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SLIT MAPS IN THE STUDY OF EQUAL-STRENGTH CAVITIES IN n-CONNECTED ELASTIC PLANAR DOMAINS*

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Abstract. The inverse problem of plane elasticity on n equal-strength cavities in a plane subjected to constant loading at infinity and in the cavities' boundaries is analyzed. By reducing the governing boundary value problem to the Riemann–Hilbert problem on a symmetric Riemann surface of genus $n-1$, a family of conformal mappings from a parametric slit domain onto the n-connected elastic domain is constructed. The conformal mappings are presented in terms of hyperelliptic integrals, and the zeros of the first derivative of the mappings are analyzed. It is shown that for any $n \geq 1$ there always exists a set of the loading parameters for which these zeros generate inadmissible poles of the solution.

Key words. conformal mapping, equal-strength cavities, Riemann–Hilbert problem, Riemann surfaces

AMS subject classifications. 74B05, 74P10, 74E05, 30C20, 30C15, 30E25

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1. Introduction.

Analysis of the stresses induced by the presence of inclusions and cavities in an elastic matrix subjected to loading has been of interest for more than a century [10], [13], [9]. Inverse problems of elasticity which concern problems of determination of the shapes of curvilinear inclusions and cavities with prescribed properties elicit particular attention due to their relevance to the material design [11], [3], [17], [2], [15]. A considerable amount of work examines equal-strain inclusions subjected to uniform loading with uniform distribution of stresses inside. By analyzing the stress distribution in composites with single elliptic and ellipsoidal inclusions in two- and three-dimensional unbounded elastic bodies, Eshelby [5] established that the stress fields are uniform in the interior of the inclusions, provided the matrix is loaded uniformly at infinity. He also conjectured that there do not exist other shapes of single inclusions with such a property. This conjecture was proved for the plane and antiplane model problems in the simply connected case in [14]. An alternative proof for the antiplane case by the method of conformal mappings was proposed in [12].

Motivated by the problem of designing perforated structures of minimum weight, Cherepanov [3] studied the inverse problem of elasticity on a plane uniformly loaded at infinity and having n holes. The boundary of the holes is subjected to constant normal and tangential traction, and the holes' profiles L_j $(j = 0, 1, \ldots, n-1)$ are determined from an extra boundary condition. It states that the tangential normal stress σ_t is the same constant, σ, in all the contours L_j. For the solution, a conformal map of an n-connected slit domain $\mathcal{D}^e$ into the elastic domain D^e, the exterior of the n holes, is applied. The map transforms the boundary value problem into two Schwarz problems of the theory of analytic functions on the n slits. The feature of the map $z = \omega(\zeta)$ employed [3] is that it maps the point $\zeta = \infty$ into the point $z = \infty$, and the exterior of n parallel slits of the parametric plane into the n-connected domain D^e. In general,

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unless $n \leq 3$, for such a map these slits do not lie in the same line. Cherepanov [3] solved by quadratures the problem in the simply and doubly connected symmetric cases and also analyzed the periodic and doubly periodic problems. Vigdergauz [17] noticed and corrected an error in the computations [3] implemented for the symmetric doubly connected case when the slit domain is the exterior of the cuts $[-2,-1]$, $[1,2]$, and the map $z = \omega(\zeta)$ has the property $\omega : \infty \to \infty$. For the symmetric case [3], the loading parameters a and b are real, and the two equal-strength holes exist if $|b/a| < 1$ [17].

To solve the Cherepanov problem for any n-connected domain, Vigdergauz [17] proposed to employ a circular map from the exterior of n-circles onto the n-connected elastic domain. Application of the Sherman integral representation reduced the resulting boundary value problem to integral equations solved numerically by the method of least squares. This method was further developed in [18] for doubly periodic structures by using an integral representation with quasi-automorphic analogues of the Cauchy kernel and the numerical method of least squares for solving the governing integral equations. An alternative explicit representation in terms of the Weierstrass elliptic function for the profile of an inclusion in the case of a doubly periodic structure was given in [6]. The antiplane shear problem for two equal-strain inclusions by means of the Weierstrass zeta function was treated in [7].

The main goal of the current paper is to derive a closed-form representation of a family of conformal mappings $z = \omega(\zeta)$, $\omega : \mathcal{D}^e \to D^e$, for $n = 2$ in the nonsymmetric case and for $n \geq 3$ and therefore to determine a family of the profiles of n equal-strength holes. In addition, we aim to derive necessary and sufficient conditions for the solution to exist. This will be achieved by studying the poles of one of the Kolosov–Muskhelishvili complex potentials due to possible zeros of the derivative $\omega'(\zeta)$ of the conformal mapping. The stress field $\{\sigma_1, \sigma_2, \tau_{12}\}$ is expressible through two functions Φ and Ψ [10] which have to be analytic everywhere in the domain D^e. One of them, $\Phi(z)$, is a constant, while the second has the form [3, p. 918]

$$\Psi(\omega(\zeta)) = \frac{F_+(\zeta) + F_-(\zeta)}{2\omega'(\zeta)}, \quad \zeta \in \mathcal{D}^e = \mathbb{C} \setminus l, \tag{1.1}$$

where $F_\pm(\zeta)$, the solutions to certain Schwarz problems, are analytic functions in $\mathbb{C}\setminus l$, and l is the union of the slits l_j, $j = 0, 1, \ldots, n-1$, in the parametric ζ-plane. If $\omega'(\zeta)$ has zeros in the slit domain $\mathcal{D}^e$, then the potential $\Psi(z)$ has inadmissible poles at the images of these zeros. In [20], under the assumption that the solution $\Psi(z)$ found is analytic in the exterior of n holes, by applying the maximum principle it was shown that the condition $|(\sigma_2^\infty - \sigma_1^\infty)/(\sigma_2^\infty + \sigma_1^\infty)| \leq 1$ is necessary for the existence of the solution. Here, σ_1^∞ and σ_2^∞ are constant stresses applied at infinity. To the best of our knowledge, no sufficient solvability conditions for the case $n \geq 3$ and for two nonsymmetric cavities and associated exact representations for equal-strength cavities' profiles are available in the literature.

