METHOD OF AUTOMORPHIC FUNCTIONS FOR AN INVERSE PROBLEM OF ANTIPLANE ELASTICITY

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Summary

A nonlinear inverse problem of antiplane elasticity for a multiply connected domain is examined. It is required to determine the profile of n uniformly stressed inclusions when the surrounding infinite body is subjected to antiplane uniform shear at infinity. A method of conformal mappings of circular multiply connected domains is employed. The conformal map is recovered by solving consequently two Riemann-Hilbert problems for piecewise holomorphic automorphic functions generated by the Schottky symmetry group. A series-form representation of a $(3n-4)$-parametric family of conformal maps solving the problem is discovered. Numerical results for two and three uniformly stressed inclusions are reported and discussed.

1. Introduction

Considerable interest in inverse boundary value problems of the theory of holomorphic functions has been developed since the work by Riabuchinsky (**1**) on recovering the boundary of a domain on the basis of the prescribed values of a harmonic function and its normal derivative on the boundary of the domain. Numerous applications of inverse problems to filtration and hydroaerodynamics, also known as free boundary problems, advanced the development of a qualitative theory and various constructive techniques (**2**). The solution (**3**) to the problem on determination of the shapes of curvilinear cavities and inclusions with prescribed properties inspired materials scientists to work on inverse problems of elasticity. It was found (**3**) that if an unbounded elastic body is uniformly loaded at infinity and the body has an elliptic or ellipsoidal inclusion with different elastic constants, then the stress field is uniform inside the inclusion. Eshelby's conjecture that in plane and antiplane cases there are no other shapes apart from ellipses was proved in (**4**). Another proof based on the method of conformal mappings was later proposed in (**5**).

An inverse problem of plane elasticity for a plane uniformly loaded at infinity and having n holes was examined by Cherepanov (**6**). In this model, the holes boundaries are subjected to constant normal and tangential traction, and the boundaries have to be determined from the condition that the tangential normal stress is constant in all the contours. Cherepanov employed the method of conformal mappings and homogeneous Schwarz problems to recover the shapes of two symmetric holes. A circular map from the exterior of n-circles onto the n-connected elastic domain, integral equations, and the method of least squares for their numerical solution were proposed in (**7**). An explicit representation in terms of the Weierstrass elliptic function for the profile of an inclusion in the case of a doubly periodic structure was found in (**8**). Recently, the theory of the Cherepanov problem for n inclusions was advanced in (**9**) by developing a method of the Riemann–Hilbert problem on a

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Riemann surface. It was shown that for any $n \geq 1$ there always exists a set of the loading parameters which generate inadmissible poles of the solution.

For the antiplane inverse problem on reconstructing the boundaries of two symmetric uniformly stressed inclusions the Weierstrass zeta function and the Schwarz–Christoffel formula were found to be effective in (**10**). A method of Laurent series and a conformal mapping from an annulus to a doubly connected domain to recover the profile of two inclusions with uniform stresses was applied in (**11**). Different numerical approaches for inverse antiplane problems were employed in (**12**), (**13**).

Two methods for nonlinear inverse problems on supercavitating flow past n hydrofoils were proposed in (**14**), (**15**). Both methods are based on the existence theorem (**16**), (**17**) of a conformal map of an n connected parametric slit or circular domain into the n-connected physical domain. These methods express the conformal map in terms of the solutions to two Riemann–Hilbert problems in a multiply connected canonical domain. In the first method, the Riemann–Hilbert problems are set in n slits and reduce to two Riemann–Hilbert problems on a genus-n Riemann surface. The parametric domain for the second technique is the exterior of n circles. The method employs linear fractional transformations, a symmetry transformation and the Schottky groups (**18**) and leads to two Riemann–Hilbert problems of the theory of symmetric automorphic functions. The theory of such problems and the Schwarz problem as their particular case has been developed and presented in many publications including (**19**), (**20**), (**21**), (**22**), (**23**), (**24**), (**25**), (**15**), (**26**), (**27**). We also note that numerous conformal maps of n-connected canonical domains into physical domains exist in the literature when the boundary conditions of the problem allow for bypassing the Hilbert problem in the circular domain. Examples include the Schwarz–Christoffel map from a multiply connected circular domain into a multiply connected polygonal region in a series form (**28**), in terms of the Schottky–Klein prime function of the Schottky group associated with the circular domain (**29**). In (**30**), the Schwarz–Christoffel map was derived by means of the Riemann–Hilbert problem and successive approximations.

In this article, we aim to propose an exact method of conformal mappings and the Riemann–Hilbert problem of the theory of automorphic functions for the inverse antiplane problem on n inclusions. The inclusions may have different shear moduli and are in ideal contact with the surrounding elastic matrix subjected at infinity to uniform antiplane shear $\tau_{13} = \tau_1^\infty$ and $\tau_{23} = \tau_2^\infty$. The profiles of the inclusions are not prescribed and have to be determined from the condition that the stress field inside all the inclusions is uniform, $\tau_{13} = \tau_1$ and $\tau_{23} = \tau_2$.

In Section 2, we formulate the problem, map the exterior of n circles into the exterior of n uniformly stressed inclusions and reduce the problem on determination of the conformal map to two inhomogeneous Schwarz problems to be solved consecutively. In Section 3, we convert the Schwarz problems to two Riemann–Hilbert problems for piecewise holomorphic symmetric automorphic functions. For their solution we employ a qusiautomorphic analog of the Cauchy kernel (**21**), (**24**). Note that the Cauchy kernel analog (**25**) in terms of the Schottky–Klein prime function of the Schottky group could also be employed.

Another possibility to solve the problem is to choose the parametric domain as the exterior of n slits lying in the same line (the real axis for example). In this case, the slit map was recently constructed (**31**) by solving two Riemann–Hilbert problems on a symmetric genus-n Riemann surface. The advantage of such an approach is in its ability to recover the conformal map by quadratures in the cases of doubly and triply connected domains. However, when $n \geq 4$, since not each n-connected domain D^e can be considered as the image by a slit map with n-slits lying in the same line, the method of slit maps is in general inapplicable. At the same time, if $n \geq 4$ and the associated Schottky group is convergent, then the method of circular maps and the Riemann–Hilbert problems of the theory of automorphic functions works and gives a series representation of the conformal map. The set of domains associated with the first class Schottky group is broader and includes not only the set of n circles whose centers fall in the same line.

In Section 4, we write down a series representation of a family of conformal maps solving the problem. The family has $3n-4$ free parameters and should satisfy the natural restriction that the inclusions contours cannot overlap. We also give some sample profiles of two and three uniformly stressed inclusions and discuss the numerical results. In Appendix, for completeness, we examine the case $n=1$ and show that the profile of a single uniformly stressed inclusion is an ellipse.

2. Formulation

Consider the following problem of antiplane elasticity.

Let $D_0, D_1, \dots, D_{n-1}$ be n finite inclusions in an infinite isotropic solid. The shear moduli of the inclusions and the solid $D^e = \text{ext}D$ $(D = \cup_{j=0}^{n-1} D_j)$ are taken to be μ_j and μ, respectively. Suppose the inclusions are in ideal contact with the matrix, and the whole solid $D^e \cup D$ is in a state of antiplane shear due to constant shear stresses applied at infinity, $\tau_{13} = \tau_1^\infty$, $\tau_{23} = \tau_2^\infty$. It is aimed to determine the boundaries of the inclusions, L_j, such that the stresses τ_{13} and τ_{23} are constant in all the inclusions D_j, $\tau_{13} = \tau_1$, $\tau_{23} = \tau_2$, $j = 0, 1, \dots, n-1$.

Let u and u_j be the x_3-components of the displacement vectors for the body D^e and the inclusions D_j, respectiely. Then $\tau_{k3} = \mu \partial u/\partial x_k$ $(k = 1, 2)$, $(x_1, x_2) \in D^e$, and $\tau_{k3} = \mu_j \partial u/\partial x_k$ $(k = 1, 2)$, $(x_1, x_2) \in D_j$, $j = 0, 1, \dots, n-1$. At infinity, the displacement u is growing as

$$u \sim \mu^{-1}(\tau_1^\infty x_1 + \tau_2^\infty x_2) + \text{const}, \quad x_1^2 + x_2^2 \to \infty. \tag{2.1}$$

Due to the fact that the stresses τ_{12} and τ_{13} are constant in the inclusions, the x_3-displacements u_j for $(x_1, x_2) \in D_j$ are linear functions

$$u_j = \mu_j^{-1}(\tau_1 x_1 + \tau_2 x_2) + d_j', \quad (x_1, x_2) \in D_j, \quad j = 0, 1, \dots, n-1, \tag{2.2}$$

and d_j' are real constants.

Denote $z = x_1 + ix_2$ and suppose v and v_j are the harmonic conjugates of the harmonic functions u and u_j in the domains D^e and D_j, respectively. Then $\phi(z) = u(x_1, x_2) + iv(x_1, x_2)$ and $\phi_j(z) = u_j(x_1, x_2) + iv_j(x_1, x_2)$ are holomorphic functions in the corresponding domains D^e and D_j. The boundary conditions of ideal contact imply that the traction component $\tau_{\nu 3}$ and the x_3-component of the displacement are continuous through the contours L_j,

$$\mu \frac{\partial u}{\partial \nu} = \mu_j \frac{\partial u_j}{\partial \nu}, \quad u = u_j, \quad (x_1, x_2) \in L_j, \quad j = 0, 1, \dots, n-1, \tag{2.3}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative. In terms of the functions $\phi_j(z)$ the ideal contact boundary conditions can be written as (**5**)

$$\frac{\kappa_j + 1}{2}\phi_j(z) - \frac{\kappa_j - 1}{2}\overline{\phi_j(z)} = \phi(z) + ib_j, \quad z \in L_j, \quad j = 0, 1, \dots, n-1, \tag{2.4}$$

where $\kappa_j = \mu_j/\mu$, and b_j are arbitrary real constants. The equivalence of the boundary conditions (2.3) and (2.4) follows from the Cauchy–Riemann condition $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial s}$, where $\frac{\partial}{\partial s}$ is the tangential derivative. Since the functions u_j are known everywhere in the domains D_j, the functions $\phi_j(z)$ are defined up to arbitrary constants and given by

$$\phi_j(z) = \frac{\bar{\tau}z}{\mu_j} + d_j, \quad z \in D_j, \quad j = 0, 1, \ldots, n-1, \tag{2.5}$$

where $\bar{\tau} = \tau_1 - i\tau_2$, $d_j = d'_j + id''_j$, d''_j are real constants. In view of the relations (2.5), instead of the function $\phi(z)$, it is convenient to deal with the function

$$f(z) = \phi(z) - \bar{\tau}z/\mu, \quad z \in D^e. \tag{2.6}$$

which is holomorphic in the domain D^e. On substituting formulas (2.5) and (2.6) into (2.4) after a simple calculation we establish the boundary condition

$$f(z) = \frac{1}{\lambda_j}\operatorname{Re}\left(\frac{\bar{\tau}}{\mu}z\right) + d'_j + ia_j, \quad z \in L_j, \quad j = 0, 1, \ldots, n-1. \tag{2.7}$$

The requirement (2.1) implies

$$f(z) \sim \frac{(\bar{\tau}^\infty - \bar{\tau})z}{\mu} + \text{const}, \quad z \to \infty. \tag{2.8}$$

Here, $\lambda_j = \kappa_j/(1-\kappa_j)$, $a_j = \kappa_j d''_j - b_j$ are real constants (therefore, without loss of generality, b_j can be dropped), and $\bar{\tau}^\infty = \tau_1^\infty - i\tau_2^\infty$. The condition (2.8) is due to the relation $\phi(z) \sim \bar{\tau}^\infty z/\mu + \text{const}$, $z \to \infty$.

