1}\sin\beta_0 \\ z\sin\beta_0 & \cos\beta_0 \end{pmatrix}, \quad z \to \infty. \tag{4.22}$$

Substituting the splitting (4.8) into the Riemann–Hilbert boundary condition (4.5) and replacing the vector $-i(\nu+1)^{-1}[X^+(t)]^{-1}\mathbf{T}(t)g(t)$ by $\mathbf{\Psi}^+(t) - \mathbf{\Psi}^-(t)$ yield

$$[X^+(t)]^{-1}\mathbf{\Phi}^+(t) - \mathbf{\Psi}^+(t) = [X^-(t)]^{-1}\mathbf{\Phi}^-(t) - \mathbf{\Psi}^-(t), \quad -k < t < k. \tag{4.23}$$

Here, $\mathbf{\Psi}^\pm(t) = \mathbf{\Psi}(t \pm i0)$ and

$$\mathbf{\Psi}(z) = -\frac{1}{2\pi(\nu+1)} \int_{-k}^{k} \frac{[X^+(\tau)]^{-1}\mathbf{T}(\tau)g(\tau)d\tau}{\tau - z}. \tag{4.24}$$

Due to (4.1) the vector $\mathbf{\Phi}(z)$ has a simple zero at the infinite point, and according to the continuity principle, the Liouville theorem and the second asymptotic relation in (4.22) the vector $[X(z)]^{-1}\mathbf{\Phi}(z) - \mathbf{\Psi}(z)$ is a polynomial vector of the form

$$[X(z)]^{-1}\mathbf{\Phi}(z) - \mathbf{\Psi}(z) = \begin{pmatrix} C_0 \cos\beta_0 \\ C_0 z \sin\beta_0 + C_1 \end{pmatrix}, \tag{4.25}$$

where C_0 and C_1 are arbitrary constants. This asserts

$$\mathbf{\Phi}(z) = X(z)\left[\mathbf{\Psi}(z) + \begin{pmatrix} C_0 \cos\beta_0 \\ C_0 z \sin\beta_0 + C_1 \end{pmatrix}\right], \quad z \in \mathbb{C} \setminus [-k, k]. \tag{4.26}$$

Analysis of the asymptotics of the vector $\mathbf{\Phi}(z)$ shows that is has an integrable singularity at the endpoint $z = -k$, while at the second end, $z = k$, it has a nonintegrable singularity. It follows from (4.17) and (4.26) that in the vicinity of the point $z = k$

$$\mathbf{\Phi}(z) = \frac{C_+(z-k)^{-1}}{(2k)^{1/4-i\gamma}} \begin{pmatrix} 1 & -\frac{1}{\sqrt{1-k^2}} \\ -\sqrt{1-k^2} & 1 \end{pmatrix} \begin{pmatrix} \Psi_1(k) + C_0 \cos\beta_0 \\ \Psi_2(k) + C_0 k \sin\beta_0 + C_1 \end{pmatrix} + \mathbf{\Phi}_0(z), \tag{4.27}$$

where the vector $\mathbf{\Phi}_0(z)$ may have at most an integrable singularity at the point $z = k$. From this relation we infer that the vector $\mathbf{\Phi}(z)$ has an integrable singularity at the point $z = k$ if and only if

$$C_1 = \sqrt{1-k^2}\,\Psi_1(k) - \Psi_2(k) + C_0(\sqrt{1-k^2}\cos\beta_0 - k \sin\beta_0). \tag{4.28}$$

At the infinite point, both components of the vector $\mathbf{\Phi}(z)$ in (4.26) have a simple zero and

$$\mathbf{\Phi}(z) \sim \frac{1}{z} \begin{pmatrix} C_0 \\ C_1 \cos\beta_0 - \Psi_1^\circ \sin\beta_0 \end{pmatrix}, \quad z \to \infty, \tag{4.29}$$

where Ψ_1° is the first component of the vector

$$\mathbf{\Psi}^\circ = \frac{1}{2\pi(\nu+1)} \int_{-k}^{k} [X^+(\tau)]^{-1} \mathbf{T}(\tau) g(\tau) d\tau. \tag{4.30}$$

Comparing (4.31) with (4.7) determines the constant C_0

$$C_0 = -\frac{P_*}{2\pi a}. \tag{4.31}$$

This completes the solution of the vector Riemann–Hilbert problem (4.5). Its exact solution is given by (4.26), (4.28) and (4.31).

5. Solution of the singular integral equation

The solution of (3.22) is expressed through the solution of the vector Riemann–Hilbert problem by the formula $\psi(at) = \Phi_1^+(t) - \Phi_1^-(t)$, $-k < t < k$. This section transforms this formula into a form convenient for computations. According to formulas (4.9) and (4.10), the limiting values of the matrix $X(z)$ admit the representation

$$X^{\pm}(t)=\begin{pmatrix}\chi_1^{\pm}(t) & \chi_2^{\pm}(t)\\ (1-t^2)\chi_2^{\pm}(t) & \chi_1^{\pm}(t)\end{pmatrix},\quad -k<t<k, \tag{5.1}$$

where the functions $\chi_j^{\pm}(t)$, $j=1,2$ are to be expressed through the functions $\beta^{\pm}(t)$ and $\Lambda^{\pm}(t)$. Utilizing (4.26) and (5.1) we write

$$\psi(at)=\Omega(t)+[\chi_2^+(t)-\chi_2^-(t)][\sqrt{1-k^2}\Psi_1(k)-\Psi_2(k)]$$

$$-\frac{P_*}{2\pi a}\{[\chi_1^+(t)-\chi_1^-(t)]\cos\beta_0+[\chi_2^+(t)-\chi_2^-(t)][(t-k)\sin\beta_0+\sqrt{1-k^2}\cos\beta_0]\}, \tag{5.2}$$

where

$$\Omega(t)=\chi_1^+(t)\Psi_1^+(t)-\chi_1^-(t)\Psi_1^-(t)+\chi_2^+(t)\Psi_2^+(t)-\chi_2^-(t)\Psi_2^-(t). \tag{5.3}$$

In order to determine the functions $\chi_j^{\pm}(t)$, we evaluate the principal value of the singular integral (4.16) that is

$$\beta(t)=\left(\frac{1}{4}-i\gamma\right)I(t), \tag{5.4}$$

where

$$I(t)=\lim_{\varepsilon\to 0^+}\left(-\int_{-k}^{t-\varepsilon}\frac{d\tau}{\sqrt{1-\tau^2}(t-\tau)}+\int_{t+\varepsilon}^{k}\frac{d\tau}{\sqrt{1-\tau^2}(\tau-t)}\right). \tag{5.5}$$

Making the substitutions $\xi=(t-\tau)^{-1}>0$ and $\xi=(\tau-t)^{-1}>0$ in the first and second integrals in (5.5), respectively, we deduce

$$I(t)=-\lim_{\varepsilon\to 0^+}\left(\int_{1/(k+t)}^{1/\varepsilon}\frac{d\xi}{\sqrt{r_+(\xi)}}+\int_{1/\varepsilon}^{1/(k-t)}\frac{d\xi}{\sqrt{r_-(\xi)}}\right), \tag{5.6}$$

where

$$r_{\pm}(\xi)=(1-t^2)\xi^2\pm 2t\xi-1,\quad r_+(\xi)>0,\quad \frac{1}{k+t}<\xi<\frac{1}{\varepsilon},\quad r_-(\xi)>0,\quad \frac{1}{\varepsilon}<\xi<\frac{1}{k-t}. \tag{5.7}$$

Employing the antiderivative

$$\int\frac{d\xi}{\sqrt{r_{\pm}(\xi)}}=\frac{1}{\sqrt{1-t^2}}\ln\left[2\sqrt{(1-t^2)r_{\pm}(\xi)}+2(1-t^2)\xi\pm 2t\right]+\text{const} \tag{5.8}$$

and passing to the limit $\varepsilon\to 0^+$ in (5.6) we eventually obtain

$$I(t)=\frac{1}{\sqrt{1-t^2}}\ln\frac{(k-t)R(t)}{k+t}, \tag{5.9}$$

where we denoted

$$R(t)=\frac{R_+(t)}{R_-(t)},$$

$$R_\pm(t)=\sqrt{(1-t^2)[(1-t^2)\pm 2t(k\pm t)-(k\pm t)^2]}+1-t^2\pm t(k\pm t)>0,\quad -k<t<k. \tag{5.10}$$

We apply now the Sokhotski–Plemelj formulas and discover the following explicit representations of the limiting values of the function $\beta(z)$:

$$\sqrt{1-t^2}\beta^\pm(t)=\left(\frac{1}{4}-i\gamma\right)\left(\pm\pi i+\ln\frac{(k-t)R(t)}{k+t}\right). \tag{5.11}$$