In section 2, we formulate the problem as two Schwarz problems for two auxiliary functions coupled by two conditions. These conditions guarantee that the potential Ψ is analytic everywhere in the domain D^e and that the conformal map is single-valued. Section 3 gives an integral representation in terms of elliptic integrals of the mapping function for $n = 2$ in the general not necessarily symmetric case. The map has two free parameters and, in addition, has two free scaling parameters. It is shown that if $\gamma = |b/a| < 1$, then the solution always exists. Here,

$$a = \frac{1}{2}(\sigma - p) + i\tau, \quad b = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty) + i\tau^\infty, \tag{1.2}$$

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σ_1^∞, σ_2^∞, and τ^∞ are constant stresses applied at infinity, p and τ are the traction components applied to the holes' boundaries, and $\sigma = \sigma_1^\infty + \sigma_2^\infty - p$. If $\gamma > 1$, then the function $\Psi(\zeta)$ has four poles, while the contours L_1 and L_2, the profiles of the holes, may (for sufficiently large values of the parameter γ) or may not intersect. If $\gamma = 1$, then the contours are two straight segments, and the function $\Psi(z)$ has removable singularities on the contours. In section 4, we analyze the triply connected case. To pursue the goal of describing the general family of equal-strength cavities, we construct the most general form of the conformal map possible in the case $n = 3$. It maps the exterior of three slits $[-1/k, -1]$, $[k_1, k_2]$, and $[1, 1/k]$ $(0 < k < 1,\ -1 < k_1 < k_2 < 1)$ into the triply connected domain D^e, while the infinite point $z = \infty$ is the image of a point $\zeta_\infty = \zeta'_\infty + i\zeta''_\infty$. Thus, in addition to two scaling parameters it has five real free parameters, k, k_1, k_2, ζ'_∞, and ζ''_∞. The governing boundary value problem reduces to a symmetric Riemann–Hilbert problem on a genus-2 Riemann surface and, similarly to [1], is solved exactly. It is shown that if ζ_∞ is a finite point, then regardless of the values of the parameter γ the function $\Psi(z)$ always has inadmissible poles in D^e, and the solution does not exist; that is, the condition $\gamma < 1$ is necessary but not sufficient. In section 5, we identify a family of n-connected domains which can be interpreted as images of a slit domain with cuts located in the same line. We choose the parametric domain $\mathcal{D}^e$ as the union of slits in the real axis and assume that $\omega : \infty \to \infty$. The conformal map with such properties has $2n - 2$ free real parameters. In particular, for the case $n = 3$ and $\zeta_\infty = \infty$, we show that the family of conformal mappings is four-parametric (the slits are $[-1, k_1]$, $[k_2, k_3]$, and $[k_4, 1]$), and if $\gamma < 1$, then the solution is free of poles. Otherwise, if $\gamma > 1$, it has six poles, and the solution does not exist. In the appendix we analyze the case $n = 1$ and show that the potential $\Psi(z)$ has two inadmissible poles if $\gamma > 1$. The same solvability condition for the case $n = 1$ by a different method was derived earlier in [19].

2. Formulation. Consider the following problem of plane elasticity [3] (Figure 1).

Let an infinite isotropic plane subjected to constant stresses at infinity, $\sigma_1 = \sigma_1^\infty$, $\sigma_2 = \sigma_2^\infty$, *and* $\tau_{12} = \tau^\infty$, *have* n *holes* $D_0, D_1, \ldots, D_{n-1}$. *Assume that constant normal and tangential traction components are applied to their boundaries* L_j, $\sigma_n = p$, $\tau_{nt} = \tau$, $j = 0, 1, \ldots, n-1$. *Find the shape and location of the holes such that the tangent normal stress* σ_t *is constant,* $\sigma_t = \sigma$, *in all the contours* L_j.

Let $\Phi(z)$ and $\Psi(z)$ $(z = x_1 + ix_2)$ be the Kolosov–Muskhelishvili potentials of the problem. These functions are analytic everywhere in the n-connected domain $D^e = \mathbb{C} \setminus D$, $D = \cup_{j=0}^{n-1} D_j$ and continuous in D^e up to its boundary. The equilibrium equations of plane elasticity are satisfied if the stresses are [10]

$$\sigma_1 + \sigma_2 = 4\,\mathrm{Re}\,\Phi(z), \quad \sigma_2 - \sigma_1 + 2i\tau_{12} = 2[\bar{z}\Phi'(z) + \Psi(z)]. \tag{2.1}$$

The stresses and the traction vector components on the boundaries are connected by the relations

$$\sigma_t + \sigma_n = \sigma_1 + \sigma_2, \quad \sigma_t - \sigma_n + 2i\tau_{nt} = e^{2i\alpha(z)}(\sigma_2 - \sigma_1 + 2i\tau_{12}). \tag{2.2}$$

Here, $\alpha(z)$ is the angle between the positive direction of the x_1-axis and the external normal n to the cavity boundary (internal with respect to the body D^e). At infinity, the functions $\Phi(z)$ and $\Psi(z)$ behave as [10]

$$\Phi(z) = b' + \frac{X + iY}{2\pi(1+\kappa)}\frac{1}{z} + O\left(\frac{1}{z^2}\right),$$

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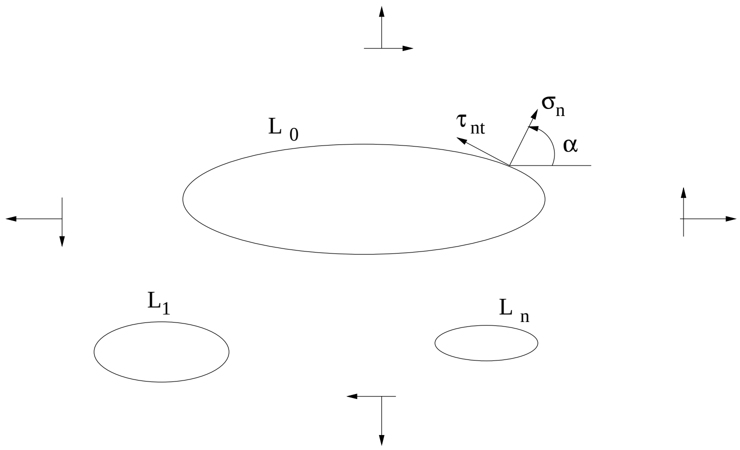