Let $z = \omega(\zeta)$ be a conformal map that transforms the exterior of n circles $\mathcal{L}_j$ ($j = 0, 1, \ldots, n-1$) into the physical domain D^e. Denote this n-connected circular domain by $\mathcal{D}^e$ ($\infty \in \mathcal{D}^e$). By scaling and rotation, it is always possible to achieve (**16**), (**17**) that one of the circles say, $\mathcal{L}_0$, is of unit radius and centered at the origin and, in addition, the center of another circle say, $\mathcal{L}_1$, falls in the real axis. If the circular map meets the condition $\omega(\infty) = \infty$ and it turns out that the original problem has a unique solution, then the radius of $\mathcal{L}_1$, the complex centers and the radii of the rest $n-2$ circles cannot be selected arbitrarily and have to be fixed by some additional conditions naturally emerging from the physical model. The actual number of free parameters in the conformal map will be determined by solving the inverse boundary value problem of the theory of holomorphic functions.

Having selected the condition $\omega(\infty) = \infty$, we can expand the conformal map in the vicinity of the infinite point

$$\omega(\zeta) = c_{-1}\zeta + c_0 + \sum_{j=1}^{\infty} \frac{c_j}{\zeta^j}, \tag{2.9}$$

where $c_{-1} = c'_{-1} + ic''_{-1}$. Denote $f(\omega(\zeta)) = F(\zeta)$. From the boundary condition (2.7), we deduce that the functions $F(\zeta)$ and $\omega(\zeta)$ satisfy the following two Schwarz problems to be solved consecutively.

Find two functions $F(\zeta)$ and $\omega(\zeta)$ holomorphic in the domain $\mathcal{D}^e$ and continuous up to the boundary $\mathcal{L} = \cup_{j=0}^{n-1}\mathcal{L}_j$ such that

$$\operatorname{Im} F(\zeta) = a_j, \quad \zeta \in \mathcal{L}_j, \quad j = 0, 1 \ldots, n-1, \tag{2.10}$$

and

$$\mathrm{Re}\left[\frac{\bar{\tau}}{\mu}\omega(\zeta)\right] = \lambda_j[\mathrm{Re}\, F(\zeta) - d_j'], \quad \zeta \in \mathcal{L}_j, \quad j = 0, 1 \ldots, n-1. \tag{2.11}$$

At the infinite point, both of the functions have a simple pole,

$$F(\zeta) \sim \frac{\bar{\tau}^{\infty} - \bar{\tau}}{\mu} c_{-1}\zeta, \quad \omega(\zeta) \sim c_{-1}\zeta, \quad \zeta \to \infty. \tag{2.12}$$

In addition, the function $\omega : \mathcal{L}_j \to L_j$ *(*$j = 0, \ldots, n-1$*) has to be univalent, and the interiors of the images of the contours* $\mathcal{L}_j$*, the domains* D_j*, are disjoint sets.*

Notice that the functions $F(\zeta)$ and $\omega(\zeta)$ have to be single-valued holomorphic functions in the n-connected domain $\mathcal{D}^e$. For arbitrarily fixed real constants a_j and d_j' ($j = 0, \ldots, n-1$) the solution to the problems (2.10) and (2.11), in general, does not exist. Since the constants are free, we shall show in the next section that it is possible to fix these constants (the constants a_0 and d_0' remain to be free) such that the functions $F(\zeta)$ and $\omega(\zeta)$ are single-valued.

Recall that although the inverse Cherepanov's problem (**6**) concerns a plane problem and two holomorphic functions are involved in its setting, one of the complex potentials turns out to be a constant. Eventually, the problem reduces to two Schwarz problems with respect to the derivative of the conformal mapping and the second complex potential. These problems are coupled by two auxiliary conditions (**9**). In the antiplane problem under consideration, we also have two Schwarz problems. However, now they are coupled by the right-hand side of the second probem. Another difference is that the second problem is formulated for the map itself, not for its derivative as in the Cherepanov's problem (**6**).

3. Riemann–Hilbert problems of the theory of automorphic functions

3.1 *Setting*

Let $\mathcal{G}$ be the symmetry group of the circular line $\mathcal{L} = \mathcal{L}_0 \cup \ldots \cup \mathcal{L}_{n-1}$ generated by the linear transformations $\sigma_j = T_j T_0(\zeta)$, $j = 0, 1, \ldots, n-1$, where T_j are linear fractional transformations

$$T_j(\zeta) = \zeta_j + \frac{r_j^2}{\bar{\zeta} - \bar{\zeta}_j}, \quad j = 0, 1, \ldots, n-1. \tag{3.1}$$

Here, r_j and ζ_j are the radius and the center of the circle $\mathcal{L}_j$. The transformation $\sigma_j(\zeta)$ ($j = 1, \ldots, n-1$) maps the exterior domain $\mathcal{D}^e$ into the exterior of $\sigma_j(\mathcal{L}_m) \subset \mathrm{int}\, \mathcal{L}_j$, $j, m = 1, \ldots, n-1$. Denote by $\tilde{\mathcal{D}}^e = T_0(\mathcal{D}^e)$ the exterior of the circles $T_0(\mathcal{L}_j)$ inside $\mathcal{L}_0$. The domains $\mathcal{D}^e$ and $\tilde{\mathcal{D}}^e$ and the contour $\mathcal{L}$ comprise a fundamental region $\mathcal{F}_{\mathcal{G}}$ of the group $\mathcal{G}$: $\mathcal{F}_{\mathcal{G}} = \mathcal{D}^e \cup \tilde{\mathcal{D}}^e \cup \mathcal{L}$. The group $\mathcal{G}$ is a symmetry Schottky group (**18**). It consists of the identical map $\sigma_0(\zeta) = T_0 T_0(\zeta) = \zeta$ and all possible compositions of the generators σ_j and the inverse maps $\sigma_j^{-1} = T_0 T_j$ $j = 1, 2, \ldots, n-1$. Therefore, each element of the group $\mathcal{G}$ is a composition of an even number of the symmetry maps $T_j(\zeta)$ ($j = 0, 1, \ldots, n-1$),

$$\sigma = T_{k_1} T_{k_2} \ldots T_{k_{2s-1}} T_{k_{2s}},$$

$$k_2 \neq k_1, k_3 \neq k_2, \ldots, k_{2s} \neq k_{2s-1}, \quad k_1, k_2, \ldots k_{2s} = 0, 1, \ldots, n-1. \tag{3.2}$$

The region $\Omega = \cup_{\sigma \in \mathcal{G}} \sigma(\mathcal{F}_{\mathcal{G}})$ is invariant with respect to the group $\mathcal{G}$: $\sigma(\Omega) = \Omega$ for any $\sigma \in \mathcal{G}$, where $\Omega = \bar{C} \setminus \Lambda$, $\bar{C} = C \cup \{\infty\}$, and Λ is the set of all limit points of the group $\mathcal{G}$. If $n = 2$, then the set Λ consists of two points, while for $n \geq 3$, the number of limit points is infinite. All maps of the group $\mathcal{G}$ are linear fractional transformations

$$\sigma(\zeta) = \frac{\alpha_\sigma \zeta + \beta_\sigma}{\gamma_\sigma \zeta + \delta_\sigma}, \quad \alpha_\sigma \delta_\sigma - \gamma_\sigma \beta_\sigma \neq 0, \tag{3.3}$$

and $\gamma_\sigma \neq 0$ if $\sigma \neq \sigma_0$. The series

$$\sum_{\sigma \in \mathcal{G} \backslash \sigma_0} \frac{|\alpha_\sigma \delta_\sigma - \beta_\sigma \gamma_\sigma|}{|\gamma_\sigma|^2} \tag{3.4}$$

is always convergent regardless of the mutual location of disjoint circles $\mathcal{L}_j$ when $n = 2$ or $n = 3$. If $n \geq 4$ there are several families of circular domains for which the associated series (3.4) and the Schottky groups are convergent. Such groups are known as the first class groups (**32**), (**33**). For $n \geq 4$, the groups are of the first class if the centers of the circles $\mathcal{L}_j$ lie in a straight line or when the domain $\mathcal{D}$ can be split into a union of triply or doubly connected domains by circles which do not intersect each other and the circles $\mathcal{L}_j$ $(j = 0, 1, \ldots, n-1)$. There are some other sufficient conditions for a group to be of the first class including those given in (**19**), (**34**). An example of the domain $\mathcal{D}^e$ that generates a Schottky symmetry group for which the series (3.4) is divergent and the corresponding Poincaré series of dimension (-2) is not absolutely convergent is given in (**24**). In what follows, it is assumed that the domain $\mathcal{D}^e$ obeys the sufficient conditions which guarantee the convergence of the series (3.4). This justifies the change of order of summation used in the representation of a quasiautomorphic analog of the Cauchy kernel. Note that the 1-dimensional measure of the singular set Λ of the first class groups is always zero (**35**), while in the case of the second class (divergent) groups this measure is positive (**36**).