Utilizing these representations determines the functions

$$\cosh[\sqrt{1-t^2}\beta^\pm(t)]=\frac{e_0^{\pm 1}}{2}\left[\frac{(k-t)R(t)}{k+t}\right]^{1/4-i\gamma}+\frac{e_0^{\mp 1}}{2}\left[\frac{(k-t)R(t)}{k+t}\right]^{-1/4+i\gamma},$$

$$\sinh[\sqrt{1-t^2}\beta^\pm(t)]=\frac{e_0^{\pm 1}}{2}\left[\frac{(k-t)R(t)}{k+t}\right]^{1/4-i\gamma}-\frac{e_0^{\mp 1}}{2}\left[\frac{(k-t)R(t)}{k+t}\right]^{-1/4+i\gamma}, \tag{5.12}$$

Here, $e_0=e^{\pi i/4+\pi\gamma}$. The limiting values of the function $\Lambda(z)$ are discontinuous through the cut $[-k,k]$. According to the choice of the branch of $\Lambda(z)$, $\arg(z-k)=\pi$ and $\arg(z+k)=\pi\mp\pi$ as $z=t\pm i0$, $-k<t<k$. We have

$$\Lambda^\pm(t)=-e_0^{\pm 1}(k-t)^{-3/4-i\gamma}(k+t)^{-1/4+i\gamma},\quad -k<t<k. \tag{5.13}$$

Combining these results we find the limiting values $\chi_j^\pm(t)$, $-k<t<k$, of the functions $\chi_j(z)$. They are given by

$$\chi_j^\pm(t)=-\frac{1}{2(\sqrt{1-t^2})^{j-1}}\left[e_0^{\pm 2}(k-t)^{-1/2-2i\gamma}(k+t)^{-1/2+2i\gamma}R^{1/4-i\gamma}(t)-\frac{(-1)^j}{k-t}R^{-1/4+i\gamma}(t)\right]. \tag{5.14}$$

Now we wish to specify the limiting values of the vector $\mathbf{\Psi}(z)$. Computing first

$$[X^+(t)]^{-1}\mathbf{T}(t)=-\frac{1}{e_0^2}(k-t)^{1/2+2i\gamma}(k+t)^{1/2-2i\gamma}R^{-1/4+i\gamma}\mathbf{T}(t) \tag{5.15}$$

and employing next the Sokhotski–Plemelj formulas we obtain the components $\Psi_1^\pm(t)$ and $\Psi_2^\pm(t)$ of the vectors $\mathbf{\Psi}^\pm(t)$ as

$$\Psi_j^\pm(t)=\pm\frac{ig(t)}{2(\nu+1)e_0^2}(\sqrt{1-t^2})^{j-1}(k-t)^{1/2+2i\gamma}(k+t)^{1/2-2i\gamma}R^{-1/4+i\gamma}(t)+\Psi_j(t), \tag{5.16}$$

where $\Psi_j(t)$ are the principal values of the integrals

$$\Psi_j(t)=\frac{1}{2\pi(\nu+1)e_0^2}\int_{-k}^{k}(k-\tau)^{1/2+2i\gamma}(k+\tau)^{1/2-2i\gamma}R^{-1/4+i\gamma}(\tau)\frac{(\sqrt{1-\tau^2})^{j-1}g(\tau)d\tau}{\tau-t}. \tag{5.17}$$

It is seen that the functions $\Psi_j^{\pm}(t)$ are bounded at the endpoints and have finite limits. At the point k,

$$\Psi_j(k) = -\frac{1}{2\pi(\nu+1)e_0^2}\int_{-k}^{k}(k-\tau)^{-1/2+2i\gamma}(k+\tau)^{1/2-2i\gamma}R^{-1/4+i\gamma}(\tau)(\sqrt{1-\tau^2})^{j-1}g(\tau)d\tau. \tag{5.18}$$

We now ready to evaluate $\Omega(t)$ and $\chi_j^{+}(t)-\chi_j^{-}(t)$. Substituting (5.14) and (5.16) into (5.3) yields, after some simplifications,

$$\Omega(t) = -\frac{i(1-\nu)g(t)}{\nu_1} + \chi(t)\left[\Psi_1(t) + \frac{\Psi_2(t)}{\sqrt{1-t^2}}\right],$$

$$\chi_j^{+}(t)-\chi_j^{-}(t) = \frac{\chi(t)}{(\sqrt{1-t^2})^{j-1}}, \quad j=1,2. \tag{5.19}$$

Here,

$$\chi(t) = -\frac{2i}{(1+\nu)\sqrt{\nu_0}}(k-t)^{-1/2-2i\gamma}(k+t)^{-1/2+2i\gamma}R^{1/4-i\gamma}(t). \tag{5.20}$$

Remembering the formula for the function $\psi(at)$, we can write down the final representation for the solution of the singular integral equation (3.22)

$$\psi(at) = -\frac{i(1-\nu)g(t)}{\nu_1} + \chi(t)\left[\Psi_1(t) + \sqrt{\frac{1-k^2}{1-t^2}}\Psi_1(k)\right.$$

$$\left. + \frac{\Psi_2(t)-\Psi_2(k)}{\sqrt{1-t^2}} - \frac{P_*}{2\pi a}\left(\cos\beta_0 + \frac{(t-k)\sin\beta_0 + \sqrt{1-k^2}\cos\beta_0}{\sqrt{1-t^2}}\right)\right]. \tag{5.21}$$

Notice that this formula is a closed-form solution and possesses only two integrals (5.17).

6. Analysis of the solution at the singular points: numerical results

6.1 *Stress intensity factors*

In section 3, we saw that the jump of the tangential derivative of the displacement vector could readily be expressed through the solution of (3.22) by (3.21). Due to formulas (5.20) and (5.21) this function has the square root singularities at the crack tips

$$\phi(x) \sim \phi_0^{\pm}(a \mp x)^{-1/2}, \quad x \to \pm a^{\mp}, \tag{6.1}$$

where $a^{\mp} = a \mp 0$ and

$$\phi_0^{\pm} = \mp\frac{1}{\pi}\sqrt{\frac{2}{a}}\left[\int_{-b}^{b}\left(\frac{a+\xi}{a-\xi}\right)^{\pm 1/2}\psi(\xi)d\xi + 2\int_{-a}^{a}\left(\frac{a+\xi}{a-\xi}\right)^{\pm 1/2}\sigma^{\circ}(\xi)d\xi \pm \frac{(1-\nu)P_*}{2}\right]. \tag{6.2}$$

At the inclusion endpoints, $x = b$ and $x = -b$, the internal singular points, the function $\phi(x)$ not only has square root singularities but also oscillates

$$\phi(x) = (x-b)^{-1/2-2i\gamma}(x+b)^{-1/2+2i\gamma}\Upsilon(x), \tag{6.3}$$

where $\Upsilon(x)$ is bounded and in general discontinuos at the points $x = \pm b$. It has definite nonzero limits as $x \to b^{\pm}$ and as $x \to -b^{\mp}$.

Now we wish to analyze the concentration of stresses in the vicinities of the crack and the inclusion tips. Assume $y = 0$ and x is outside the segment $[-b, b]$. Applying (3.12) we have

$$\sigma_{12}(x,0) + i\sigma_{22}(x,0) = \frac{i(1-\nu)}{4\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi - x} + \frac{1}{4\pi}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi - x}, \quad |x| > b. \tag{6.4}$$

Analysis of the Cauchy integral with the density $\phi(\xi)$ in (6.4) yields

$$\sigma_{12}(x,0) + i\sigma_{22}(x,0) \sim \mp\frac{\phi_0^{\pm}}{4\sqrt{\pm x - a}}, \quad x \to \pm a^{\pm}. \tag{6.5}$$

On the other hand, on employing the conventional notations of the stress intensity factors (SIFs) we have

$$\sigma_{12}(x,0) + i\sigma_{22}(x,0) \sim \frac{K_{II}^{\pm} + iK_I^{\pm}}{\sqrt{2\pi(\pm x - a)}}, \quad x \to \pm a^{\pm}. \tag{6.6}$$

Combining (6.5) and (6.6) we express the SIFs through two integrals

$$K_{II}^{\pm} + iK_I^{\pm} = \frac{1}{2\sqrt{\pi a}}\left[\int_{-b}^{b}\left(\frac{a+\xi}{a-\xi}\right)^{\pm 1/2}\psi(\xi)d\xi + 2\int_{-a}^{a}\left(\frac{a+\xi}{a-\xi}\right)^{\pm 1/2}\sigma^{\circ}(\xi)d\xi \pm \frac{(1-\nu)P_*}{2}\right]. \tag{6.7}$$

Next we determine the contact stresses (the traction vector components)

$$\sigma_{12}(x,0^-) + i\sigma_{22}(x,0^-) = \sigma_+(x) - \psi(x) \tag{6.8}$$

acting in the contact zone $-b < x < b$. After substituting (5.21) and (3.23) into (6.8) and since $\sigma_+(x) = -\sigma^{\circ}(x)$, we have

$$\sigma_{12}(at,0^-) + i\sigma_{22}(at,0^-) = -\frac{4\sigma^{\circ}(at)}{\nu_1} + \frac{1-\nu}{\nu_1}\Big[Eh'(at) + iEw^{\circ}(at)$$

$$-\frac{2i}{\pi\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau - t}\sigma^{\circ}(a\tau)d\tau + \frac{i(1-\nu)P_*}{2\pi a\sqrt{1-t^2}}\Big] - \chi(t)\left[\Psi_1(t) + \sqrt{\frac{1-k^2}{1-t^2}}\Psi_1(k)\right.$$

$$\left.+\frac{\Psi_2(t) - \Psi_2(k)}{\sqrt{1-t^2}} - \frac{P_*}{2\pi a}\left(\cos\beta_0 + \frac{(t-k)\sin\beta_0 + \sqrt{1-k^2}\cos\beta_0}{\sqrt{1-t^2}}\right)\right], \quad -k < t < k. \tag{6.9}$$