FIG. 1. *Geometry of the problem.*

$$\Psi(z) = b - \frac{\kappa(X - iY)}{2\pi(1+\kappa)}\frac{1}{z} + O\left(\frac{1}{z^2}\right), \tag{2.3}$$

where

$$b' = \frac{\sigma_1^\infty + \sigma_2^\infty}{4} + iC', \quad b = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty) + i\tau^\infty, \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu},$$

$$X + iY = \sum_{j=0}^{n-1}(X_j + iY_j), \quad X_j + iY_j = \int_{L_j}(X_{jn} + iY_{jn})ds, \tag{2.4}$$

λ and μ are the Lamé constants, (X_j, Y_j) is the total force applied to the boundary L_j $(j = 0, \ldots, n-1)$ from the side of a normal directed towards the body D^e, and X_{jn} and Y_{jn} are the x_1- and x_2-projections of the force at a point of the boundary. Since the traction components σ_n and τ_{nt} are constant on the boundary, $X = Y = 0$.

The boundary condition $\operatorname{Re}\Phi(z) = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty)$ on L_j, $j = 0, \ldots, n-1$, and the condition at infinity (2.3) imply that the function $\Phi(z)$ is a constant everywhere in D^e, $\Phi(z) = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + iC'$. The constant C' does not affect the stresses. It is expressed through the rotation ε_∞ of the plane at infinity, $C' = 2\mu(1+\kappa)^{-1}\varepsilon_\infty$ [10].

Due to (2.1)–(2.3) the analytic function $\Psi(z)$ has to satisfy the conditions

$$\Psi(z) = ae^{-2i\alpha(z)}, \quad z \in L = \cup_{j=0}^{n-1} L_j, \quad \Psi(z) = b + O(z^{-2}), \quad z \to \infty, \quad a = \frac{1}{2}(\sigma - p) + i\tau. \tag{2.5}$$

It is known [4], [8] that there exists an analytic function $z = \omega(\zeta)$ that conformally maps the extended complex ζ-plane $\mathbb{C} \cup \infty$ cut along n segments parallel to the real ζ-axis onto the n-connected domain D^e in the z-plane. Such a function is a one-to-one map. The infinite point $z = \infty$ is the image of a certain point $\zeta = \zeta_\infty$, and in the vicinity of that point, the conformal map $\omega(\zeta)$ can be represented as

$$\omega(\zeta) = \frac{c_{-1}}{\zeta - \zeta_\infty} + c_0 + \sum_{j=1}^{\infty} c_j(\zeta - \zeta_\infty)^j. \tag{2.6}$$

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If $\zeta_\infty = \infty$, then

$$\omega(\zeta) = c_{-1}\zeta + c_0 + \sum_{j=1}^{\infty} \frac{c_j}{\zeta^j}. \tag{2.7}$$

The boundary condition in (2.5) written in the ζ-plane reads [3]

$$\psi(\zeta)\omega'(\zeta) + a\overline{\omega'(\zeta)} = 0, \quad \zeta \in l = \cup_{j=0}^{n-1} l_j, \tag{2.8}$$

where $\psi(\zeta) = \Psi(z)$. The problem is significantly simplified for the two analytic functions

$$F_\pm(\zeta) = [\psi(\zeta) \pm \bar{a}]\omega'(\zeta). \tag{2.9}$$

The new functions solve the following boundary value problem.

Find two functions $F_\pm(\zeta)$ analytic in the domain $\mathbb{C} \setminus l$ and continuous up to the boundary l, having at most integrable singularities at the endpoints of the contour l and satisfying the Schwarz boundary conditions on the cuts l,

$$\operatorname{Re} F_+(\zeta) = 0, \quad \operatorname{Im} F_-(\zeta) = 0, \quad \zeta \in l. \tag{2.10}$$

If ζ_∞ is a finite point, then the functions $F_\pm(\zeta)$ behave at ζ_∞ and ∞ as

$$F_\pm(\zeta) = -\frac{c_{-1}(b \pm \bar{a})}{(\zeta - \zeta_\infty)^2} + O(1), \quad \zeta \to \zeta_\infty, \quad F_\pm(\zeta) = O(\zeta^{-2}), \quad \zeta \to \infty. \tag{2.11}$$

In the case $\zeta_\infty = \infty$,

$$F_\pm(\zeta) = c_{-1}(b \pm \bar{a}) + O(\zeta^{-2}), \quad \zeta \to \infty. \tag{2.12}$$

The two Schwarz problems are coupled by the conditions

(i) *the difference $F_+(\zeta) - F_-(\zeta)$ may not have zeros in the domain $\mathbb{C} \setminus l$;*

(ii) *the following n integrals over the loops l_j vanish:*

$$\int_{l_j} [F_+(\zeta) - F_-(\zeta)] d\zeta = 0, \quad j = 0, 1, \dots, n-1. \tag{2.13}$$

If the two functions $F_+(\zeta)$ and $F_-(\zeta)$ are known, then the functions $\omega'(\zeta)$ and $\psi(\zeta)$ are determined by

$$\omega'(\zeta) = \frac{F_+(\zeta) - F_-(\zeta)}{2\bar{a}}, \quad \psi(\zeta) = \frac{F_+(\zeta) + F_-(\zeta)}{2\omega'(\zeta)}, \quad \zeta \in \mathbb{C} \setminus l. \tag{2.14}$$

The condition (i) is necessary and sufficient for the function $\psi(\zeta)$ to be analytic in the domain $\mathbb{C}\setminus l$, while the condition (ii) guarantees that the function $\omega(\zeta)$ is single-valued.