Introduce next two functions, $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$, holomorphic in the domain $\mathcal{D}^e$ by

$$\Phi_1(\zeta) = F(\zeta) - c\zeta, \quad \Phi_2(\zeta) = \frac{i\bar{\tau}}{\mu}[\omega(\zeta) - c_{-1}\zeta], \quad \zeta \in \mathcal{D}^e, \tag{3.5}$$

where

$$c = \frac{\bar{\tau}^\infty - \bar{\tau}}{\mu} c_{-1}. \tag{3.6}$$

To extend their definition inside the domain $\mathcal{D}$, we set

$$\begin{aligned} &\Phi_m(\zeta) = \overline{\Phi_m(T_0(\zeta))}, \quad \sigma \in T_0(\mathcal{D}^e), \\ &\Phi_m(\zeta) = \Phi_m(\sigma^{-1}(\zeta)), \quad \zeta \in \sigma(\mathcal{D}^e \cup T_0(\mathcal{D}^e)), \quad \sigma \in \mathcal{G}. \end{aligned} \tag{3.7}$$

Then $\Phi_m(\zeta)$ are piecewise holomorphic and $\mathcal{G}$-automorphic functions which satisfy the symmetry condition

$$\overline{\Phi_m(T_j(\zeta))} = \Phi_m(\zeta), \quad \zeta \in T_j(\mathcal{D}^e) = \sigma_j(T_0(\mathcal{D}^e)), \quad j = 1, 2, \ldots n-1. \tag{3.8}$$

All circles $\sigma(\mathcal{L})$ including $\mathcal{L}$ are discontinuity lines for the functions $\Phi_m(\zeta)$. Let $\Phi_m^+(\xi)$ and $\Phi_m^-(\xi)$ be the boundary values of the functions $\Phi_m(\zeta)$ from the interior and the exterior of the circles $\sigma(\mathcal{L})$, $\sigma \in \mathcal{G}$, respectively. Then the functions $\Phi_m(\zeta)$ solve the following two Riemann–Hilbert problems.

Find all piecewise holomorphic and $\mathcal{G}$-automorphic functions bounded at infinity which meet the symmetry condition (3.8) and satisfy the linear relation in the circles $\mathcal{L}_j$

$$\Phi_m^+(\xi) - \Phi_m^-(\xi) = g_{mj}(\xi), \quad \xi \in \mathcal{L}_j, \quad j = 0, 1, \ldots, n-1, \quad m = 1, 2, \tag{3.9}$$

where

$$g_{1j}(\xi) = 2i[\mathrm{Im}(c\xi) - a_j], \quad g_{2j}(\xi) = -2i\,\mathrm{Re}\left\{\frac{\kappa_j}{1-\kappa_j}[F(\xi) - d_j'] - \frac{\bar{\tau}}{\mu}c_{-1}\xi\right\}. \tag{3.10}$$

Due to formulas (3.5), to determine the conformal mapping $\omega(\zeta)$, it is required to find the function $\Phi_2(\zeta)$. This can be done only if the first function $\Phi_1(\zeta)$ is known. To solve the two Riemann–Hilbert problems (3.9), we employ the following singular integral:

$$\Psi(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{K}(\zeta, \eta) g(\eta) d\eta, \tag{3.11}$$

where $\mathcal{K}(\zeta, \eta)$ is the series (**21**), (**24**)

$$\mathcal{K}(\zeta, \eta) = \sum_{\sigma \in \mathcal{G}} \left(\frac{1}{\sigma(\eta) - \zeta} - \frac{1}{\sigma(\eta) - \zeta_*}\right)\sigma'(\eta), \tag{3.12}$$

$\zeta_* \in \mathcal{D}^e$ is an arbitrary fixed point, and $g(\eta)$ is a density. Alternatively, because of the identity

$$\left(\frac{1}{\sigma(\eta) - \zeta} - \frac{1}{\sigma(\eta) - \zeta_*}\right)\sigma'(\eta) = \frac{1}{\eta - \sigma^{-1}(\zeta)} - \frac{1}{\eta - \sigma^{-1}(\zeta_*)}, \tag{3.13}$$

the kernel $\mathcal{K}(\zeta, \eta)$ can be written as

$$\mathcal{K}(\zeta, \eta) = \sum_{\omega \in \mathcal{G}} \left(\frac{1}{\eta - \omega(\zeta)} - \frac{1}{\eta - \omega(\zeta_*)}\right). \tag{3.14}$$

Here, since $\sigma \in \mathcal{G}$ implies $\sigma^{-1} \in \mathcal{G}$, we made the substitution $\omega = \sigma^{-1}$. The absolute convergence of the series in (3.13) and (3.14) is guaranteed by the convergence of the series (3.4). This circumstance allows for changing the order of summation and integration in the expression (3.11). Since the identity transformation $\sigma_0 \in \mathcal{G}$, the series admits the representation

$$\mathcal{K}(\zeta, \eta) = \frac{1}{\eta - \zeta} + \mathcal{K}_0(\zeta, \eta), \tag{3.15}$$

where $\mathcal{K}_0(\zeta, \eta)$ is an holomorphic function of ζ in the domain Ω

$$\mathcal{K}_0(\zeta, \eta) = -\frac{1}{\eta - \zeta_*} + \sum_{\sigma \in \mathcal{G} \setminus \sigma_0} \left(\frac{1}{\eta - \sigma(\zeta)} - \frac{1}{\eta - \sigma(\zeta_*)}\right). \tag{3.16}$$

For our next step, we need to use the following property of the kernel $\mathcal{K}(\zeta,\eta)$ (**21**), (**24**):

$$\mathcal{K}(\sigma(\zeta),\eta)=\mathcal{K}(\zeta,\eta)+\chi_\sigma(\eta), \quad \sigma\in\mathcal{G}, \tag{3.17}$$

where

$$\chi_\sigma(\eta)=\mathcal{K}(\sigma(\zeta_*),\eta). \tag{3.18}$$

The two relations (3.15) and (3.17) classify the function $\mathcal{K}(\zeta,\eta)$ as a quasiauthomorphic analog of the Cauchy kernel. As a function of τ, the kernel $\mathcal{K}(\zeta,\tau)$ is a Poincaré series of dimension (-2),

$$\sigma'(\tau)\mathcal{K}(z,\sigma(\tau))=\mathcal{K}(\zeta,\tau). \tag{3.19}$$

3.2 *Solution $\Phi_1(\zeta)$ of the first Riemann–Hilbert problem*

We next claim that under a certain choice of the real constants a_j the function

$$\Phi_1(\zeta)=\Psi_1(\zeta)+\overline{\Psi_1(T_0(\zeta))}+C_1, \tag{3.20}$$

provides the general solution of the first Riemann–Hilbert problem (3.9). Here, C_1 is an arbitrary real constant and

$$\Psi_1(\zeta)=\frac{1}{2\pi}\sum_{j=0}^{n-1}\int_{\mathcal{L}_j}[\mathrm{Im}(c\eta)-a_j]\mathcal{K}(\zeta,\eta)d\eta. \tag{3.21}$$

Apparently, the function $\Phi_1(\zeta)$ satisfies the symmetry condition $\Phi_1(\zeta)=\overline{\Phi_1(T_0(\zeta))}$, $\sigma\in T_0(\mathcal{D}^e)$. Because of the quasiautomorphicity of the kernel $\mathcal{K}(\zeta,\eta)$, in general, the function $\Phi_1(\zeta)$ is not automorphic. However, since the constants a_j are free, it is possible to fix them such that the conditions $\Phi_1(\zeta)=\Phi_1(\sigma_j(\zeta))$, $j=1,2,\dots,n-1$, are satisfied and the solution becomes automorphic. Indeed, the relation (3.17) implies

$$\mathcal{K}(\sigma_j(\zeta),\eta)=\mathcal{K}(\zeta,\eta)+\chi_{\sigma_j}(\eta), \quad \chi_{\sigma_j}(\eta)=\mathcal{K}(\sigma_j(\zeta_*),\eta), \quad j=1,2,\dots,n-1. \tag{3.22}$$

Hence

$$\Psi_1(\sigma_j(\zeta))=\Psi_1(\zeta)+e_j, \quad j=1,2,\dots,n-1, \tag{3.23}$$

where

$$e_j=\frac{1}{2\pi}\sum_{l=0}^{n-1}\int_{\mathcal{L}_l}[\mathrm{Im}(c\eta)-a_l]\chi_{\sigma_j}(\eta)d\eta. \tag{3.24}$$

On writing the relation (3.23) as $\Psi_1(\zeta)=\Psi_1(\sigma_j^{-1}(\zeta))+e_j$, $j=1,2,\dots,n-1$, there is no difficulty in verifying that

$$\overline{\Psi_1(T_0\sigma_j(\zeta))}=\overline{\Psi_1(\sigma_j^{-1}T_0(\zeta))}=\overline{\Psi_1(T_0(\zeta))}-\overline{e_j}. \tag{3.25}$$

In view of (3.23) and (3.25) we obtain

$$\Phi_1(\sigma_j(\zeta))=\Phi_1(\zeta)+e_j-\overline{e_j}, \quad j=1,2,\dots,n-1. \tag{3.26}$$

Hence the solution $\Phi_1(\zeta)$ is a $\mathcal{G}$-automorphic function if and only if $\mathrm{Im}\, e_j = 0, j = 1, 2, \ldots, n-1$, that is

$$\mathrm{Im}\left\{\sum_{l=0}^{n-1}\int_{\mathcal{L}_l}[a_l - \mathrm{Im}(c\eta)]\chi_{\sigma_j}(\eta)d\eta\right\} = 0, \quad j = 1, 2, \ldots, n-1. \tag{3.27}$$