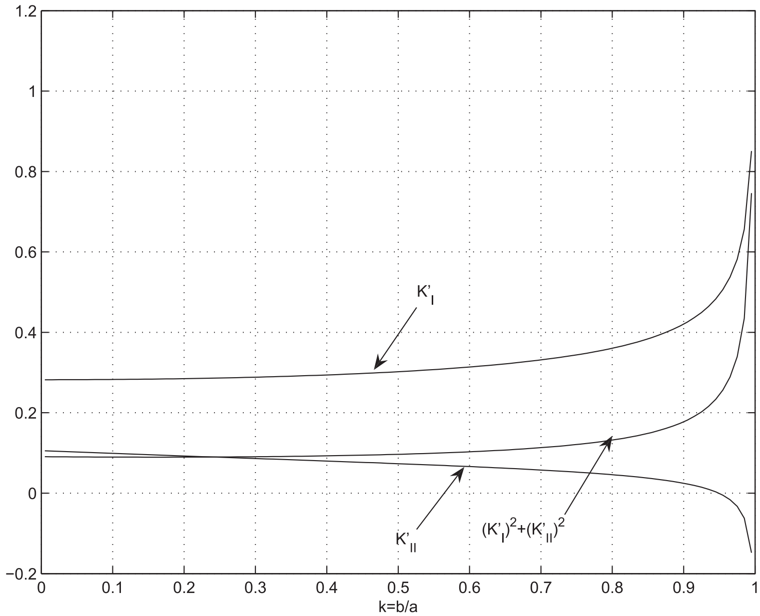

Fig. 2 Model 1: the dimensionless SIFs K_I' and K_{II}' and the dimensionless energy release rate $Ed/dr\delta U = (K_I^+)^2 + (K_{II}^+)^2$ at the crack tip $x = a$ versus $k = b/a$ for $\nu = 0.25$

By examining this formula we see that due to the existence of the singularities of the function $\chi(t)$ in the vicinity of the inclusion tips, $t \to \pm k^{\mp}$ ($x \to \pm b^{\mp}$), the traction admits the representation

$$\sigma_{12}(at, 0^-) + i\sigma_{22}(at, 0^-) = (k - t)^{-1/2-2i\gamma}(k + t)^{-1/2+2i\gamma}\,\Xi(t), \tag{6.10}$$

where $\Xi(t)$ is a bounded function having definite limits as $t \to \pm k^{\mp}$.

6.2 *Computational formulas for the SIFs*

For a numerical example, we take $w^\circ(x) = 0$, $h'(x) = 0$, $-b < x < b$, and $\sigma^\circ(x) = 0$, $-a < x < a$. Then the function $g(t)$ has a simple form,

$$g(t) = \frac{(1-\nu)P}{2\pi a\sqrt{1-t^2}}. \tag{6.11}$$

To evaluate the function $\psi(at)$ given by (5.21) we need to compute the functions $\Psi_j(t)$. It is convenient to represent them as

$$\Psi_j(t) = P_0 \int_{-k}^{k} \sqrt{k^2 - \tau^2}\, g_j(\tau) \frac{d\tau}{\tau - t}, \quad -k < t < k, \quad j = 1, 2, \tag{6.12}$$

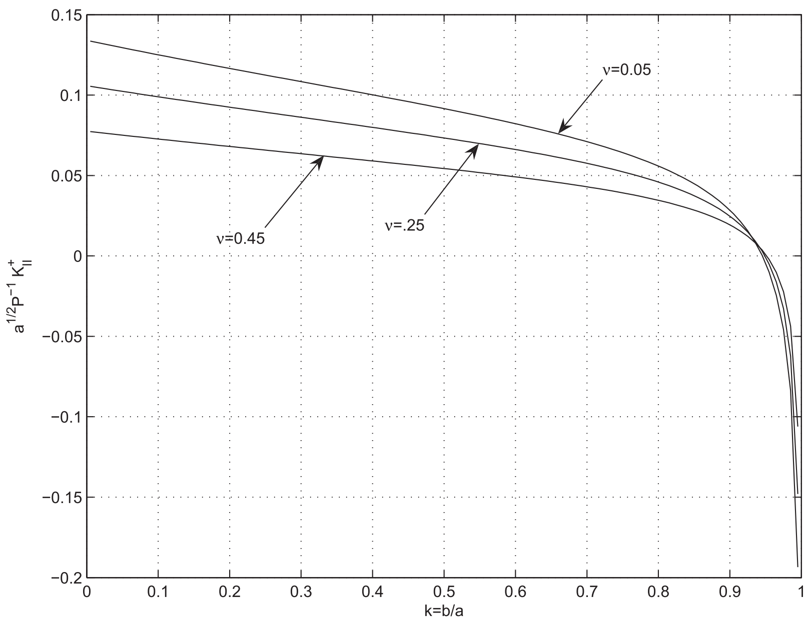

Fig. 3 Model 1: the dimensionless SIF K'_{II} at the crack tip $x = a$ versus $k = b/a$ for $\nu = 0.05$, $\nu = 0.25$, and $\nu = 0.45$

where

$$g_1(\tau) = \left(\frac{k-\tau}{k+\tau}\right)^{2i\gamma} \frac{R^{-1/4+i\gamma}(\tau)}{\sqrt{1-\tau^2}}, \quad g_2(\tau) = \sqrt{1-\tau^2}\, g_1(\tau), \quad P_0 = \frac{(1-\nu)P}{4\pi^2 a(1+\nu)e_0^2}. \tag{6.13}$$

The functions $g_j(\tau)$ can be expanded in terms of the Chebyshev polynomials of the second kind as

$$g_j(\tau) = \sum_{m=0}^{\infty} c_{jm} U_m\left(\frac{\tau}{k}\right), \quad -k < t < k, \tag{6.14}$$

with the coefficients c_{jm} defined by

$$c_{jm} = \frac{2}{\pi k^2} \int_{-k}^{k} g_j(\tau)\sqrt{k^2-\tau^2}\, U_m\left(\frac{\tau}{k}\right) d\tau. \tag{6.15}$$

On employing next the spectral relations for the Chebyshev polynomials of the second and first kind

$$\int_{-k}^{k} \sqrt{k^2-\tau^2}\, U_m\left(\frac{\tau}{k}\right) \frac{d\tau}{\tau - t} = -\pi k T_{m+1}\left(\frac{t}{k}\right), \quad -k < t < k, \quad m = 0, 1, \ldots, \tag{6.16}$$

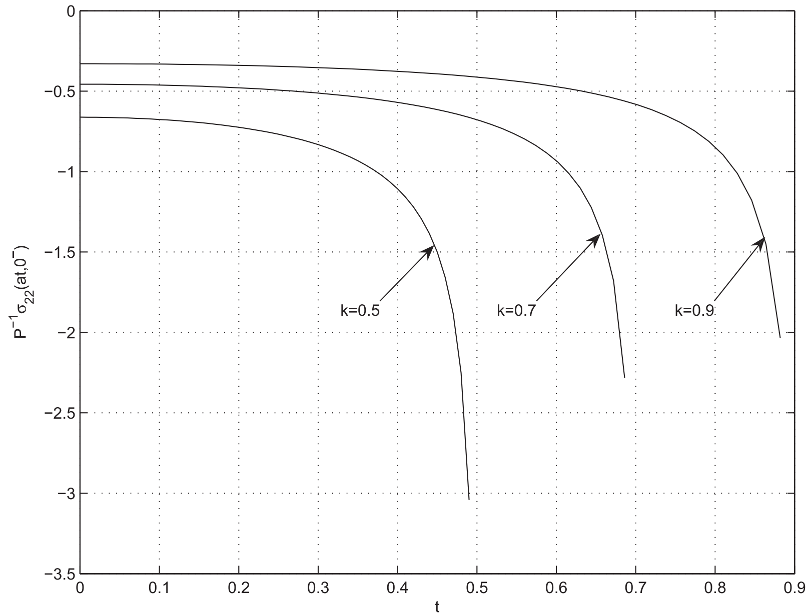

Fig. 4 Model 1: the normal traction component $P^{-1}\sigma_{22}(at, 0^-)$ versus t for $k = 0.5$, $k = 0.7$ and $k = 0.9$, when $\nu = 0.3$

we obtain the series representation of the functions $\Psi_j(t)$

$$\Psi_j(t) = -\pi k P_0 \sum_{m=0}^{\infty} c_{jm} T_{m+1}\left(\frac{t}{k}\right), \quad -k < t < k. \tag{6.17}$$