To justify the relations (2.11) in the case $|\zeta_\infty| < \infty$, we note that $\zeta = \infty$ is the ω-image of a finite point of the domain D^e, the function $\psi(\zeta)$ is bounded at infinity, and also

$$\omega(\zeta) = d_0 + \frac{d_1}{\zeta} + \frac{d_2}{\zeta^2} + \cdots, \quad \omega'(\zeta) = -\frac{d_1}{\zeta^2} - \frac{2d_2}{\zeta^3} - \cdots, \quad \zeta \to \infty. \tag{2.15}$$

In a neighborhood of the point $\zeta_\infty \in (-1, 1)$ the function $\omega(\zeta)$ admits the expansion (2.6) and therefore

$$\psi(\zeta) = b + O((\zeta - \zeta_\infty)^2), \quad \omega'(\zeta) = -\frac{c_{-1}}{(\zeta - \zeta_\infty)^2} + c_1 + 2c_2(\zeta - \zeta_\infty) + \cdots, \quad \zeta \to \zeta_\infty. \tag{2.16}$$

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3. Two cavities.

3.1. Two cavities when ζ_∞ is a finite point: The general case. Every doubly connected domain D^e may be conformally mapped by a function $\zeta = \omega^{-1}(z)$ onto a slit domain $\mathcal{D}^e$, the extended complex ζ-plane cut along the two cuts $l_0 = [-1/k, -1]$ and $l_1 = [1, 1/k]$, where $k \in (0, 1)$. Moreover, it is possible to choose the map such that the infinite point $z = \infty$ falls into a point ζ_∞ in the open segment $(-1, 1)$ of the real ζ-axis. Such a map can be expressed through elliptic integrals.

We introduce an elliptic surface $\mathcal{R}$ of the algebraic function $u^2 = p_2(\zeta)$, where $p_2(\zeta) = (\zeta^2 - 1)(\zeta^2 - 1/k^2)$. A single branch $f(\zeta)$ of the function $p_2^{1/2}(\zeta)$ is fixed in the ζ-plane cut along the two segments l_0 and l_1 by the condition $p_2^{1/2}(\zeta) \sim \zeta^2$, $\zeta \to \infty$. This branch is pure imaginary on the sides of the cuts, $f^\pm(\zeta) = \mp i(-1)^j \sqrt{|p_2(\zeta)|}$, $\zeta \in l_j$, $j = 0, 1$, and is real for $\zeta = \xi$ lying outside the cuts in the real axis, $f(\xi) > 0$, $|\xi| > 1/k$, and $f(\xi) < 0$, $-1 < \xi < 1$.

Since $\operatorname{Re} F_+(\zeta) = 0$ and $\operatorname{Im} F_-(\zeta) = 0$ on l, the functions $iF_+(\zeta)$ and $F_-(\zeta)$ can be analytically and symmetrically continued onto the whole Riemann surface. The new functions, $F_1(\zeta, u)$ and $F_2(\zeta, u)$, given by

$$(3.1) \quad F_1(\zeta, u) = \begin{cases} iF_+(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ -i\overline{F_+(\bar{\zeta})}, & (\zeta, u) \in \mathbb{C}_2, \end{cases} \qquad F_2(\zeta, u) = \begin{cases} F_-(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ \overline{F_-(\bar{\zeta})}, & (\zeta, u) \in \mathbb{C}_2, \end{cases}$$

are rational on the surface and satisfy the symmetry condition

$$(3.2) \qquad \overline{F_j(\zeta_*, u_*)} = F_j(\zeta, u), \quad (\zeta, u) \in \mathcal{R},$$

where $(\zeta_*, u_*) = (\bar{\zeta}, -u(\bar{\zeta}))$ is the point symmetrical to the point (ζ, u) with respect to the line along which the two sheets $\mathbb{C}_1$ and $\mathbb{C}_2$ of the surface are connected. Note that if $(\zeta, u) \in \mathbb{C}_1$, then $(\zeta_*, u_*) \in \mathbb{C}_2$.

At the points $(\zeta_\infty, u_\infty) \in \mathbb{C}_1$ and $(\zeta_\infty^*, u_\infty^*) \in \mathbb{C}_2$ ($\zeta_\infty^* = \bar{\zeta}_\infty$), both of the functions $F_1(\zeta, u)$ and $F_2(\zeta, u)$ have order-2 poles with zero residues. At the branch points of the surface, they have simple poles (in the sense of Riemann surfaces). At the two infinite points of the surface they have order-2 zeros. The Schwarz boundary conditions (2.10) can be rewritten as the two symmetric Riemann–Hilbert problems on the elliptic surface $\mathcal{R}$:

$$(3.3) \qquad F_j^+(\xi, u) - F_j^-(\xi) = 0, \quad \xi \in l = l_0 \cup l_1.$$

The solutions of these problems are rational functions in the surface $\mathcal{R}$. When written in the first sheet, they have the form

$$F_-(\zeta) = \frac{1}{(\zeta - \zeta_\infty)^2} \left(A_0^- + \frac{i(A_1^- + A_2^- \zeta + A_3^- \zeta^2)}{f(\zeta)} \right), \quad \zeta \in \mathbb{C} \setminus l,$$

$$(3.4) \qquad F_+(\zeta) = \frac{1}{(\zeta - \zeta_\infty)^2} \left(iA_0^+ + \frac{A_1^+ + A_2^+ \zeta + A_3^+ \zeta^2}{f(\zeta)} \right), \quad \zeta \in \mathbb{C} \setminus l,$$

where $A_j^\pm$ are real constants to be determined. Denote

$$(3.5) \qquad c_{-1} = c' + ic'', \quad b \pm \bar{a} = \alpha^\pm + i\beta^\pm.$$

Due to (2.5) the four parameters $\alpha^\pm$ and $\beta^\pm$ are expressed through the loading data as

$$(3.6) \qquad \alpha^+ = \sigma_2^\infty - p, \quad \alpha^- = p - \sigma_1^\infty, \quad \beta^+ = \tau^\infty - \tau, \quad \beta^- = \tau^\infty + \tau.$$