The integrals

$$\int_{\mathcal{L}_l}\chi_{\sigma_j}(\eta)d\eta = \sum_{\sigma\in\mathcal{G}}\int_{\mathcal{L}_l}\left(\frac{1}{\eta - \sigma\sigma_j(\zeta_*)} - \frac{1}{\eta - \sigma(\zeta_*)}\right)d\eta, \quad j = 1, 2, \ldots, n-1, \tag{3.28}$$

can be evaluated by the theory of residues. Assume first that $l = 0$. If $\sigma = \sigma_0$, then $\sigma\sigma_j(\zeta_*) \in \mathrm{int}\,\mathcal{L}_j$ $(j = 1, 2, \ldots, n-1)$ and $\sigma(\zeta_*) = \zeta_* \in \mathcal{D}^e$. Hence,

$$\int_{\mathcal{L}_0}\left(\frac{1}{\eta - \sigma\sigma_j(\zeta_*)} - \frac{1}{\eta - \sigma(\zeta_*)}\right)d\eta = 0. \tag{3.29}$$

If $\sigma = \sigma_j^{-1}$ and since $\sigma_j^{-1} = T_0T_j$, then $\sigma\sigma_j(\zeta_*) \in \mathcal{D}^e$ and $\sigma(\zeta_*) = T_0T_j(\zeta_*) \in \mathrm{int}\,\mathcal{L}_0$. Thus,

$$\int_{\mathcal{L}_0}\left(\frac{1}{\eta - \sigma\sigma_j(\zeta_*)} - \frac{1}{\eta - \sigma(\zeta_*)}\right)d\eta = -2\pi i. \tag{3.30}$$

Let now $\sigma \neq \sigma_0$, $\sigma \neq \sigma_j^{-1}$, and $\sigma = T_k \ldots T_\nu$. If $k \neq 0$, then

$$\int_{\mathcal{L}_0}\frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \int_{\mathcal{L}_0}\frac{d\eta}{\eta - \sigma(\zeta_*)} = 0. \tag{3.31}$$

In the case $k = 0$ we have $\sigma\sigma_j(\zeta_*) \in \mathrm{int}\,\mathcal{L}_0$ and $\sigma(\zeta_*) \in \mathrm{int}\,\mathcal{L}_0$. That is why

$$\int_{\mathcal{L}_0}\frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \int_{\mathcal{L}_0}\frac{d\eta}{\eta - \sigma_j(\zeta_*)} = 2\pi i. \tag{3.32}$$

Summing up the results obtained we evaluate the integral (3.28) for $l = 0$

$$\int_{\mathcal{L}_0}\chi_{\sigma_j}(\eta)d\eta = -2\pi i \quad j = 1, 2, \ldots, n-1. \tag{3.33}$$

Assume next that $l = 1, 2, \ldots, n-1$ and evaluate the integrals (3.28). If $\sigma = \sigma_0$, then $\sigma\sigma_j(\zeta_*) \in \mathrm{int}\,\mathcal{L}_j$, while $\sigma(\zeta_*) = \zeta_* \notin \mathrm{int}\,\mathcal{L}_j$. This implies

$$\int_{\mathcal{L}_l}\frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \begin{cases} 2\pi i, & j = l, \\ 0, & j \neq l, \end{cases} \quad \int_{\mathcal{L}_l}\frac{d\eta}{\eta - \sigma_j(\zeta_*)} = 0, \quad l = 1, 2, \ldots, n-1. \tag{3.34}$$

In the case $\sigma = \sigma_j^{-1}$ we have $\sigma\sigma_j(\zeta_*) = \zeta_* \in \mathcal{D}^e$ and $\sigma(\zeta_*) \in \mathrm{int}\,\mathcal{L}_0$, and therefore

$$\int_{\mathcal{L}_l}\frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \int_{\mathcal{L}_l}\frac{d\eta}{\eta - \sigma(\zeta_*)} = 0. \tag{3.35}$$

Suppose $\sigma \neq \sigma_0, \sigma \neq \sigma_j^{-1}$, and $\sigma = T_l \dots T_\nu$. This implies that $\sigma\sigma_j(\zeta_*)) \in \text{int}\,\mathcal{L}_l$ and $\sigma(\zeta_*) \in \text{int}\,\mathcal{L}_l$. Therefore

$$\int_{\mathcal{L}_l} \frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \int_{\mathcal{L}_l} \frac{d\eta}{\eta - \sigma(\zeta_*)} = 2\pi i. \tag{3.36}$$

If σ takes on the value $T_k \dots T_\nu$ and $k \neq l$, then

$$\int_{\mathcal{L}_l} \frac{d\eta}{\eta - \sigma\sigma_j(\zeta_*)} = \int_{\mathcal{L}_l} \frac{d\eta}{\eta - \sigma(\zeta_*)} = 0. \tag{3.37}$$

Combining all these cases we discover

$$\int_{\mathcal{L}_l} \chi_{\sigma_j}(\eta) d\eta = 2\pi i \delta_{lj}, \quad l, j = 1, 2, \dots, n-1. \tag{3.38}$$

On substituting the integrals (3.33) and (3.38) into equations (3.27) we determine all the constants $a_1, a_2, \dots, a_{n-1}$

$$a_j = a_0 + \frac{1}{2\pi} \operatorname{Im} \sum_{l=0}^{n-1} \int_{\mathcal{D}_l} \operatorname{Im}(c\eta) \chi_{\sigma_j}(\eta) d\eta, \quad j = 1, 2, \dots, n-1. \tag{3.39}$$

The constant a_0 remains to be free. We thus proved that if the constants a_j are chosen as in (3.39), then $\Phi_1(\zeta)$ is a $\mathcal{G}$-automorphic function. Show finally that it satisfies the Riemann–Hilbert boundary condition (3.9). On splitting $\mathcal{G}$ into σ_0 and $\mathcal{G} \setminus \sigma_0$ we represent the function $\Psi_1(\zeta)$ in the form

$$\begin{aligned}\Psi_1(\zeta) = {} & \frac{1}{2\pi} \sum_{l=0}^{n-1} \int_{\mathcal{L}_l} \left(\frac{1}{\eta - \zeta} - \frac{1}{\eta - \zeta_*} \right) [\operatorname{Im}(c\eta) - a_l] d\eta \\ & + \frac{1}{2\pi} \sum_{l=0}^{n-1} \sum_{\sigma \in \mathcal{G} \setminus \sigma_0} \int_{\mathcal{L}_l} \left(\frac{1}{\eta - \sigma(\zeta)} - \frac{1}{\eta - \sigma(\zeta_*)} \right) [\operatorname{Im}(c\eta) - a_l] d\eta. \end{aligned} \tag{3.40}$$

Passing to the limit $\zeta \to \xi \in \mathcal{L}_j$ and employing the Sokhotski–Plemelj formulas we obtain

$$\Psi_1^-(\xi) = -\frac{i}{2}[\operatorname{Im}(c\xi) - a_j] + \Psi_1(\xi). \tag{3.41}$$

In a similar fashion we next analyze the function $\overline{\Psi_1(T_0(\zeta))}$. We have

$$\lim_{\zeta \to \xi \in \mathcal{L}_j, \zeta \in \mathcal{D}^e} \overline{\Psi_1(T_0(\zeta))} = -\frac{i}{2}[\operatorname{Im}(c\xi) - a_j] + \overline{\Psi_1(T_0(\xi))}, \tag{3.42}$$

and then the limit values $\Phi_1^{\pm}(\xi)$ of the solution become

$$\Phi_1^{\pm}(\xi) = \pm i[\operatorname{Im}(c\xi) - a_j] + \Psi_1(\xi) + \overline{\Psi_1(T_0(\xi))}, \quad \xi \in \mathcal{L}_j. \tag{3.43}$$

This verifies that the $\mathcal{G}$-automorphic symmetric function $\Phi_1(\zeta)$ bounded at infinity and given by (3.20) with the constants a_j defined by (3.39) solves the first Riemann–Hilbert problem (3.9). Any other solution of this problem differs from the function (3.20) by a constant. This may be proved in a manner standard in the theory of boundary value problems of the theory of holomorphic functions (**37**).

3.3 *The function* $\Phi_2(\zeta)$

We begin with rewriting the second Riemann–Hilbert problem (3.9) in the following form:

$$\Phi_2^+(\xi) - \Phi_2^-(\xi) = -2i\tilde{d}_j + 2ig_{2j}^\circ(\xi), \quad \zeta \in \mathcal{L}_j, \quad j = 0, 1, \ldots, n-1, \tag{3.44}$$

where

$$\tilde{d}_j = \frac{\kappa_j}{\kappa_j - 1} d_j', \quad g_{2j}^\circ(\xi) = \mathrm{Re}\left(\frac{\kappa_j}{\kappa_j - 1} F(\xi) + \frac{\bar{\tau}}{\mu} c_{-1}\xi\right), \tag{3.45}$$

and

$$F(\xi) = c\xi - i[\mathrm{Im}(c\xi) - a_j] + \Psi_1(\xi) + \overline{\Psi_1(T_0(\xi))}, \quad \xi \in \mathcal{L}_j. \tag{3.46}$$

The Riemann–Hilbert problem is solved in the same fashion as the first problem for the function $\Phi_1(\zeta)$. Its solution is given by

$$\Phi_2(\zeta) = \Psi_2(\zeta) + \overline{\Psi_2(T_0(\zeta))} + C_2, \tag{3.47}$$

where C_2 is an arbitrary real constant and

$$\Psi_2(\zeta) = \frac{1}{2\pi} \sum_{j=0}^{n-1} \int_{\mathcal{L}_j} [g_{2j}^\circ(\eta) - \tilde{d}_j] \mathcal{K}(\zeta, \eta) d\eta, \quad \zeta \notin \mathcal{L}. \tag{3.48}$$

This functions is a $\mathcal{G}$-automorphic function if and only if the constants $\tilde{d}_j$ are given by

$$\tilde{d}_j = \tilde{d}_0 + \frac{1}{2\pi} \mathrm{Im} \sum_{l=0}^{n-1} \int_{\mathcal{L}_l} g_{2l}^\circ(\eta) \chi_{\sigma_j}(\eta) d\eta, \quad j = 1, 2, \ldots, n-1. \tag{3.49}$$

The constant $\tilde{d}_0$ may be fixed arbitrarily. As before, it is directly verified that the function $\Phi_2(\xi)$ satisfies the symmetry condition (3.8). On the circles $\mathcal{L}_j$, the limiting values of the function $\Phi_2(\xi)$ are determined according to the Sokhotski–Plemelj formulas

$$\Phi_2^\pm(\xi) = \pm i[g_{2j}^\circ(\xi) - \tilde{d}_j] + \Psi_2(\xi) + \overline{\Psi_2(T_0(\xi))}, \quad \xi \in \mathcal{L}_j. \tag{3.50}$$

where $\Psi_2(\xi)$ and $\overline{\Psi_2(T_0(\xi))}$ are the Cauchy principal values of the integrals $\Psi_2(\zeta)$ and $\overline{\Psi_2(T_0(\zeta))}$, respectively, with $\zeta = \xi \in \mathcal{L}_j$.