Since the functions $g_j(\tau)$ oscillate at the points $\tau = \pm k$, instead of using formula (6.17) we evaluate $\Psi_j(k)$ directly

$$\Psi_j(k) = -P_0 \int_{-k}^{k} \left(\frac{k+\tau}{k-\tau}\right)^{1/2} g_j(\tau) d\tau. \tag{6.18}$$

To write down the final formula used for the SIFs, we denote

$$\chi_{\pm}(t) = -\frac{-2ia}{(1+\nu)\sqrt{\nu_0}g_2(t)}\left(\frac{1+t}{1-t}\right)^{\pm 1/2}\left[\Psi_1(t) + \sqrt{\frac{1-k^2}{1-t^2}}\Psi_1(k)\right.$$
$$\left. + \frac{\Psi_2(t) - \Psi_2(k)}{\sqrt{1-t^2}} - \frac{P}{2\pi a}\left(\cos\beta_0 + \frac{(t-k)\sin\beta_0 + \sqrt{1-k^2}\cos\beta_0}{\sqrt{1-t^2}}\right)\right]. \tag{6.19}$$

In terms of these functions, after a rearrangement, formula (6.7) reads

$$K_{II}^{\pm} + iK_I^{\pm} = \frac{1}{2\sqrt{\pi a}}\left[\pm\frac{(1-\nu)P}{2} - \frac{i(1-\nu)^2 P}{2\pi\nu_1}\ln\frac{1+k}{1-k} + \int_{-k}^{k}\frac{\chi_{\pm}(\tau)d\tau}{\sqrt{k^2-\tau^2}}\right]. \tag{6.20}$$

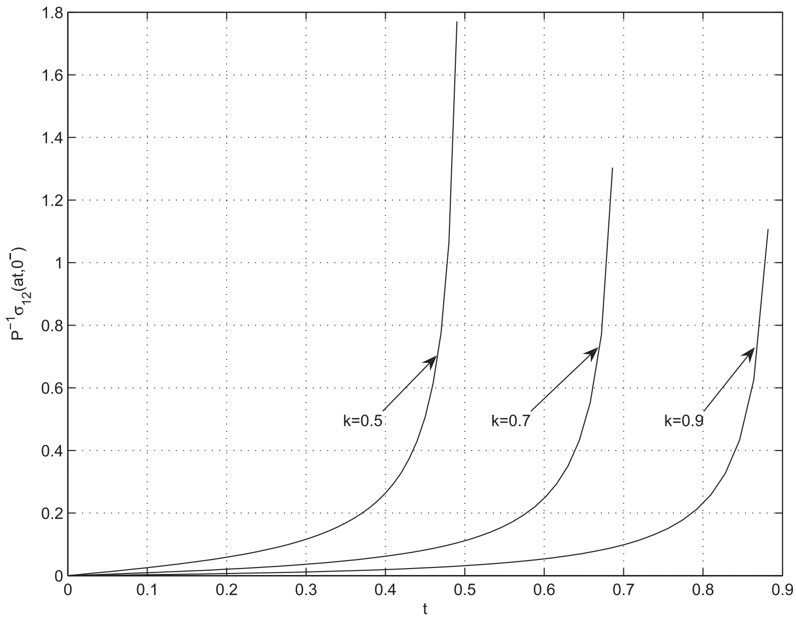

Fig. 5 Model 1: the tangential traction component $P^{-1}\sigma_{12}(at, 0^-)$ versus t for $k = 0.5$, $k = 0.7$ and $k = 0.9$, when $\nu = 0.3$

The integrals (6.15), (6.18) and (6.20) are computed by employing the corresponding Gauss quadrature formulas

$$c_{jm} \approx \frac{2}{N+1}\sum_{l=1}^{N} \sin\frac{l\pi}{N+1} \sin\frac{l\pi(m+1)}{N+1} g_j\left(k\cos\frac{l\pi}{N+1}\right),$$

$$\Psi_j(k) \approx -\frac{4\pi k P_0}{2M+1}\sum_{l=1}^{M} x_l g_j(-k+2kx_l), \quad x_l = \cos^2\frac{(2l-1)\pi}{2(2M+1)},$$

$$\int_{-k}^{k} \frac{\chi_\pm(\tau)d\tau}{\sqrt{k^2-\tau^2}} \approx \frac{\pi}{L}\sum_{l=1}^{L} \chi_\pm\left(k\cos\frac{(2l-1)\pi}{2L}\right), \tag{6.21}$$

where N, M and L are the orders of the Gauss integration formulas.

Figure 2 represents the dimensionless SIFs $K_I' = \sqrt{a}P^{-1}K_I^+$ and $K_{II}' = \sqrt{a}P^{-1}K_{II}^+$ and the dimensionless energy release rate $(K_I')^2 + (K_{II}')^2$ at the crack tip $x = a$ for $\nu = 0.25$. Figure 3 shows the dependence of the SIF K_{II}' on the parameter k for some values of the Poisson ratio. At the left crack tip, $x = -a$, for the symmetric case (6.11), $K_I^- = K_I^+$, $K_{II}^- = -K_{II}^+$. Referring to these figures, we observe the effect of the ratio $k = b/a$ on the SIFs. As the inclusion ends approach the crack tips, the Mode-I SIFs $K_I^\pm$ grow to $+\infty$, while the Mode-II SIF K_{II}^+ decreases to zero for k close to 0.95 and then, as $k \to 1$, $K_{II}^+ \to -\infty$.

We next write down expressions for the traction in the contact line between the inclusion and the matrix and the tangential derivative of the displacement vector on the upper and lower sides of the crack. For the case (6.11), they are

$$\sigma_-(at) = \frac{i(1-\nu)^2 P}{2\pi a \nu_1 \sqrt{1-t^2}} - \chi(t)\left[\Psi_1(t) + \sqrt{\frac{1-k^2}{1-t^2}}\Psi_1(k)\right.$$

$$\left. + \frac{\Psi_2(t)-\Psi_2(k)}{\sqrt{1-t^2}} - \frac{P}{2\pi a}\left(\cos\beta_0 + \frac{(t-k)\sin\beta_0 + \sqrt{1-k^2}\cos\beta_0}{\sqrt{1-t^2}}\right)\right], \quad |t| < k. \tag{6.22}$$

and

$$Ew_+(at) = \frac{1}{\pi}\int_{-k}^{k}\left(-1 + \sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\psi(a\tau)d\tau}{\tau - t} - \frac{(1-\nu)P}{2\pi a\sqrt{1-t^2}}, \quad |t| < 1,$$

$$Ew_-(at) = i(1-\nu)\psi(at) - \frac{1}{\pi}\int_{-k}^{k}\left(1 + \sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\psi(a\tau)d\tau}{\tau - t} + \frac{(1-\nu)P}{2\pi a\sqrt{1-t^2}}, \quad |t| < 1, \tag{6.23}$$

where $\chi(t)$ is given by (5.20). Due to the factor $-1+\sqrt{(1-\tau^2)/(1-t^2)}$ the displacement derivative $w_+(at)$ on the upper side of the crack is not singular, and the upper profile of the crack is smooth. The traction $\sigma_-(at)$ and the displacement derivative $w_-(at)$ have the square root singularity at the points $t = \pm k$ and oscillate in a neighborhood of these points. This oscillation occurs due to the presence of the function $\chi(t)$ in (6.22) and (5.21). Analysis of the function $\chi(t)$ shows that it is monotonic everywhere in the interval $(-k+\varepsilon, k-\varepsilon)$, where $0 < \varepsilon \leq 10^{-6}$, and oscillates in a small neighborhood of the points $\pm k$. This implies the oscillation of the functions $\sigma_-(at)$ and and $w_-(at)$ only in small neighborhoods $k - \varepsilon < |t| < k$ and $k < |t| < k + \varepsilon$, respectively.

Figures 4 and 5 show sample curves of the traction components in the right-hand contact zone $0 < x < b, y = 0^-$. Due to the symmetry, $\sigma_{12}(x, 0^-) = -\sigma_{12}(-x, 0^-)$ and $\sigma_{12}(x, 0^-) = \sigma_{22}(-x, 0^-)$. It is seen that the absolute values of both contact stresses attain minima at the point $x = 0$ and grow as $x \to b^-$ $(t \to k^-)$.