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On expanding the functions (3.4) in a neighborhood of the point $\zeta = \zeta_\infty$ and satisfying the first condition in (2.11) we determine $A_0^\pm$ as

$$A_0^+ = -c'\beta^+ - c''\alpha^+, \quad A_0^- = -c'\alpha^- + c''\beta^-, \tag{3.7}$$

and derive four real equations for the other six coefficients:

$$A_1^\pm + \zeta_\infty A_2^\pm + \zeta_\infty^2 A_3^\pm = d^\pm,$$

$$A_2^\pm + 2\zeta_\infty A_3^\pm = \frac{d^\pm p_2'(\zeta_\infty)}{2p_2(\zeta_\infty)}, \tag{3.8}$$

where

$$d^+ = \sqrt{|p_2(\zeta_\infty)|}(c'\alpha^+ - c''\beta^+), \quad d^- = \sqrt{|p_2(\zeta_\infty)|}(c'\beta^- + c''\alpha^-). \tag{3.9}$$

Denoting

$$A_0 = iA_0^+ - A_0^-, \quad A_j = A_j^+ - iA_j^-, \quad j = 1, 2, 3, \tag{3.10}$$

and applying formula (2.14), we obtain the derivative of the conformal map

$$\omega'(\zeta) = \frac{1}{2\bar{a}(\zeta - \zeta_\infty)^2}\left(A_0 + \frac{A_1 + A_2\zeta + A_3\zeta^2}{f(\zeta)}\right). \tag{3.11}$$

In general, the map $z = \omega(\zeta)$ given by (3.11) is a multivalued function. It is a one-to-one map if

$$\int_{l_0} \omega'(\zeta)d\zeta = 0, \quad \int_{l_1} \omega'(\zeta)d\zeta = 0. \tag{3.12}$$

The two integrals over the loops l_0 and l_1 vanish if the coefficients $A_j^\pm$ ($j = 1, 2, 3$) solve the four equations

$$A_1^\pm I_0^- + A_2^\pm I_1^- + A_3^\pm I_2^- = 0,$$

$$A_1^\pm I_0^+ - A_2^\pm I_1^+ + A_3^\pm I_2^+ = 0, \tag{3.13}$$

where

$$I_j^\pm = \int_1^{1/k} \frac{\xi^j d\xi}{(\xi \pm \zeta_\infty)^2\sqrt{|p_2(\xi)|}}, \quad j = 0, 1, 2. \tag{3.14}$$

The system of eight equations (3.8), (3.13) for the six unknowns $A_j^\pm$ ($j = 1, 2, 3$) has rank 6: the third and fourth equations in (3.13) are identically satisfied, provided $A_j^\pm$ solve equations (3.8) and the first and second equations in (3.13). Upon solving the system we express the coefficients $A_j^\pm$ through the four problem parameters $\alpha^\pm$ and $\beta^\pm$ and the four conformal map parameters c', c'', ζ_∞, and k in the form

$$A_1^\pm = \frac{d^\pm}{\lambda_0}[\zeta_\infty(\lambda_1\zeta_\infty - 2)I_1^- + (1 - \lambda_1\zeta_\infty)I_2^-],$$

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$$A_2^{\pm} = \frac{d^{\pm}}{\lambda_0}[\zeta_\infty(2-\lambda_1\zeta_\infty)I_0^- + \lambda_1 I_2^-], \quad A_3^{\pm} = -\frac{d^{\pm}}{\lambda_0}[(1-\lambda_1\zeta_\infty)I_0^- + \lambda_1 I_1^-]. \tag{3.15}$$

Here,

$$\lambda_0 = \zeta_\infty^2 I_0^- - 2\zeta_\infty I_1^- + I_2^-, \quad \lambda_1 = \frac{p_2'(\zeta_\infty)}{2p_2(\zeta_\infty)}. \tag{3.16}$$

The map itself, in addition to the four real parameters c', c'', ζ_∞, and k, has an additive constant, B:

$$\omega(\zeta) = \frac{1}{2\bar{a}}\left[-\frac{A_0}{\zeta-\zeta_\infty} + \int_{\zeta_0}^{\zeta}\frac{(A_1+A_2\xi+A_3\xi^2)d\xi}{(\xi-\zeta_\infty)^2\sqrt{p_2(\xi)}}\right] + B, \tag{3.17}$$

where the path of integration $\zeta_0\zeta$ does not pass through the point $\zeta_\infty \in (-1,1)$. When a point ζ traverses the contours l_0 or l_1, the point $z=\omega(\zeta)$ traverses the contours L_0 or L_1, respectively.

The function $\psi(\zeta)$ may have inadmissible poles in the exterior of l. They coincide with the zeros of the derivative $\omega'(\zeta)$ or, equivalently, with the zeros of the function

$$\eta(\zeta) = A_1 + A_2\zeta + A_3\zeta^2 + A_0 p_2^{1/2}(\zeta). \tag{3.18}$$

The number of inadmissible poles of the function $\psi(\zeta)$ is determined by

$$Z = \frac{1}{2\pi i}\left(\int_{l_0} + \int_{l_1} + \lim_{R\to\infty}\int_{\Gamma_R}\right)\frac{\eta'(\zeta)d\zeta}{\eta(\zeta)}, \tag{3.19}$$

where the positive direction is chosen such that the interior of circle $\Gamma_R = \{|\zeta| = R\}$ and the exterior of the cuts l is on the left. Noticing that

$$\eta'(\zeta) = A_2 + 2A_3\zeta + \frac{A_0\zeta(2\zeta^2-1-1/k^2)}{p_2^{1/2}(\zeta)} \sim 2(A_0+A_3)\zeta, \quad \zeta\to\infty, \tag{3.20}$$

we can transform formula (3.19) as

$$Z = 2 + \frac{1}{2\pi i}\left(\int_{-1/k}^{-1} - \int_{1}^{1/k}\right)\left(\frac{\eta_0^-(\xi)}{\eta^-(\xi)} - \frac{\eta_0^+(\xi)}{\eta^+(\xi)}\right)\frac{d\xi}{\sqrt{|p_2(\xi)|}}, \tag{3.21}$$

where

$$\eta^{\pm}(\xi) = \pm i\sqrt{|p_2(\xi)|}A_0 + A_1 + A_2\xi + A_3\xi^2,$$

$$\eta_0^{\pm}(\xi) = \mp i A_0\xi(2\xi^2-1-1/k^2) + \sqrt{|p_2(\xi)|}(A_2+2A_3\xi). \tag{3.22}$$