4. Solution to the inverse problem

4.1 *Conformal mapping*

The conformal mapping $\omega(\zeta)$ can be expressed in terms of the solution of the second Riemann–Hilbert problem. From (3.5) and (3.50) we have

$$\omega(\xi) = c_{-1}\xi - \frac{\mu}{\bar{\tau}} \left[g_{2j}^\circ(\xi) - \tilde{d}_j + i\Psi_2(\xi) + i\overline{\Psi_2(T_0(\xi))}\right], \quad \xi \in \mathcal{L}_j, \quad j = 0, 1, \ldots, n-1. \tag{4.1}$$

When a point ξ traverses the circle $\mathcal{L}_j$, the point $z = \omega(\xi)$ describes the circumference of the inclusion D_j ($j = 0, 1, \ldots, n-1$). The conformal mapping is independent of b_j and the point ζ_*.

Select $b_j = 0, j = 0, \ldots, n-1$. Up to transformations of translation and rotation it is invariant with respect to the real constants a_0 and $\tilde{d}_0$. This means that only the real constants d_0' and d_0'' in (2.5) can be chosen arbitrarily. The other $2n-2$ constants are defined by

$$d_j' = \frac{\kappa_j - 1}{\kappa_j}\tilde{d}_j, \quad d_j'' = \frac{a_j}{\kappa_j}, \quad j = 1, \ldots, n-1, \tag{4.2}$$

and the constants a_j and $\tilde{d}_j$ are determined by the conditions (3.39) and (3.49), respectively. These conditions guarantee the $\mathcal{G}$-automorphicity of the functions $\Phi_1(\zeta)$ and $\Phi_2(z)$ in the complex plane and the single valuedness of the solution of the Schwarz problems (2.10) to (2.12) in the n-connected circular domain $\mathcal{D}^e$.

Formula (4.1) represents a $(3n-4)$-parametric family of conformal mappings of the n-connected circular domain $\mathcal{D}^e$ into the n-connected physical domain D^e. The free parameters of this family are the radii of the circles $\mathcal{L}_j$, r_j, $(j = 1, 2, \ldots, n-1)$, the real center ζ_1 of the circle $\mathcal{L}_1$, the complex centers $\zeta_j = \zeta_j' + i\zeta_j''$ of the circles $\mathcal{L}_j$ $(j = 2, 3, \ldots, n-1)$. Changing the complex parameter c_{-1} not only scales and rotates the inclusions but also changes their shape due to coupling of the parameter c_{-1} with the elastic parameters of the model. The inverse model problem of antiplane elasticity itself has $n+4$ parameters, τ_1/μ, τ_2/μ, τ_1^∞/μ τ_2^∞/μ, and κ_j $(j = 0, 1, \ldots, n-1)$,

To implement the method, one needs to compute the principal value of the integrals $\Psi_2(\zeta)$ and $\overline{\Psi_2(T_0(\zeta))}$ on the contours $\mathcal{L}_j$ given by

$$\Psi_2(\xi) = \frac{1}{2\pi}\sum_{l=0}^{n-1}\int_{\mathcal{L}_l}[g_{2l}^\circ(\eta) - \tilde{d}_l]\mathcal{K}(\xi,\eta)d\eta, \quad \xi \in \mathcal{L}_j, \quad j = 0, 1, \ldots, n-1. \tag{4.3}$$

Here, the functions $g_{2l}^\circ(\eta)$ are represented through the principal value of the integral $\Psi_1(\zeta)$

$$\Psi_1(\xi) = \frac{1}{2\pi}\sum_{l=0}^{n-1}\int_{\mathcal{L}_l}[\mathrm{Im}(c\eta) - a_l]\mathcal{K}(\xi,\eta)d\eta, \quad \xi \in \mathcal{L}_j, \quad j = 0, 1, \ldots, n-1, \tag{4.4}$$

by the formula

$$g_{2j}^\circ(\xi) = \frac{\kappa_j}{\kappa_j - 1}[\Psi_1(\xi) + \overline{\Psi_1(T_0(\xi))}] + \frac{c_{-1}}{\mu(\kappa_j - 1)}\mathrm{Re}\{(\kappa_j\bar{\tau}^\infty - \bar{\tau})\xi\}. \tag{4.5}$$

The kernel $\mathcal{K}(\xi,\eta)$ can be written in the form

$$\begin{aligned}\mathcal{K}(\xi,\eta) = {} & \frac{1}{\eta - \xi} - \frac{1}{\eta - \zeta_*} + \sum_{k_1=0}^{n-1}\sum_{k_2=0,k_2\neq k_1}^{n-1}\left(\frac{1}{\eta - T_{k_1}T_{k_2}(\xi)} - \frac{1}{\eta - T_{k_1}T_{k_2}(\zeta_*)}\right) \\ & + \sum_{k_1=0}^{n-1}\sum_{k_2=0,k_2\neq k_1}^{n-1}\sum_{k_3=0,k_3\neq k_2}^{n-1}\sum_{k_4=0,k_4\neq k_3}^{n-1}\left(\frac{1}{\eta - T_{k_1}T_{k_2}T_{k_3}T_{k_4}(\xi)}\right. \\ & \left. - \frac{1}{\eta - T_{k_1}T_{k_2}T_{k_3}T_{k_4}(\zeta_*)}\right) + \ldots, \quad \xi \in \mathcal{L}_j, \quad \eta \in \mathcal{L}_l.\end{aligned} \tag{4.6}$$

Recall that $T_j(\xi) = \xi, \xi \in \mathcal{L}_j$ and $T_jT_j(\xi) = \xi$. After a rearrangement possible due to the absolute convergence of the series (4.6) we have

$$\mathcal{K}(T_0\xi, \eta) = \mathcal{K}(T_0T_j\xi, \eta) = \mathcal{K}(\xi, \eta), \quad \xi \in \mathcal{L}_j. \tag{4.7}$$

Clearly, $\mathcal{K}(\xi, \eta)$ is a regular kernel if $j \neq l$, and a singular kernel otherwise.

4.2 *Two symmetric inclusions*

Consider the special case of two inclusions, $n = 2$, when they are symmetric with respect to the origin. We choose $\zeta_* = 0$, $r_0 = r_1 = 1$, $\zeta_0 = -\zeta_1$ $\kappa_0 = \kappa_1 \neq 1$. Since the constants a_0 and $\tilde{d}_0$ can be fixed arbitrarily, take $a_0 = -a_1$ and $\tilde{d}_0 = -\tilde{d}_1$. Denote the density of the integrals (4.3) and (4.4) as

$$g^*_{1j}(\xi) = \mathrm{Im}(c\xi) - a_j, \quad g^*_{2j}(\xi) = g^\circ_{2j}(\xi) - \tilde{d}_j, \quad \xi \in \mathcal{L}_j, \quad j = 0, 1. \tag{4.8}$$

Then

$$g^*_{m0}(\xi) = -g^*_{m1}(-\xi), \quad m = 0, 1. \tag{4.9}$$

It is directly verified that

$$T_0(\zeta) = -T_1(-\zeta), \quad T_0T_1(\zeta) = -T_1T_0(-\zeta). \tag{4.10}$$

The series representation (4.6) of the kernel $\mathcal{K}(\zeta, \eta)$ in our case reads

$$\mathcal{K}(\xi, \eta) = \frac{1}{\eta - \xi} - \frac{1}{\eta} + \frac{1}{\eta - T_0T_1(\xi)} - \frac{1}{T_0T_1(0)} + \frac{1}{\eta - T_1T_0(\xi)} - \frac{1}{T_1T_0(0)} + \ldots = -\mathcal{K}(-\xi, -\eta), \quad \xi \in \mathcal{L}_0 \cup \mathcal{L}_1. \tag{4.11}$$

Since in the contour $\mathcal{L}_0$, $T_0(\xi) = \xi$, we have $\Phi_j(\xi) = \Psi_j(\xi) + \overline{\Psi_j(\xi)}, \xi \in \mathcal{L}_0$. Analyze now the principal value of the singular integrals $\Psi_j(\xi)$ on the second circle $\mathcal{L}_1$ and show that we also have the relation $\Phi_j(\xi) = \Psi_j(\xi) + \overline{\Psi_j(\xi)}, \xi \in \mathcal{L}_1$. SInce $T_1(\xi) = \xi$, $\xi \in \mathcal{L}_1$, it is suffices to prove that $\mathcal{K}(T_0T_1(\xi), \eta) = \mathcal{K}(\xi, \eta)$. The series representation (4.6) implies

$$\mathcal{K}(T_0T_1(\xi), \eta) = \frac{1}{\eta - T_0T_1(\xi)} - \frac{1}{\eta} + \frac{1}{\eta - T_0T_1T_0T_1(\xi)} - \frac{1}{T_0T_1(0)} + \frac{1}{\eta - T_1T_0T_0T_1(\xi)} - \frac{1}{T_1T_0(0)} + \ldots, \quad \xi \in \mathcal{L}_1. \tag{4.12}$$

By taking into account that $T_1T_0T_0T_1(\xi) = \xi$, and making the following permutation of the terms at the odd places while keeping the other terms at the same places:

$$1 \to 3, \quad 3 \to 7, \quad 5 \to 1, \quad 7 \to 9, \quad 9 \to 5, \ldots, \quad (2m) \to (2m), \quad m = 1, 2, \ldots, \tag{4.13}$$

we obtain $\mathcal{K}(T_0T_1(\xi), \eta) = \mathcal{K}(\xi, \eta)$ in the contour $\mathcal{L}_1$. We emphasize that the rearrangement made is possible due to the absolute convergence of the Poincaré series associated with the convergent first-class symmetry Schottky group.