6.3 *Limiting case* k → *0*

To test the numerical results obtained, we derive explicit formulas for the SIFs in the limiting case $k \to 0$ by passing to the limit in (6.7). In the case (6.11), formula (6.7) becomes

$$K_{II}^{\pm} + iK_I^{\pm} = \frac{1}{2\sqrt{\pi a}}\left[a\int_{-1}^{1}\left(\frac{1+k\xi}{1-k\xi}\right)^{\pm 1/2} k\psi(b\xi)d\xi \pm \frac{(1-\nu)P}{2}\right], \tag{6.24}$$

where due to (5.21) the function $k\psi(b\xi)$ has the form

$$k\psi(b\xi) = -\frac{i(1-\nu)kg(k\xi)}{\nu_1} + \chi^\circ(\xi)\left[\Psi_1(k\xi) + \sqrt{\frac{1-k^2}{1-k^2\xi^2}}\Psi_1(k)\right.$$

$$\left. + \frac{\Psi_2(k\xi)-\Psi_2(k)}{\sqrt{1-k^2\xi^2}} - \frac{P}{2\pi a}\left(\cos\beta_0 + \frac{k(\xi-1)\sin\beta_0 + \sqrt{1-k^2}\cos\beta_0}{\sqrt{1-k^2\xi^2}}\right)\right], \tag{6.25}$$

and

$$\chi^\circ(\xi) = -\frac{2i}{(1+\nu)\sqrt{\nu_0}}(1-\xi)^{-1/2-2i\gamma}(1+\xi)^{-1/2+2i\gamma}R^{1/4-i\gamma}(k\xi). \tag{6.26}$$

It is directly verified that $\beta_0 \to 1$ and $\Psi_j(k) \to 0$ as $k \to 0^+$ and also that the functions $R(k\xi)$ and $\Psi_j(k\xi)$ uniformly with respect to $\xi \in [-1, 1]$ tend to 1 and 0, respectively. Passing to the limit $k \to 0^+$ in (6.24) we deduce

$$\lim_{k\to 0^+}(K_{II}^\pm + iK_I^\pm) = \frac{iP}{\pi\sqrt{\pi a\nu_0}(1+\nu)}\int_{-1}^{1}(1-\xi)^{-1/2-2i\gamma}(1+\xi)^{-1/2+2i\gamma}d\xi \pm \frac{(1-\nu)P}{4\sqrt{\pi a}}, \tag{6.27}$$

This ultimately yields that the normalized SIFs $K_I^\pm$ in the limiting case $k \to 0$ are independent of the Poisson ratio and given by

$$\sqrt{a}P^{-1}K_I^\pm = \frac{1}{2\sqrt{\pi}} = 0.2820948\ldots. \tag{6.28}$$

The SIFs $K_{II}^\pm$ as $k \to 0$ become

$$\sqrt{a}P^{-1}K_{II}^\pm = \pm\frac{1-\nu}{4\sqrt{\pi}}. \tag{6.29}$$

Formulas (6.28) and (6.29) coincide with the formulas for the SIFs derived in (**15**) for a crack $\{-a < x < a, y = 0^\pm\}$ whose upper side is free of traction and the lower side is subjected to the normal concentrated load P applied at the center, $\sigma_{12}(x, 0^-) = 0$, $\sigma_{22}(x, 0^-) = -P\delta(x)$.

Notice that formulas (6.28) and (6.29) are consistent with the values of the SIF K_I^+ and K_{II}^+ for small values of the parameter k obtained from formula (6.7) in the case (6.11): if $k = 0.005$, then $\sqrt{a}P^{-1}K_I^+ = 0.2820933$ for $\nu = 0.05$ and $\sqrt{a}P^{-1}K_I^+ = 0.2821035$ for $\nu = 0.45$. For $\nu = 0.05$, $\nu = 0.25$ and $\nu = 0.45$, formula, (6.29) gives the following values of the dimensionless SIF $\sqrt{a}P^{-1}K_{II}^+$: 0.1339950, 0.1057855 and 0.07757607, respectively. On the other hand, the corresponding values of this factor for $k = 0.005$ computed from formula (6.7) in the case (6.11) are 0.1335321, 0.1054322 and 0.07732310; they are in good agreement with the limiting values for $k = 0$.

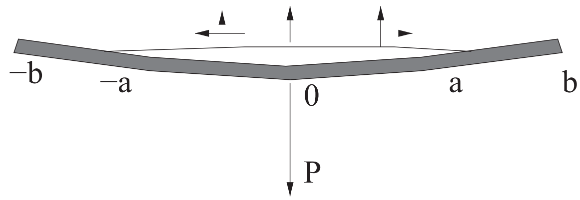

Fig. 6 The geometry of Model 2

7. Cases $a < b$ and $a = b$

7.1 *Model 2: an inclusion whose upper surface is partly separated from the matrix*

Suppose now that the crack length is less than that of the inclusion that is $a < b$ (Fig. 6). In the notations (3.11), the boundary conditions (2.1) should be replaced by

$$w_-(x) = ih'(x) - w^\circ(x), \quad -b < x < b,$$
$$w_+(x) = ih'(x) - w^\circ(x), \quad a < |x| < b,$$
$$\sigma_+(x) = -\sigma^\circ(x), \quad -a < x < a. \tag{7.1}$$

We will show that this problem reduces to a single singular integral equation with respect to the function $\phi(x)$ in the segment $(-a, a)$, and the structure of the equation is the same as that of (3.22). Rewrite first the representations (3.13) as

$$\frac{1}{\pi}\int_{-b}^{b}\frac{\psi(\xi)d\xi}{\xi - x} = -\frac{2}{\nu_1}[\phi(x) + 2Ew_-(x)] - \frac{i(1-\nu)}{\pi\nu_1}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi - x}, \quad -b < x < b,$$

$$2\sigma^\circ(x) = -\psi(x) + \frac{i(1-\nu)}{\nu_1}[\phi(x) + 2Ew_-(x)] - \frac{2}{\pi\nu_1}\int_{-a}^{a}\frac{\phi(\xi)d\xi}{\xi - x}, \quad -a < x < a, \tag{7.2}$$

where $\nu_1 = (3-\nu)(1+\nu)$ is the parameter introduced in (3.10). Inverting the Cauchy integral in the left-hand side of the first equation in (7.2) and applying the Poincaré–Bertrand formula and the inclusion equilibrium condition (3.24) similarly to section 3 we express the function $\psi(x)$ through $\phi(x)$. We have

$$\psi(x) = \frac{2}{\pi\nu_1\sqrt{b^2 - x^2}}\int_{-b}^{b}\frac{\sqrt{b^2-\xi^2}}{\xi - x}[\phi(\xi) + 2Ew_-(\xi)]d\xi - \frac{i(1-\nu)}{\nu_1}\phi(x) + \frac{iP_*}{\pi\sqrt{b^2-x^2}}. \tag{7.3}$$

Upon substituting (7.3) into the second equation in (7.2) we eventually derive the governing integral equation for the function $\phi(x)$. It reads

$$\frac{1}{\pi}\int_{-k_0}^{k_0}\left(1 + \sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\phi(b\tau)d\tau}{\tau - t} - i(1-\nu)\phi(bt) = g_0(t), \quad -k_0 < t < k_0, \tag{7.4}$$

where $k_0 = 1/k = a/b < 1$,

$$g_0(t) = -\nu_1\sigma^\circ(bt) - E(1-\nu)[h'(bt) + iw^\circ(bt)]$$

$$-\frac{2E}{\pi\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-t}[ih'(b\tau) - w^\circ(b\tau)]d\tau - \frac{iP_*\nu_1}{2\pi b\sqrt{1-t^2}}. \tag{7.5}$$

The function $\phi(at)$ has to satisfy the additional condition (3.16). It is seen that the governing integral equation (7.4) coincides with (3.22) if we replace $\psi(at)$ by $\phi(bt)$, k by $1/k$, and $g(t)$ by $g_1(t)$. Instead of the additional condition (3.24) for the function ψ we now have the condition (3.16) for the function ϕ. This circumstance changes the condition (4.7) for

$$\Phi_1(z) = O(z^{-2}), \quad z \to \infty, \tag{7.6}$$

and, therefore, from (4.28) and (4.29) we derive

$$C_0 = 0, \quad C_1 = \sqrt{1-k_0^2}\,\Psi_1(k_0) - \Psi_2(k_0). \tag{7.7}$$

The solution of the singular integral equation (7.4), the derivative of the complex displacement vector on the upper side of the crack, becomes

$$\phi(bt) = \frac{(1-\nu)g_0(t)}{i\nu_1} + \chi(t)\left[\Psi_1(t) + \sqrt{\frac{1-k_0^2}{1-t^2}}\,\Psi_1(k_0) + \frac{\Psi_2(t) - \Psi_2(k_0)}{\sqrt{1-t^2}}\right], \quad -k_0 < t < k_0. \tag{7.8}$$

7.2 *An inclusion whose upper surface is completely separated from the matrix: passage to the limit $k \to 1$*

Consider now the case when the whole upper side of the inclusion is debonded from the matrix that is when there is a crack $-a < x < a$, $y = 0^+$ between the inclusion $-a < x < a$, $y = 0^-$ and the matrix $\mathbb{R}^2 \setminus \{-a < x < a, y = 0\}$. The governing equation in this case can be obtained from (3.22):

$$\frac{1}{\pi}\int_{-1}^{1}\left(1 + \sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\psi(a\tau)d\tau}{\tau-t} - i(1-\nu)\psi(at) = g(t), \quad -1 < t < 1, \tag{7.9}$$

or from (7.4) by putting $k = 1$

$$\frac{1}{\pi}\int_{-1}^{1}\left(1 + \sqrt{\frac{1-\tau^2}{1-t^2}}\right)\frac{\phi(a\tau)d\tau}{\tau-t} - i(1-\nu)\phi(at) = g_1(t), \quad -1 < t < 1. \tag{7.10}$$