On evaluating the integrals in (3.21) we conclude that if $\gamma = |b/a| < 1$, then $Z=0$, and the function $\psi(\zeta)$ is analytic in the exterior of the cuts $l = l_0 \cup l_1$ and continuous up to the boundary l. When $\gamma > 1$, the function $\psi(\zeta)$ has four inadmissible poles; the solution does not exist. Finally, if $\gamma = 1$, then the contours L_0 and L_1 are two straight segments, and the four poles become removable singularities of the function $\psi(\zeta)$ lying in the loops l_0 and l_1.

The profiles L_j of equal-strength holes are determined from formula (3.17) as

$$z = \frac{1}{2\bar{a}}\left[-\frac{A_0}{\zeta-\zeta_\infty} + I_- \pm iJ(-1/k,\zeta)\right], \quad \zeta\in l_0^{\pm}, \quad z\in L_0,$$

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$$z = \frac{1}{2\bar{a}}\left[-\frac{A_0}{\zeta - \zeta_\infty} + I_+ \mp iJ(1,\zeta)\right], \quad \zeta \in l_1^\pm, \quad z \in L_1, \tag{3.23}$$

where

$$I_\pm = \int_{\Gamma_\pm} \frac{(A_1 + A_2\xi + A_3\xi^2)d\xi}{(\xi - \zeta_\infty)^2 p_2^{1/2}(\xi)}, \quad J(d,\zeta) = \int_d^\zeta \frac{(A_1 + A_2\xi + A_3\xi^2)d\xi}{(\xi - \zeta_\infty)^2\sqrt{|p_2(\xi)|}}. \tag{3.24}$$

Here, $\Gamma_\pm$ are the segments with the starting and terminal points ζ_0 and ± 1, respectively, and with no loss we assume $B = 0$ and $\zeta_0 = -i$.

Some typical shapes of two nonsymmetric equal-strength holes when ζ_∞ is a finite point are shown in Figures 2 and 3. In Figures 2 and 3(b) and 3(c), the cavities are of different area and not symmetric due to the shift of the point ζ_∞ with respect to the center $\zeta = 0$. In Figure 3(a), $\zeta_\infty = 0$, and the cavities have the same shape and area but are not symmetric with respect to the real and imaginary axes due to the nonzero tangential stress τ^∞ applied at infinity. In the cases shown in Figures 2 and 3(a)–(c), $\gamma < 1$, and the function $\Psi(z)$ does not have poles in the exterior of the holes. Referring to Figure 3(d), we observe that the contours L_0 and L_1 intersect each other; that is, in the case $\gamma > 1$ the presence of inadmissible poles of the function $\Psi(z)$ may not be the only feature which indicates that the solution does not exist. When γ approaches 1 and either $\gamma < 1$ or $\gamma > 1$, the contours become slim, and in the limit, when $\gamma = 1$, the contours L_0 and L_1 become segments, and the function $\Psi(z)$ is continuous everywhere in D^e up to the boundary.

3.2. Two cavities when ζ_∞ is a finite point: The symmetric case. The solution may be significantly simplified in the symmetric case when $c'' = 0$, $\zeta_\infty = 0$, $\tau = \tau^\infty = 0$. Then

$$a = \frac{\sigma - p}{2}, \quad b = \frac{\sigma_2^\infty - \sigma_1^\infty}{2}, \quad \alpha^\pm = b \pm a, \quad \beta^\pm = 0,$$

$$d^+ = \frac{c'\alpha^+}{k}, \quad d^- = 0, \quad A_0^+ = 0, \quad A_0^- = -c'\alpha^-,$$

$$A_1^+ = \frac{c'\alpha^+}{k}, \quad A_1^- = 0, \quad A_2^\pm = 0, \quad A_3^- = 0, \quad A_3^+ = -\frac{c'\alpha^+ I_0^-}{kI_2^-}, \tag{3.25}$$

and the derivative of the conformal map has the form

$$\omega'(\zeta) = \frac{c'}{2a\zeta^2}\left[\alpha^- + \frac{\alpha^+(1 - \zeta^2 I_0^-/I_2^-)}{kp_2^{1/2}(\zeta)}\right]. \tag{3.26}$$

Two sample symmetric holes are represented in Figure 4. For the parameters chosen, as $k \to 0$, ellipse-like holes deform into "kidney"-like cavities known in the literature [3], [17], [7].

It can be directly verified that $\sigma_t = \sigma$, $\sigma_n = p$, and $\tau_{nt} = \tau$, $z \in L_j$, $j = 0, 1$. Indeed, on using (2.1)–(2.5) we have

$$\sigma_t = \frac{\sigma + p}{2} + \operatorname{Re} a, \quad \sigma_n = \sigma + p - \sigma_t, \quad \tau_{tn} = \operatorname{Im} a. \tag{3.27}$$

For the lower half of the boundary of the right hole shown in Figure 4, the variation of the stresses σ_1, σ_2, and τ_{12} is given by

$$\sigma_1 = \frac{1}{2}(\sigma + p) - \operatorname{Re}\Psi(z), \quad \sigma_2 = \frac{1}{2}(\sigma + p) + \operatorname{Re}\Psi(z), \quad \tau_{12} = \operatorname{Im}\Psi(z), \tag{3.28}$$