On using these relations we derive the symmetry of the solution of the Riemann–Hilbert problems and therefore the symmetry of the conformal mapping

$$\omega(\xi) = -\omega(-\xi), \quad \xi \in \mathcal{L}_j, \quad j = 0, 1. \tag{4.14}$$

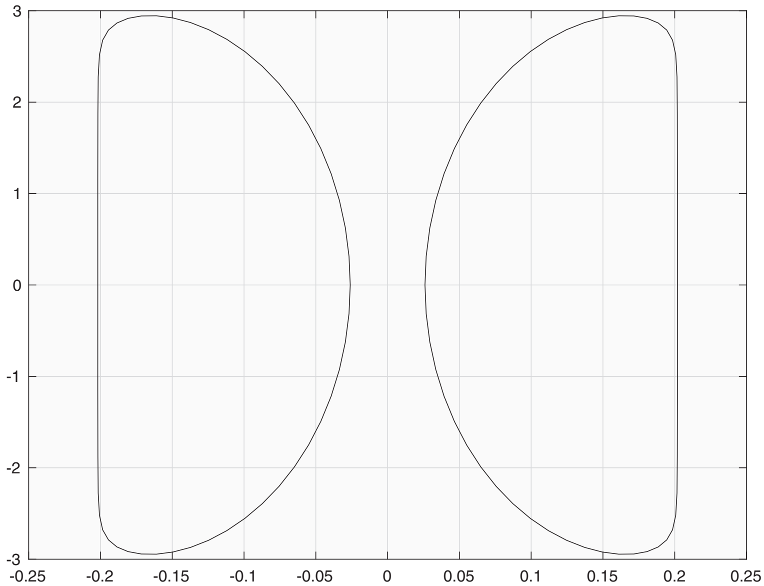

Fig. 1 Two symmetric inclusions ($n = 2$) when $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = \tau_2^\infty = 0$, $\kappa_0 = \kappa_1 = 0.1$, $c_{-1} = 1$, $r_0 = r_1 = 1$, $\zeta_0 = -\zeta_1 = -1.02$, $\zeta_* = 0$, $a_0 = -a_1$, and $\tilde{d}_0 = -\tilde{d}_1$

4.3 *Numerical results*

The singular part of the integrals (4.3) and (4.4) are evaluated numerically by the formula

$$\frac{1}{2\pi}\int_{\mathcal{L}_l}\frac{\phi(\eta)d\eta}{\eta-\xi} = \frac{i}{2(2N+1)}\sum_{j=-N}^{N}\phi(\zeta_l + r_l e^{i\theta_j}) \times\left[1+\frac{2i\sin\frac{N}{2}(\theta-\theta_j)\,\sin\frac{N+1}{2}(\theta-\theta_j)}{\sin\frac{\theta-\theta_j}{2}}\right], \quad \xi\in\mathcal{L}_l, \tag{4.15}$$

where

$$\theta_j = \frac{2\pi j}{2N+1}, \quad \theta = -i\ln\frac{\xi-\zeta_l}{r_l}, \tag{4.16}$$

and N is a sufficiently large positive integer. The regular integrals in (3.39), (3.49), (4.3) and (4.4) are computed by the Simpson rule.

We numerically verified that the solution derived satisfies the boundary condition (2.5). Although the conformal mapping possesses free parameters this does not mean that any set of parameters will deliver nonintersecting inclusions. In what follows we report on selected sets of parameters when the solution meets all the requirements of the model.

Figures 1 and 2 provide samples of two inclusions symmetric with respect of the origin. The conformal map has two free parameters, the x-coordinate of the center and the radius of the circle $\mathcal{L}_1$. The circles $\mathcal{L}_1$ and $\mathcal{L}_2$ are taken of unit radius, $r_0 = r_1 = 1$, while the distance between the

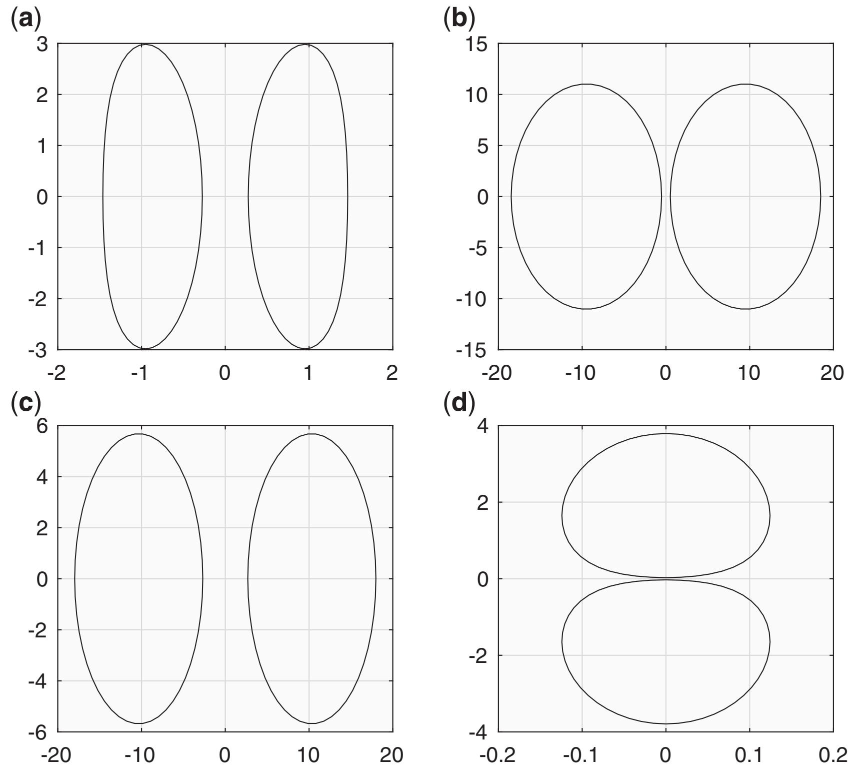

Fig. 2 Samples of two symmetric inclusions L_0 and L_1 when $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = \tau_2^\infty = 0$, $r_0 = r_1 = 1$, $\zeta_* = 0$, $a_0 = -a_1$, and $\tilde{d}_0 = -\tilde{d}_1$. (a) $\kappa_0 = \kappa_1 = 0.4$, $\zeta_0 = -\zeta_1 = -1.5$, $c_{-1} = 1$. (b) $\kappa_0 = \kappa_1 = 0.9$, $\zeta_0 = -\zeta_1 = -10$, $c_{-1} = 1$. (c) $\kappa_0 = \kappa_1 = 1.15$, $\zeta_0 = -\zeta_1 = -10$, $c_{-1} = 1$. (d) $\kappa_0 = \kappa_1 = 0.1$, $\zeta_0 = -\zeta_1 = -1.5$, $c_{-1} = i$

centers ζ_0 and ζ_1 varies. When the two circles are close to each other and the ratio of the shear moduli $\kappa_0 = \kappa_1 = \mu_0/\mu$ (μ is the matrix shear modulus) is small, the inclusions resemble halves of an ellipse (Fig. 1). If the distance $|\zeta_1 - \zeta_0|$ is growing with κ_j being fixed or when the parameter $\kappa_0 = \kappa_1$ increases while $|\zeta_1 - \zeta_0|$ is fixed, the shape of uniformly stressed inclusions has a tendency to look like two ellipses (Fig. 2). This is of course relates to the cases when the map $\omega(\zeta)$ does not recover intersecting contours that is when the mapping is univalent. However, even when they look like two ellipses, they are not perfect ellipses, and the external or internal halves of the inclusions boundary are flatter than the the ones facing each other. The same pattern is recovered when the method of slit conformal maps and Riemann–Hilbert problems on Riemann surfaces of algebraic functions is employed (**31**).

Numerical tests show that both functions $\Phi_1(\zeta$ and $\Phi_2(\zeta)$, the auxiliary function $F(\zeta)$, and the complex potential $\phi(z)$ are single-valued. The function $\phi(z) = u(x_1, x_2) + iv(x_1, x_2)$, when $z \in L_0$ and $z \in L_1$, is plotted in Fig. 3. Numerical results also demostrate strong dependence of the inclusion profiles on the values of the parameters a_1 and $\tilde{d}_1$ which never simultaneously vanish. For example,

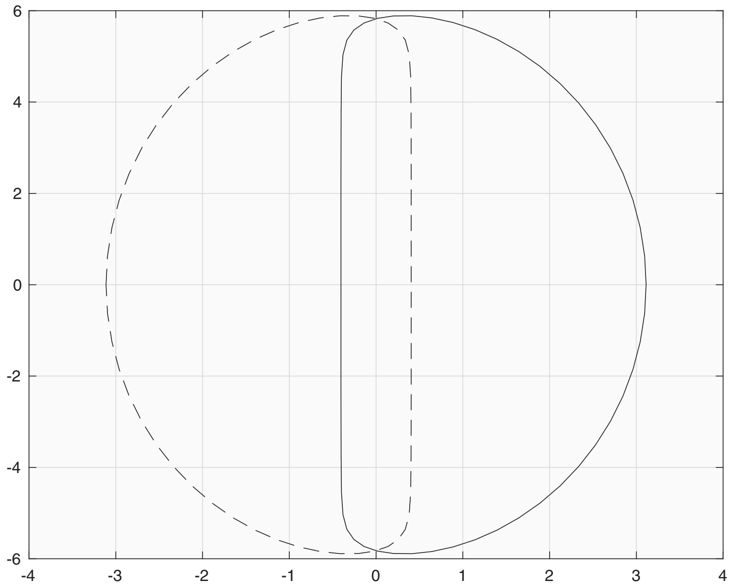

Fig. 3 The boundary values of the function $\phi(z)$ as $z \in L_0$ (the solid line) and L_1 (the broken line) in the case of the two symmetric inclusions given in Fig. 1 (L_0 is on the left)

in Fig. 1, $a_0 = a_1 = 0$, while $\tilde{d}_0 = -\tilde{d}_1 = -5.19615$. In the case of Fig. 2b, $a_0 = a_1 = 0$, while $\tilde{d}_0 = -\tilde{d}_1 = -109.448618$. The parameters a_0 and a_1 are not necessarily zero in the symmetric case: $a_0 = -a_1 = 1.118034$ and $\tilde{d}_0 = -\tilde{d}_1 = 0$ in Fig. 2d.