The former equation, when solved, provides the jump of the traction vector, while the latter yields the function $\phi(x)$, the jump of the tangential derivative of the displacement vector. There are other ways to solve the problem in this case. Before we implement passage to the limit $k \to 1$ in the solution derived in sections 4 and 5, we briefly describe a method for the system (3.13) based on its decoupling. Since the traction vector is prescribed in the whole segment $-a < x < a$, $\sigma_+(x) = -\sigma^\circ(x)$, and the displacement vector is known on the whole lower side of the inclusion, $w_-(x) = ih'(x) - w^\circ(x)$, $-a < x < a$, the relations (3.13) allow us to deduce the following governing system:

$$\boldsymbol{\omega}(x) + \frac{A}{2\pi}\int_{-a}^{a}\frac{\boldsymbol{\omega}(\xi)d\xi}{\xi - x} = \mathbf{g}(x), \quad -a < x < a, \tag{7.11}$$

where

$$A = \begin{pmatrix} i(1-\nu) & 1 \\ \nu_1 & i(1-\nu)\end{pmatrix}, \quad \boldsymbol{\omega}(x) = \begin{pmatrix}\psi(x) \\ \phi(x)\end{pmatrix}, \quad \mathbf{g}(x) = -2\begin{pmatrix}\sigma^\circ(x)(x) \\ E[ih'(x) - w^\circ(x)]\end{pmatrix}. \tag{7.12}$$

The matrix A has constant entries, and it is diagonalizable, $A = BDB^{-1}$, where D is a diagonal matrix, and B is a nonsingular matrix of transformation. Therefore, the system (7.11) reduces to two scalar singular integral equations that admit a closed-form solution by reducing them to the associated scalar Riemann–Hilbert problems in the segment $-1 < t < 1$. This method was applied for example in (**4**) to solve the problem in the case $a = b$ for an inclusion in the interface of a composite plane. Here, we give an alternative solution by solving the integral equation (7.9) by passing to the limit $k \to 1$ in the representation formulas for the solution in the case $k \in (0, 1)$.

Following the scheme of section 4 we reduce the singular integral equation (7.9) to the vector Riemann–Hilbert problem (4.5), (4.6) on the interval $(-1, 1)$ and factorize the matrix $G(t)$ by formulas (4.8)–(4.10) with the functions $\Lambda(z)$ and $\beta(z)$ being the solutions of the scalar Riemann problems (4.12) and (4.13), respectively. The solution of the former problem is the one given by (4.15) with $k = 1$,

$$\Lambda(z) = (z-1)^{-3/4-i\gamma}(z+1)^{-1/4+i\gamma}, \quad \gamma = \frac{\ln \nu_0}{4\pi}, \tag{7.13}$$

and the single branch is fixed by the condition $\Lambda(z) \sim z^{-1}$, $z \to \infty$, in the plane cut along the segment $[-1, 1]$ passing through the point $z = 0$. In particular, when $z = t \pm i0$ and $-1 < t < 1$, we have

$$\Lambda^\pm(t) = -e_0^{\pm 1}(1-t)^{-3/4-i\gamma}(1+t)^{-1/4+i\gamma}, \quad -1 < t < 1, \tag{7.14}$$

$\arg(1 \pm t) = 0$, $t \in (-1, 1)$, and $e_0 = e^{\pi i/4 + \pi\gamma}$ is the parameter introduced in section 5.

Let now $k \to 1^-$ in (5.11). Since $R(t) \to (1+t)/(1-t)$ as $k \to 1^-$ and $-1 < t < 1$, we deduce

$$\sqrt{1-t^2}\,\beta^\pm(t) = \pm\pi i\left(\frac{1}{4} - i\gamma\right), \tag{7.15}$$

and the Wiener–Hopf matrix factors in (4.8) become

$$X^\pm(t) = \Lambda^\pm(t)\begin{pmatrix} e_+ & \pm e_-(1-t^2)^{-1/2} \\ \pm e_-(1-t^2)^{1/2} & e_+\end{pmatrix}, \quad -1 < t < 1, \quad e_\pm = \frac{1}{2}(e_0 \pm e_0^{-1}). \tag{7.16}$$

We assert next that as $k \to 1^-$ in (4.20), $\beta_0 \to \pi(-\frac{1}{4} + i\gamma)$, $\cos\beta_0 \to e_+$, $\sin\beta_0 \to ie_-$, and, therefore, the matrices $[X(z)]^{\pm 1}$ have the following asymptotics at infinity:

$$[X(z)]^{\pm 1} \sim z^{\mp 1}\begin{pmatrix} e_+ & \pm z^{-1} ie_- \\ \mp z ie_- & e_+\end{pmatrix}, \quad z \to \infty. \tag{7.17}$$

Since the behavior of the matrix $X(z)$ at the infinite point is the same as before, the solution of the vector Riemann–Hilbert problem is given by (4.26) that reads in the limiting case $k \to 1^-$

$$\mathbf{\Phi}(z) = X(z)\left[\mathbf{\Psi}(z) + \begin{pmatrix} C_0 e_+ \\ iC_0 e_- z + C_1 \end{pmatrix}\right], \quad z \in \mathbb{C} \setminus [-1, 1], \tag{7.18}$$

where $C_0 = -P_*/(2\pi a)$, and $\mathbf{\Psi}(z)$ is simplified to the form

$$\mathbf{\Psi}(z) = -\frac{1}{2\pi(\nu+1)e_0}\int_{-1}^{1}\frac{\mathbf{T}(\tau)g(\tau)d\tau}{\Lambda^+(\tau)(\tau - z)}. \tag{7.19}$$

It is seen that the first component of the vector $\mathbf{\Phi}(z)$, the function $\Phi_1(z)$, has a nonintegrable singularity of order $-5/4$ at the point $z = 1$ unless

$$C_1 = -\Psi_2(1) + \frac{ie_- P_*}{2\pi a}. \tag{7.20}$$

This condition is necessary and sufficient for the function $\Phi_1(t)$ being integrable in the vicinity of the point $z = 1$. Notice that the condition (7.20) coincides with (4.28) when $k = 1$. Now taking the limit $k \to 1$ in the representation formula (5.21) for the solution of the integral equation (3.22) we obtain

$$\psi(at) = \frac{i(\nu-1)g(t)}{\nu_1} - \frac{2i}{(1+\nu)\sqrt{\nu_0}}(1-t)^{-3/4-i\gamma}(1+t)^{-1/4+i\gamma}$$
$$\times\left[\Psi_1(t) + \frac{\Psi_2(t)-\Psi_2(1)}{\sqrt{1-t^2}} - \frac{P_*}{2\pi a}\left(e_+ - ie_-\sqrt{\frac{1-t}{1+t}}\right)\right], \quad -1 < t < 1. \tag{7.21}$$

Utilizing formula (7.19) we find the principal values of the integrals $\Psi_1(t)$ and $(1-t^2)^{-1/2}[\Psi_2(t) - \Psi_2(1)]$. They read

$$\Psi_1(t) = \frac{1}{2\pi(\nu+1)e_0^2}\int_{-1}^{1}\frac{\omega_1(\tau)g(\tau)d\tau}{\tau - t},$$
$$\frac{\Psi_2(t)-\Psi_2(1)}{\sqrt{1-t^2}} = \frac{1}{2\pi(\nu+1)e_0^2}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\frac{\omega_2(\tau)g(\tau)d\tau}{\tau - t}, \quad -1 < t < 1, \tag{7.22}$$

where

$$\omega_1(t) = (1-t)^{3/4+i\gamma}(1+t)^{1/4-i\gamma}, \quad \omega_2(t) = (1-t)^{1/4+i\gamma}(1+t)^{3/4-i\gamma}. \tag{7.23}$$

We next substitute formulas (7.22) into (7.21). The result,

$$\psi(at) = \frac{i(\nu-1)g(t)}{\nu_1} - \frac{1}{\pi\omega_1(t)}\left[\frac{1}{\nu_1}\int_{-1}^{1}\frac{\omega_1(\tau)g(\tau)d\tau}{\tau - t} - \frac{iP_* e_+}{a(1+\nu)\sqrt{\nu_0}}\right]$$
$$-\frac{1}{\pi\omega_2(t)}\left[\frac{1}{\nu_1}\int_{-1}^{1}\frac{\omega_2(\tau)g(\tau)d\tau}{\tau - t} - \frac{P_* e_-}{a(1+\nu)\sqrt{\nu_0}}\right], \tag{7.24}$$

is the exact solution to the integral equation (7.9), the limit of the solution (5.21) as $k \to 1$.