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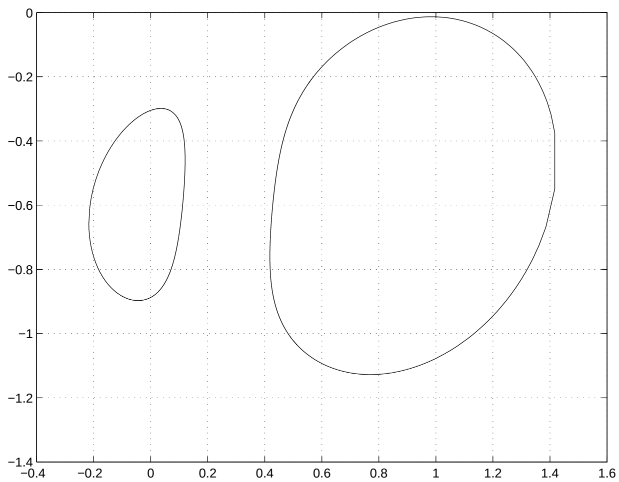

FIG. 2. *Two equal-strength holes when* $\zeta_\infty = 0.5$, $p = \tau = 0$, $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = 0.1$ *(*$\gamma = 0.1$*),* $k = 0.01$, $c' = 1$, $c'' = 0$.

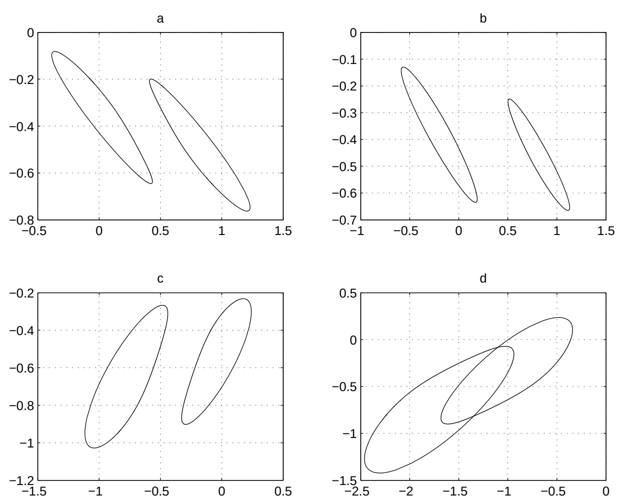

FIG. 3. *Two equal-strength holes when* ζ_∞ *is a finite point,* $p = \tau = 0$, $c' = 1$, *and* $c'' = 0$. (a) $\sigma_1^\infty = 2$, $\sigma_2^\infty = 1$, $\tau^\infty = -1$ *(*$\gamma = 0.745360$*),* $k = 0.01$, $\zeta_\infty = 0$; (b) $\sigma_1^\infty = 2$, $\sigma_2^\infty = 1$, $\tau^\infty = -1$ *(*$\gamma = 0.745360$*),* $k = 0.1$, $\zeta_\infty = -0.1$; (c) $\sigma_1^\infty = \sigma_2^\infty = 2$, $\tau^\infty = 1$ *(*$\gamma = 0.5$*),* $k = 0.01$, $\zeta_\infty = -0.1$; (d) $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = 2$ *(*$\gamma = 2$*),* $k = 0.01$, $\zeta_\infty = -0.1$. *In case* (d), $\gamma > 1$, *the function* $\Psi(z)$ *has four inadmissible poles, and the solution does not exist.*

with the arc length s represented in Figure 5. The point z traverses the contour L_1 in the clockwise direction, while its preimage ξ traverses the upper side of the loop l_1 with the starting and terminal points $\xi = 1$ and $\xi = 1/k$, respectively.

3.3. Two symmetric cavities when $\boldsymbol{\zeta_\infty = \infty}$. If it is assumed that as in [3] the function $z = \omega(\zeta)$ maps the infinite point $\zeta = \infty$ into the infinite point $z = \infty$, and the case is symmetric, that is, $c'' = 0$, $\tau = \tau^\infty = 0$, then a and b are real,

$$F_-(\zeta) = c'\alpha^-, \quad F_+(\zeta) = \frac{c'\alpha^+(\zeta^2 - I_2/I_0)}{p^{1/2}(\zeta)}, \tag{3.29}$$

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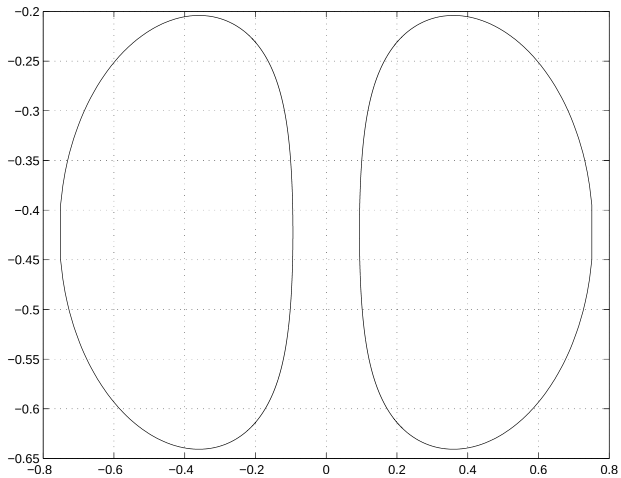

FIG. 4. *Two equal-strength symmetric holes for* $\zeta_\infty = 0$, $p = \tau = \tau^\infty = 0$, $\sigma_1^\infty = 2$, $\sigma_2^\infty = 1$ *(*$\gamma = 1/3$*),* $\kappa = 0.01$, $c' = 1$, *and* $c'' = 0$.