In Fig. 4, we show the profiles of two nonsymmetric inclusions when the symmetry is not achievable for either nonsymmetric loading, or distinct values of the ratios κ_0 and κ_1, or distinct values of the circles radii and their centers. Figures 5 and 6 give samples of three uniformly stressed inclusions.

5. Conclusion

To solve the inverse problem of antiplane elasticity on recovering the shape of n uniformly stressed inclusions, we proposed to apply the method of conformal mappings from an n-connected external circular domain to the exterior of n inclusions. The reconstruction of the conformal map requires solving two Schwarz problems on n circles when the right hand-side of the boundary condition of the second problem is expressed through the solution of the first Schwarz problem. To solve these Schwarz problems, we applied the method of symmetry and linear fractional transformations. This approach brought us to two Riemann–Hilbert problems of the theory of automorphic functions generated by a Schottky group of symmetric linear fractional transformations. By employing a quasiautomorphic analog of the Cauchy kernel we derived a series representation of a family of conformal mappings meeting the requirements of the inverse problem of antiplane elasticity. This family possesses $3n - 4$ free real parameters which have to be chosen in a sensible manner to avoid possible overlapping of the inclusions.

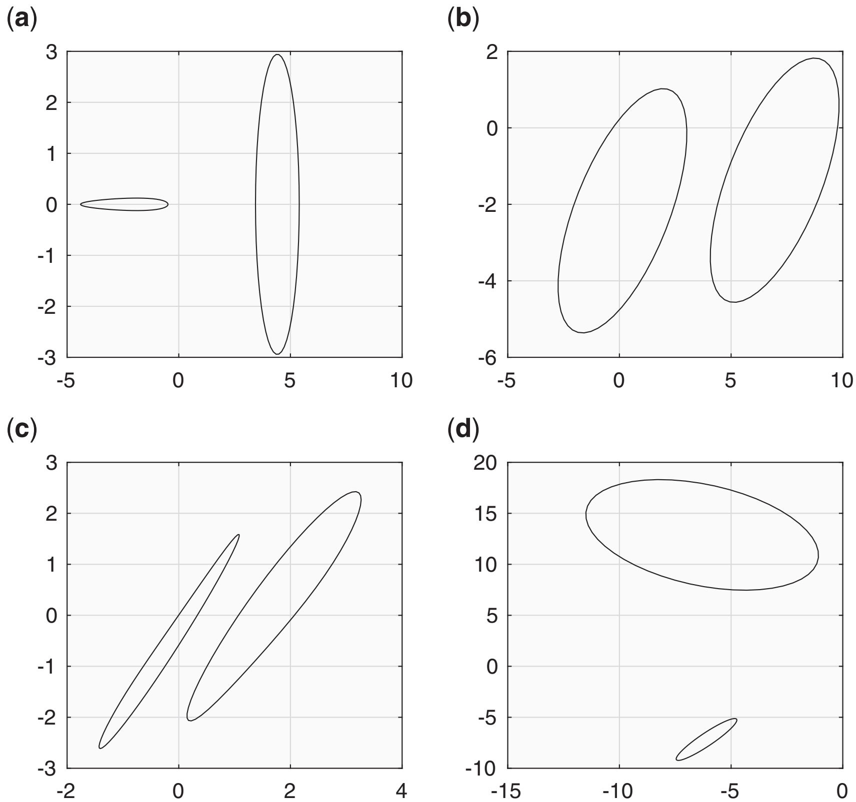

Fig. 4 Samples of two nonsymmetric inclusions L_0 and L_1 when $r_0 = r_1 = 1$, $\zeta_* = 0$, $a_0 = 0$, and $\tilde{d}_0 = 0$. a. $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = \tau_2^\infty = 0$, $\kappa_0 = 2$, $\kappa_1 = 0.5$, $\zeta_0 = -2$, $\zeta_1 = 5$, $c_{-1} = 1$. b-d. $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = -1$, $\tau_2^\infty = 1$. b. $\kappa_0 = \kappa_1 = 0.5$, $\zeta_0 = -2$, $\zeta_1 = 5$, $c_{-1} = 1$. c. $\kappa_0 = 0.1$, $\kappa_1 = 0.2$, $\zeta_0 = -1.1$, $\zeta_1 = 1.5$, $c_{-1} = 1$. d. $\kappa_0 = 0.2$, $\kappa_1 = 1.5$, $\zeta_0 = -\zeta_1 = -10$, $c_{-1} = i$

This article does not study the conditions on the geometric and elastic parameters which guarantee that the holomorphic map $\omega(\zeta)$ is univalent. In addition to the $3n-4$ geometric parameters the map possesses $n+4$ elastic parameters. A random choice of these $4n$ geometric and elastic parameters may result in a map that generates intersecting contours. Without computations we cannot recover the inclusions and conclude if the map found is univalent or not.

Numerical results obtained for two-connected domains revealed the tendency of the uniformly stressed inclusions to look like ellipses when the distance $|\zeta_1 - \zeta_0|$ is growing with κ_j $(j = 0, 1)$ being fixed or when the parameters κ_j are increasing while $|\zeta_1 - \zeta_0|$ is fixed. Here, ζ_0 and ζ_1 are the centers of the preimage circles, $\kappa_j = \mu_j/\mu$ and μ_j and μ are the shear moduli of the inclusions and the matrix, respectively. When the two circles are close to each other and the ratio of the shear moduli κ_j is small, the inclusions resemble halves of an ellipse. Even when the inclusions look like two ellipses, they are not perfect ellipses, and one of the halves of the inclusions boundary is flatter than another one. It was also discovered that the inclusions profiles are quite sensitive to the values

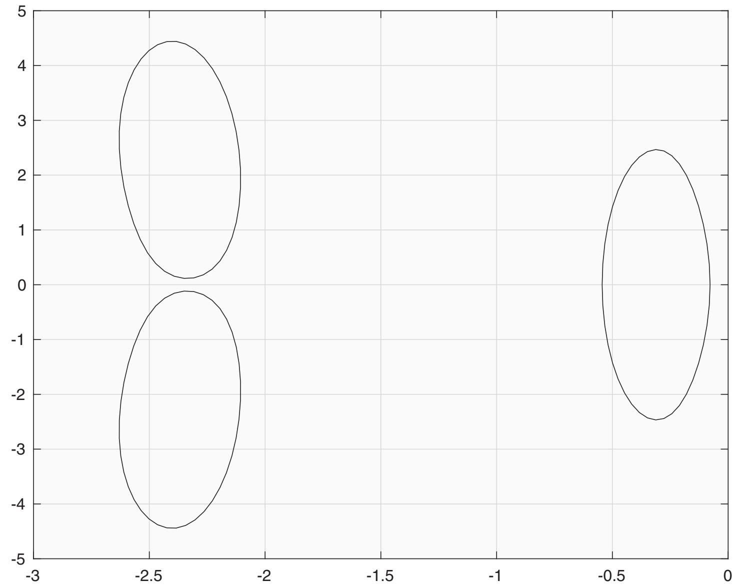

Fig. 5 Three uniformly stressed inclusions when $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = \tau_2^\infty = 0$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.2$, $c_{-1} = 1$, $r_0 = r_1 = r_2 = 1$, $\zeta_0 = 2$, $\zeta_1 = 2e^{2\pi i/3}$, $\zeta_2 = 2e^{-2\pi i/3}$, $\zeta_* = 0$, and $a_0 = \tilde{d}_0 = 0$

of the constants a_j and d_j' $(j = 1, \ldots, n-1)$ or, equivalently, to the values of the $n-1$ complex constant $d_j = d_j' + id_j''$ $(j = 1, \ldots, n-1)$ in (2.5). The complex constant d_0 and the real constants b_j in (2.4) $(j = 0, 1, \ldots, n-1)$ can be chosen arbitrarily. If the meromorphicity conditions (3.39) and (3.49) are ignored, that is if the constants a_j and d_j' $(j = 1, \ldots, n-1)$ are fixed not by these conditions, then the solution is not single-valued, and the inclusions significantly diverge from their actual shape.

The restriction imposed on the preimage circular domain to generate the symmetric Schottky group of the first class does not relate to the cases of two and three inclusions. When $n \geq 4$ and the associated group satisfies some sufficient conditions, the absolute convergence of the quasiautomorphic kernel or, equivalently, the associated Poincaré theta series of dimension (-2) is guaranteed. Burnside conjectured (**32**) that the Poincaré theta series of dimension (-2) generated by the Schottky group always converges. This conjecture was negatively solved in (**38**) and later by a different method in (**36**). The Akaza results (**36**) were employed in (**24**) to give an example of a divergent Schottky symmetry group and therefore absolutely divergent dimension (-2) Poincaré theta series used in the theory of Riemann–Hilbert problems for piecewise holomorphic automorphic functions. This example concerns the case of nine circles. One of the circles, L_0, is of infinite radius, the real axis, while the others are placed in two rows in a rectangle with the distance $\varepsilon < 0.019021$ between the circles such that the fundamental domain is symmetric with respect to the real axis.

All these examples relate to the absolute divergence of the series. The absolute convergence is needed in order to justify the change of the order of summation in Poincaré series that, in its turn, is needed to verify the automorphicity of the solution. There is another issue. Due to the Schottky result (**35**), if the associated group is of the first class, then the 1-dimensional measure of

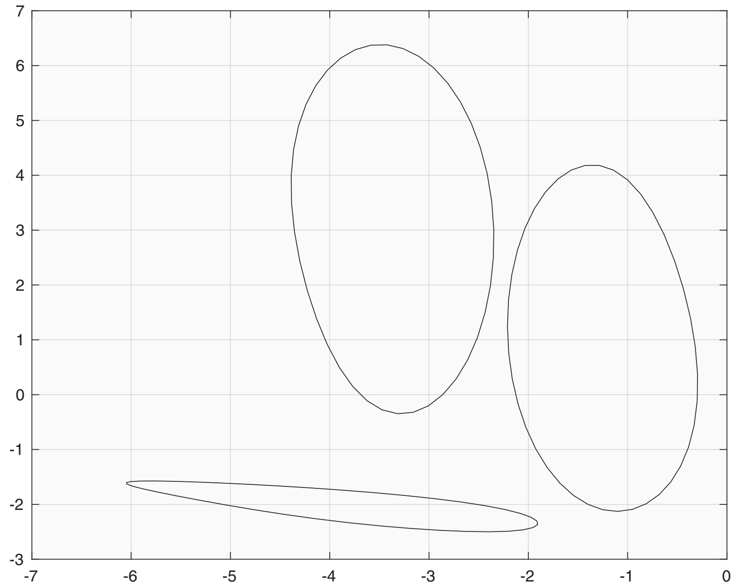

Fig. 6 Three uniformly stressed inclusions when $\tau_1/\mu = 2$, $\tau_1^\infty/\mu = 1$, $\tau_2 = \tau_2^\infty = 0$, $\kappa_0 = \kappa_1 = 0.5$, $\kappa_2 = 2$, $c_{-1} = 1$, $r_0 = r_1 = r_2 = 1$, $\zeta_0 = 2$, $\zeta_1 = 2e^{2\pi i/3}$, $\zeta_3 = 3e^{-2\pi i/3}$, $\zeta_* = 0$, and $a_0 = \tilde{d}_0 = 0$

the singular set is always zero. Akaza showed the existence of Schottky groups (**36**) and Kleinian groups (**39**) with fundamental domains bounded by 32 and 4 circles, respectively, whose singular sets have positive 1-dimensional measure. In this case, the Riemann–Hilbert problem for piecewise holomorphic automorphic functions might need a special treatment.