7.3 *The limiting case $k = 1$ when $w^\circ = 0$, $\sigma^\circ = 0$ and $h = const$*

Set now $w^\circ(x) = 0$, $h'(x) = 0$, $-b < x < b$ and $\sigma^\circ(x) = 0$, $-a < x < a$. This case was treated in (**2**) by the method of complex potentials. We aim to simplify the representation formula for the solution (7.24) derived for the limiting case $k = 1$. We have $P_* = P$, and formula (7.24) reads

$$\psi(at) = -\frac{P}{2\pi a}\left[\frac{i(1-\nu)^2}{\nu_1\sqrt{1-t^2}} - \frac{2}{(1+\nu)\sqrt{\nu_0}}\left(\frac{ie_+}{\omega_1(\tau)} + \frac{e_-}{\omega_2(\tau)}\right)\right.$$

$$\left. + \frac{1-\nu}{\pi\nu_1}\left(\frac{1}{\omega_1(t)}\int_{-1}^{1}\frac{\omega_1(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-t)} + \frac{1}{\omega_2(t)}\int_{-1}^{1}\frac{\omega_2(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-t)}\right)\right]. \tag{7.25}$$

The two integrals in (7.25) can be evaluated by means of the relation

$$\int_{-1}^{1}\frac{(1-\tau)^\alpha(1+\tau)^{-\alpha}d\tau}{\tau-t} = \pi\cot\pi\alpha\left(\frac{1-t}{1+t}\right)^\alpha - \frac{\pi}{\sin\pi\alpha}, \quad -1 < t < 1, \tag{7.26}$$

a particular case of the more general formula (**16**)

$$\frac{1}{\pi}\int_{-1}^{1}(1-\tau)^\alpha(1+\tau)^\beta P_n^{(\alpha,\beta)}(\tau)\frac{d\tau}{\tau-t} = \cot\pi\alpha(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)$$

$$-\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(n+\beta+1)}{\pi\Gamma(n+\alpha+\beta+1)}F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-t}{2}\right),$$

$$\alpha > -1, \quad \beta > -1, \quad \alpha \neq 0, 1, 2, \ldots, \quad -1 < t < 1, \tag{7.27}$$

obtained by employing the integral representation of the Jacobi functions of the second kind in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$. Here, F is the hypergeometric function. Alternatively, formula (7.27) can be derived by converting the left-hand side into an integral of the Mellin convolution type and applying the theory of residues (**4**).

On plugging the expressions obtained into formula (7.25) and using the identities

$$\frac{e_-}{e_+} + \frac{e_+}{e_-} = 1 - \nu,$$

$$\frac{\nu-1}{\nu_1 e_\pm} - \frac{2ie_\pm}{(1+\nu)\sqrt{\nu_0}} = \pm\frac{1}{(1+\nu)e_0^2 e_\pm} \tag{7.28}$$

we deduce

$$\psi(at) = -\frac{P}{2\pi a(1+\nu)\sqrt{\nu_0}}\left[\frac{1}{e_-}(1-t)^{-1/4-i\gamma}(1+t)^{-3/4+i\gamma} - \frac{i}{e_+}(1-t)^{-3/4-i\gamma}(1+t)^{-1/4+i\gamma}\right], \tag{7.29}$$

where the parameters $e_\pm$ and γ are given by (7.16) and (4.15), respectively. It is directly verified that this function satisfies the equilibrium condition (3.24) ($k = 1$). To show that the function (7.29) is a solution of (7.9), in addition to the relation (7.26), we employ the formula

$$\int_{-1}^{1}\frac{(1-\tau)^{\alpha}(1+\tau)^{-\alpha-1}d\tau}{\tau-t}=\pi\cot\pi\alpha(1-t)^{\alpha}(1+t)^{-\alpha-1},\quad -1<t<1, \tag{7.30}$$

that can be deduced from (7.27). Then, upon substitution, one can check that the function (7.29) solves the integral equation (7.9) with the function $g(t)$ given by (6.11).

8. Energy release rate

The potential energy $2\delta U$ released when the crack $C=\{|x|<a, y=0\}$ extends symmetrically to $C+\delta C=\{|x|<a+r, y=0\}$, and r is small, due to the symmetry, can be expressed as

$$\delta U=\frac{1}{2}\int_{a}^{a+r}\{\sigma_{12}(x,0)\delta[u_1](x)+\sigma_{22}(x,0)\delta[u_2](x)\}dx, \tag{8.1}$$

where $[u_1]+\delta[u_1]$ and $[u_2]+\delta[u_2]$ are the displacement jumps related to the extended crack. For Model 1, when the crack is spread out into the matrix ($a>b$), the stresses have the square root singularity at the crack tip $x=a$,

$$\sigma_{12}(x,0)\sim\frac{K_{II}^{+}}{\sqrt{2\pi(x-a)}},\quad \sigma_{22}(x,0)\sim\frac{K_{I}^{+}}{\sqrt{2\pi(x-a)}},\quad x\to a^{+}, \tag{8.2}$$

while the displacement jumps vanish at the extended crack tip $x=a+r$. Due to (6.5) we have

$$\delta[u_1](x)+i\delta[u_2](x)\sim\frac{8(K_{II}^{+}+iK_{I}^{+})}{\sqrt{2\pi}E}(a+r-x)^{1/2},\quad x-a\to r^{-}. \tag{8.3}$$

On substituting these relations into (8.1) and computing the integral we obtain the classical result

$$\delta U\sim[(K_{I}^{+})^2+(K_{II}^{+})^2]\frac{r}{E},\quad r\to 0^{+}, \tag{8.4}$$

and according to the Griffith criterion the crack starts growing if the energy $\delta U\geq 2Tr$, where $2Tr$ is the increase of the surface energy, and T is the Griffith material constant. Therefore, the crack growth criterion has the form

$$(K_{I}^{+})^2+(K_{II}^{+})^2\geq 2TE. \tag{8.5}$$

In Fig. 2, the dimensionless energy release rate $aEP^{-2}d/dr\delta U=aP^{-2}[(K_{I}^{+})^2+(K_{II}^{+})^2]$ is plotted as a function of $k=b/a$. It is seen that it is decaying as a is growing while b is fixed.

In the case of Model 2, the determination of the energy release rate is not so simple. Analysis of the function $\phi(x)$ given by (7.8) as $x\to a^{-}$ ($t\to k_0^{-}$) yields

$$\phi(x)\sim a^{2i\gamma}\phi'(a-x)^{-2i\gamma-1/2},\quad x\to a^{-}, \tag{8.6}$$

where

$$\phi'=-\frac{4i(1-k_0^2)^{-1/4+i\gamma}2^{2i\gamma}\sqrt{b}}{(1+\nu)\sqrt{2k_0\nu_0}}\Psi_1(k_0). \tag{8.7}$$

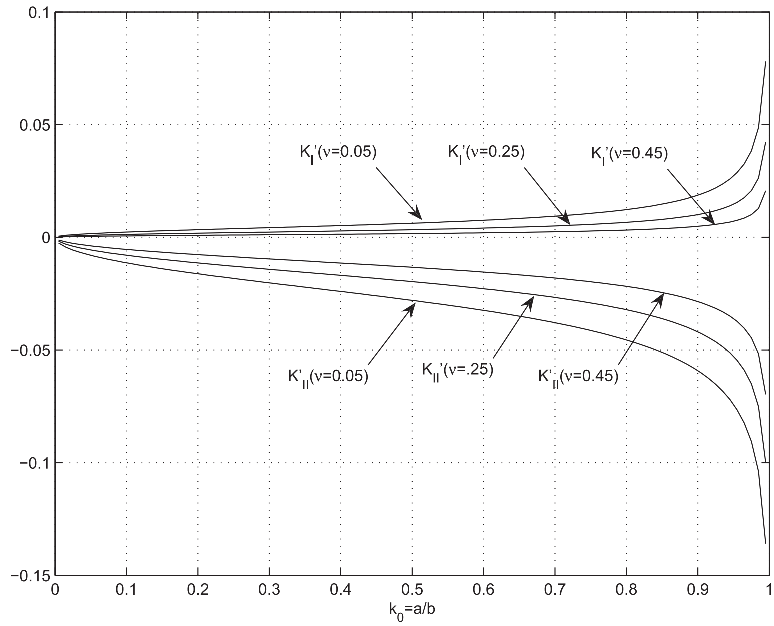

Fig. 7 Model 2: the dimensionless SIFs K'_I and K'_{II} at the crack tip $x = a$ versus $k = b/a$ for some values of ν

By integrating (8.6) we establish

$$\delta[u_1](x) + \delta[u_2](x) \sim \frac{(r + a - x)^{-2i\gamma+1/2} a^{2i\gamma} \phi'}{(2i\gamma - 1/2)E}, \quad x - a \to r^-. \tag{8.8}$$

Determine now the asymptotics of the stresses at $x = a$. Referring to (7.3) we have

$$\sigma_+(x) \sim \frac{2a^{2i\gamma}\phi'}{\pi \nu_1} \int_{-a}^{a} \frac{(a - \xi)^{-2i\gamma-1/2} d\xi}{\xi - x}, \quad x \to a^+. \tag{8.9}$$

If we employ the asymptotic relation for the singular integral

$$\int_0^1 \frac{\tau^\beta d\tau}{\tau + t} = -\frac{\pi t^\beta}{\sin \pi\beta} + \Omega(t), \quad t \to 0^-, \tag{8.10}$$

where $-1 < \operatorname{Re} \beta < 0$ and $\Omega(t)$ is a function bounded in a neighborhood of the point $t = 0$ and having a finite limit as $t \to 0^-$. Consequently, the complex stress $\sigma_+(x)$ has the asymptotics

$$\sigma_+(x) \sim -\frac{2a^{2i\gamma}\phi'(x - a)^{-2\gamma-1/2}}{\nu_1 \cosh 2\pi\gamma}, \quad x \to a^+. \tag{8.11}$$