Finally, we note that the scheme proposed can be easily applied to mathematically equivalent inverse problems of heat transfer and electrostatics governed by the harmonic equation with the corresponding boundary conditions when the field inside the inclusions is prescribed accordingly.

APPENDIX: SINGLE INCLUSION

Without loss of generality $\mathcal{L}_0$ is the unit circle centered at the origin and $a_0 = 0$. The solution of the Schwarz problem (2.10), (2.12) for the unit circle $\mathcal{L}_0$ is given by

$$F(\zeta) = \beta_0 - i\beta_1\zeta + i\bar{\beta}_1\zeta^{-1}, \tag{A.1}$$

where

$$\beta_1 = \beta_1' + i\beta_1'' = \frac{\bar{\tau}^\infty - \bar{\tau}}{\mu} ic_{-1}, \tag{A.2}$$

and c_{-1} and β_0 are real constants. The solution of the second Schwarz problem (2.11), (2.12) can be represented in the form

$$\bar{\tau}\omega(\zeta) = \gamma_{-1}\zeta^{-1} + \gamma_0 + \gamma_1\zeta, \tag{A.3}$$

where $\gamma_j = \gamma_j' + i\gamma_j''$, $j = -1, 0, 1$. On substituting the expressions (A.3) and (A.1) into the boundary condition (2.11) and replacing ζ by $e^{i\varphi}$, $0 \le \varphi \le 2\pi$, we derive

$$\gamma_0' = (\beta_0 - d_0')\frac{\mu_0}{1-\kappa_0}, \quad \gamma_{-1}' + \gamma_1' = \frac{2\beta_1''\mu_0}{1-\kappa_0}, \quad \gamma_{-1}'' - \gamma_1'' = \frac{2\beta_1'\mu_0}{1-\kappa_0}. \tag{A.4}$$

Finally, by using the second formula in (2.12) and the relations (A.4) we determine the function $\omega(\zeta)$ up to an additive complex constant γ

$$\omega(\zeta) = c_{-1}\left(\zeta + \frac{\delta}{\zeta}\right) + \gamma, \tag{A.5}$$

where

$$\delta = \frac{2\kappa_0\tau^\infty - (\kappa_0+1)\tau}{(1-\kappa_0)\bar{\tau}}. \tag{A.6}$$

Let $\kappa_0 \ne 1$, $\tau \ne 0$, and $\delta \ne \pm 1$. Then a point $z = \omega(\zeta)$ traces an ellipse L_0 whenever the point ζ traverses the unit circle $\mathcal{L}_0$. This result is consistent with (**5**)

References

1. D. Riabuchinsky, Sur la détermination d'une surface d'aprés les données qu'elle porte, *C.-R. Paris* **189** (1929) 629–632.

2. L. A. Aksent'ev, N. B. Il'inskii, M. T. Nuzhin, R. B. Salimov and G. G. Tumashev, The theory of inverse boundary value problems for holomorphic functions and its applications, *Mathematical Analysis, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii Moscow* **18** (1980) 67–124.

3. J. D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems, *Proc. Roy. Soc. London A* **241** (1957) 376–396.

4. G. P. Sendeckyj, Elastic inclusion problems in plane elastostatics, *Int. J. Solids Structures* **6** (1970) 1535–1543.

5. C.-Q. Ru and P. Schiavone, On the elliptic inclusion in anti-plane shear, *Math. Mech. Solids* **1** (1996) 327–333.

6. G. P. Cherepanov, Inverse problems of the plane theory of elasticity, *J. Appl. Math. Mech.* **38** (1974) 915–931.

7. S. B. Vigdergauz, Integral equation of the inverse problem of the plane theory of elasticity, *J. Appl. Math. Mech.* **40** (1976) 518–522.

8. Y. Grabovsky and R. V. Kohn, Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II.: The Vigdergauz microstructure, *J. Mech. Phys. Solids* **43** (1995) 949–972.

9. Y. A. Antipov, Slit maps in the study of equal-strength cavities in n-connected elastic planar domains, *SIAM J. Appl. Math.* **78** (2018) 320–342.

10. H. Kang, E. Kim and G. W. Milton, Inclusion pairs satisfying Eshelby's uniformity property, *SIAM, J. Appl. Math.* **69** (2008) 577–595.

11. X. Wang, Uniform fields inside two non-elliptical inclusions, *Math. Mech. Solids* **17** (2012) 736–761.

12. L. P. Liu, Solutions to the Eshelby Conjectures, *Proc. Roy. Soc. London A* **464** (2008) 573–594.

13. M. Dai, C.-Q. Ru and C.-F. Gao, Uniform strain fields inside multiple inclusions in an elastic infinite plane under anti-plane shear, *Math. Mech. Solids* **17** (2017) 114–128.

14. Y. A. Antipov and V. V. Silvestrov, Method of Riemann surfaces in the study of supercavitating flow around two hydrofoils in a channel, *Physica D* **235** (2007) 72–81.

15. Y. A. Antipov and V. V. Silvestrov, Circular map for supercavitating flow in a multiply connected domain, *Quart. J. Mech. Appl. Math.* **62** (2009) 167–200.

16. M. V. Keldysh, Conformal mappings of multiply connected domains on canonical domains, *Uspekhi Matem. Nauk* **6** (1939) 90–119.

17. R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces* (Interscience Publishers, Inc. New York 1950).

18. L. R. Ford, *Automorphic Functions* (McGraw-Hill Book Company, New York 1929).

19. L. A. Aksent'ev, Construction of the Schwarz operator by the symmetry method, *Trudy Sem. Kraev. Zadacham (Kazan)* **4** (1967) 3–10.

20. I. A. Aleksandrov and A. S. Sorokin, The problem of Schwarz for multiply connected circular domains, *Siberian Math. J.* **13** (1973) 671–692.

21. L. I. Chibrikova and V. V. Silvestrov, On the question of the effectiveness of the solution of Riemann's boundary value problem for automorphic functions, *Soviet Math. (Iz. VUZ)* **12** (1978) 117–121.

22. V. V. Silvestrov, The Riemann boundary value problem for symmetric automorphic functions and its application, *Theory of Functions of a Complex Variable and Boundary Value Problems* (Chuvash. Gos. Univ., Cheboksary 1982) 93–107.

23. V. V. Mityushev and S. V. Rogosin, *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Holomorphic Functions* (Chapman & Hall, Boca Raton 2000).

24. Y. A. Antipov and V. V. Silvestrov, Method of automorphic functions in the study of flow around a stack of porous cylinders, *Quart. J. Mech. Appl. Math.* **60** (2007) 337–366.

25. Y. A. Antipov and D. G. Crowdy, Riemann-Hilbert problem for automorphic functions and the Schottky-Klein prime function, *Complex Anal. Oper. Theory* **1** (2007) 317–334.

26. Y. A. Antipov and V. V. Silvestrov, Hilbert problem for a multiply connected circular domain and the analysis of the Hall effect in a plate, *Quart. Appl. Math.* **68** (2010) 563–590.

27. A. Kazarin and Y. Obnosov, An exact holomorphical solution of an R-linear conjugation problem for a n-phased concentric circular heterogeneous structure, *Appl. Math. Modeling* **40** (2016) 5292–5300.

28. T. K. Delillo, A. R. Elcrat and J. A. Pfaltzgraff, Schwarz-Christoffel mappings of multiply connected domains, *J. d'Analyse* **94** (2004) 17–47.

29. D. Crowdy, The Schwarz-Christoffel mapping to multiply-connected polygonal domains, *Proc. R. Soc.* A **461** (2005) 2653–2678.

30. V. Mityushev, Schwarz-Christoffel formula for multiply connected domains, *Comp. Methods Function Theory* **12** (2012), 449–463.

31. Y. A. Antipov, Inverse antiplane problem on n uniformly stressed inclusions, submitted for publication, 2018, arXiv:1705.06627v2.

32. W. Burnside, On a class of automorphic functions, *Proc. London Math. Soc.* **23** (1891) 49–88.

33. V. V. Golubev, *Single-Valued Analytic Functions. Automorphic Functions* (Fizmalit, Moscow 1961).

34. T. K. DeLillo, T. A. Driscoll, A. R. Elcrat and J. A. Pfaltzgraff, Radial and circular slit maps of unbounded multiply connected circle domains, *Proc. R. Soc. London* A **464** (2008) 1719–1737.

35. F. Schottky, Ueber eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt, *J. Reine Angew. Math.* **101** (1887) 227–272.

36. T. Akaza, Poincaré theta series and singular sets of Schottky groups, *Nagoya Math. J.* **24** (1964) 43–65.

37. F. D. Gakhov, *Boundary Value Problems* (Pergamon Press, Oxford 1966).

38. P. J. Myrberg, Zur Theorie der Konvergenz der Poincare'eschen Reihen, *Ann. Acad. Sci. Fennicae* (A) **9** (1916) 1–75.

39. T. Akaza, Singular sets of some Kleinian groups, *Nagoya Math. J.* **26** (1966) 127–143.