Define the SIFs in the oscillatory case as

$$\sigma_+(x) \sim \frac{K_{II}^+ + iK_I^+}{\sqrt{x - a}} \left(\frac{x}{a} - 1\right)^{-2i\gamma}, \quad x \to a^+. \tag{8.12}$$

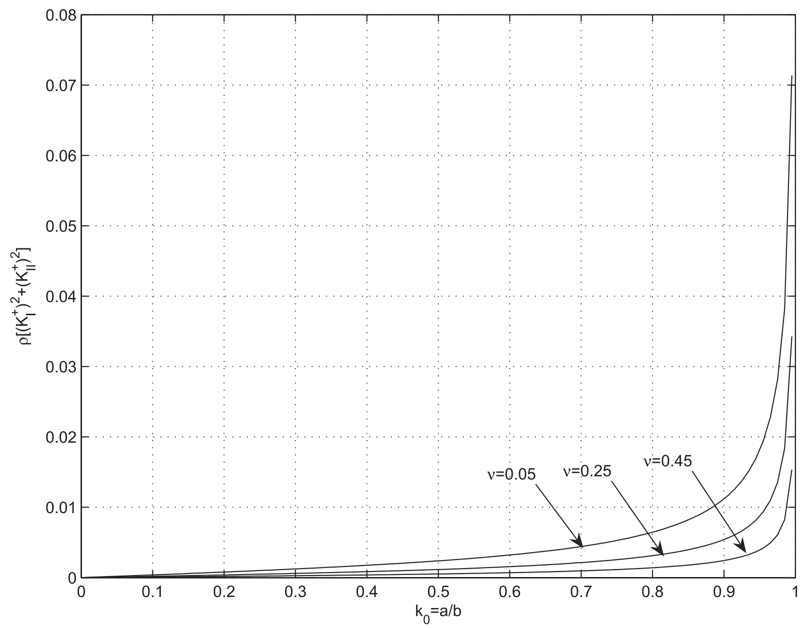

Fig. 8 Model 2: the dimensionless energy release rate $Ed/dr\delta U = \rho[(K_I^+)^2 + (K_{II}^+)^2]$ at the crack tip $x = a$ versus $k = b/a$ for some values of ν

Then the coefficient ϕ' can be expressed through the complex SIF $K^+ = K_{II}^+ + iK_I^+$ as

$$\phi' = -\frac{\nu_1 \cosh 2\pi\gamma}{2} K^+. \tag{8.13}$$

Introduce, for brevity, the notations

$$A(x) = 2\gamma \ln \frac{x-a}{a}, \quad B(x) = 2\gamma \ln \frac{r+a-x}{a}. \tag{8.14}$$

Then

$$\sigma_{12}(x,0) + i\sigma_{22}(x,0) \sim [(K_I^+ \sin A + K_{II}^+ \cos A) + i(K_I^+ \cos A - K_{II}^+ \sin A)](x-a)^{-1/2}, \quad x \to a^+,$$

$$\delta[u_1](x) + i\delta[u_2](x) \sim \{[K_I^+(\sin B - 4\gamma \cos B) + K_{II}^+(\cos B + 4\gamma \sin B)]$$

$$+i[K_I^+(\cos B + 4\gamma \sin B) - K_{II}^+(\sin B - 4\gamma \cos B)]\} \frac{\nu_1 \cosh 2\pi\gamma}{(16\gamma^2+1)E}(r+a-x)^{1/2}, \quad x - a \to r^-. \tag{8.15}$$

Table 1 The values of the parameters M and ρ for some values of the Poisson ratio ν

ν	0.05	0.15	0.25	0.35	0.45
M	1.53172	1.54054	1.54790	1.55401	1.55902
ρ	2.43276	2.57416	2.69984	2.80980	2.90405

On substituting these expressions into formula (8.1) for the potential energy increment we finally discover

$$\delta U \sim \frac{\rho r}{E}[(K_I^+)^2 + (K_{II}^+)^2], \quad r \to 0^+, \tag{8.16}$$

where ρ is the dimensionless parameter given by

$$\rho = \frac{\nu_1 \cosh 2\pi\gamma}{2(16\gamma^2+1)} M,$$

$$M = \int_0^1 \sqrt{\frac{1-t}{t}} \left[\cos\left(2\gamma \ln \frac{1-t}{t}\right) + 4\gamma \sin\left(2\gamma \ln \frac{1-t}{t}\right)\right] dt. \tag{8.17}$$

It turns out that the integral M and, therefore, the parameter ρ are positive for all $\nu \in (0, \frac{1}{2}]$. Some of their values are given in Table 1.

Consequently, although the traction oscillates at a small neighborhood of the points $\pm a$, the potential energy release rate is always positive $\frac{d}{dr}\delta U = \rho E^{-1}[(K_I^+)^2 + (K_{II}^+)^2]$, and the Griffiths criterion of crack growth in the case of Model 2 becomes

$$(K_I^+)^2 + (K_{II}^+)^2 \geq \frac{2TE}{\rho}. \tag{8.18}$$

Figure 7 shows sample curves for the dimensionless SIFs $K_I' = \sqrt{b}P^{-1}K_I^+$ and $K_{II}' = \sqrt{b}P^{-1}K_{II}^+$ introduced in (8.12) and given by

$$K_{II}^+ + iK_I^+ = \frac{8i(1-k_0^2)^{-1/4+i\gamma} 2^{2i\gamma}\sqrt{b}}{\nu_1(1+\nu)\sqrt{2k_0\nu_0}\cosh 2\pi\gamma} \Psi_1(k_0). \tag{8.19}$$

In Fig. 8, we present the dependence of the dimensionless energy release rate normalized by E, $Ed/dr\delta U = \rho[(K_I^+)^2 + (K_{II}^+)^2]$, upon the parameter $k_0 = a/b$ for some values of the Poisson ratio ν.

9. Conclusions

In this work, we have analyzed two model contact problems on a rigid inclusion debonded from an elastic medium. Model 1 concerns an inclusion completely debonded from the matrix, and the crack formed in the upper side of the inclusion penetrates into the medium. In Model 2, the crack length

$2a$ is less than the inclusion length $2b$. Each model is governed by a singular integral equation with the same kernel but a different right-hand side. We have developed a method that ultimately leads to a closed-form solution of the integral equation. The main feature of the method is the solution of the associated order-2 vector Riemann–Hilbert problem with the Chebotarev–Khrapkov matrix coefficient in a finite segment, not in a closed contour (an infinite line) as the classical Khrapkov scheme (**10**) requires. We have examined the behavior of the solution at the crack and inclusion tips, determined the SIFs and the contact stresses and reported sample numerical results for them. To verify the numerical results, we obtained the SIFs for the limiting case $k = b/a \to 0$ independently. It turns out that the numerical values of both factors, K_I and K_{II}, computed from the general formulas for $k > 0$ tend to the limiting value for $k = 0^+$.

By passing to the limit $k \to 1$ in the solution found we have managed to derive a closed-form solution for the particular case $b = a$. That solution coincides with the one known in the literature (**2**). It turns out that the traction components have square root singularities at the crack and inclusion tips in the cases $a < b$ and $a > b$, while they have a stronger singularity of order $-3/4$ in the limiting case $a = b$. Also, when $a \to b^{\pm}$, the absolute values of the Mode-I and Mode-II SIFs grow to infinity. This fact allows us to conclude that the case $a = b$ is unstable: when the crack formed on the upper surface of the inclusion ($a < b$) starts growing and its tips approach the inclusion ends ($a \to b$ and $a < b$), the SIFs grow unboundly, and the crack tends to speed up to pass the inclusion tips and penetrate into the matrix. After that moment the SIFs decrease, and eventually the crack stops at some distance from the inclusion tips.

We have examined the behavior of the stresses and the tangential derivative of the displacement at the internal singular points, that is at the point $x = b$ for Model 1 and $x = a$ for Model 2. For both models, the stresses and the displacement derivative have a square root singularity at these points. In the first problem, the stresses are monotonic everywhere in the intervals $(-b, b)$ apart from small neighborhoods $k - \varepsilon < |x/a| < k$, $\varepsilon \in (0, 10^{-6})$, where they rapidly oscillate. The displacement derivative on the upper side of the crack is monotonic everywhere, while on the lower side it is monotonic everywhere in the intervals $b < |x| < a$ except for the small zones $k < |x/a| < k + \varepsilon$. For Model 2, when a crack $(-a, a)$ is located in the upper side of the inclusion $(-b, b)$ and $b > a$, the traction oscillates when $|x| \to a^+$, and the displacement derivative oscillates when $|x| \to a^-$. Despite the oscillatory behavior of the traction and the displacement derivative at the crack tips in Model 2 the potential energy $2\delta U$ released when the crack extends symmetrically by a small distance r is proportional to r. The energy release rate $d/dr\delta U$ has the form $E^{-1}\rho[(K_I^+)^2 + (K_{II}^+)^2]$ (ρ is a dimensionless parameter dependent on the Poisson ratio only), and the Griffith criterion modified accordingly may be applied to predict the crack growth.

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