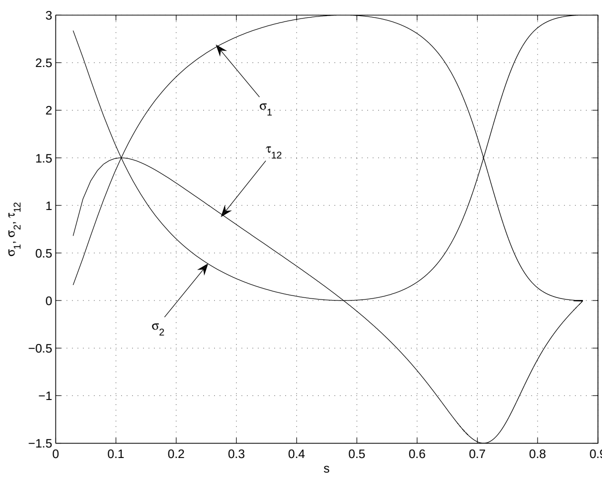

FIG. 5. *The stresses* σ_1, σ_2, *and* σ_{12} *versus the arc length* s *on the lower half of the boundary of the right hole shown in Figure* 4.

and the function $\omega'(\zeta)$ can be represented as

$$\omega'(\zeta) = \frac{c'}{2a}\left[-\alpha^- + \frac{\alpha^+(\zeta^2 - I_2/I_0)}{p_2^{1/2}(\zeta)}\right]. \tag{3.30}$$

Here,

$$I_j = \int_1^{1/k} \frac{\xi^j d\xi}{\sqrt{|p_2(\xi)|}}, \quad j = 0, 2. \tag{3.31}$$

It is directly verified that the function (3.30) satisfies the conditions (3.12) and is a one-to-one map. Possible poles of the function $\psi(\zeta)$ coincide with the zeros of the

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derivative $\omega'(\zeta)$ or, equivalently, with the zeros of the function

$$\eta(\zeta) = \alpha_1(\zeta^2 - \rho) - \alpha_0 p_2^{1/2}(\zeta), \quad \alpha_0 = \alpha^-, \quad \alpha_1 = \alpha^+, \quad \rho = I_2/I_0. \tag{3.32}$$

Due to the symmetry of the contours l_0 and l_1 the number of inadmissible poles of the function $\psi(\zeta)$ is determined by

$$Z = 2 + \frac{1}{\pi i}\int_1^{1/k} \frac{\eta^-(\xi)[\eta'(\xi)]^+ - \eta^+(\xi)[\eta'(\xi)]^-}{\eta^+(\xi)\eta^-(\xi)} d\xi, \tag{3.33}$$

where $\eta^\pm(\xi) = \eta(\xi \pm i0)$, $[\eta'(\xi)]^\pm = \eta'(\xi \pm i0)$. On substituting the limiting values $\eta^\pm(\xi)$ and $[\eta'(\xi)]^\pm$ into the last formula we simplify it to the form $Z = 2 - \alpha_0\alpha_1 I_0/\pi$, where we denote

$$I = 2\int_1^{1/k} \frac{[(\xi^2 - \rho)(2\xi^2 - 1 - 1/k^2) + 2|p_2(\xi)|]\xi d\xi}{[\alpha_1^2(\xi^2 - \rho)^2 + \alpha_0^2|p_2(\xi)|]\sqrt{|p_2(\xi)|}}. \tag{3.34}$$

By denoting $\mu_\pm = (1/k^2 \pm 1)/2$ and making substitutions, first $\zeta^2 = \mu_- \cos\theta + \mu_+$ and then $w = e^{i\theta}$, we derive

$$I = -2i\int_{|w|=1} \frac{\{[\mu_-(w^2+1) + 2\mu_+ w](\mu_+ - \rho) + 2w(\rho\mu_+ - 1/k^2)\}dw}{g_0(w)}, \tag{3.35}$$

where

$$g_0(w) = \mu_-^2(\alpha_1^2 - \alpha_0^2)(w^2+1)^2 + 4\alpha_1^2(\mu_+ - \rho)\mu_- w(w^2+1) + 4[\alpha_1^2(\mu_+ - \rho)^2 + \alpha_0^2\mu_-^2]w^2. \tag{3.36}$$

This integral is evaluated by the theory of residues. The four zeros of the function $g_0(w)$ can be easily determined; they are

$$w_{1,2} = \delta_\pm + \sqrt{\delta_\pm^2 - 1}, \quad w_{3,4} = \delta_\pm - \sqrt{\delta_\pm^2 - 1}, \tag{3.37}$$

and

$$\delta_\pm = \frac{\alpha_1^2(\rho - \mu_+) \pm \alpha_0\sqrt{\alpha_1^2(\mu_+ - \rho)^2 - \mu_-^2(\alpha_1^2 - \alpha_0^2)}}{\mu_-(\alpha_1^2 - \alpha_0^2)}. \tag{3.38}$$

The final formula for the number of inadmissible poles of the function $\psi(\zeta)$ becomes

$$Z = 2 - \alpha_* \sum_{j=1,\ldots,4; |w_j|<1} \frac{[\mu_-(w_j^2+1) + 2\mu_+ w_j](\mu_+ - \rho) + 2w_j(\rho\mu_+ - 1/k^2)}{g_j}, \tag{3.39}$$

where $\alpha_* = \alpha_0/\alpha_1$ and

$$g_j = \mu_-^2(1 - \alpha_*^2)w_j(w_j^2+1) + \mu_-(\mu_+ - \rho)(3w_j^2+1) + 2w_j[(\mu_+ - \rho)^2 + \alpha_*^2\mu_-^2]. \tag{3.40}$$

It turns out that two and only two zeros out of the four zeros w_j $(j = 1, 2, 3, 4)$ lie inside the unit disc $|w| < 1$. As in the case when ζ_∞ is a finite point in the segment $(-1, 1)$, if $\gamma < 1$, then $Z = 0$, and the function $\Psi(z)$ is analytic everywhere in the domain D^e. If $\gamma > 1$, then $Z = 4$, and the function $\Psi(z)$ has four simple poles in the domain D^e. In the limiting case $\gamma = 1$, the function $\Psi(z)$ has removable singularities in the boundary of the domain D^e, and the contours L_0 and L_1 are straight segments.

Sample contours of symmetric equal-strength holes when $\zeta_\infty = \infty$ and when $\gamma < 1$ are given in Figures 6(a)–(c). In Figure 6(d), the parameter $\gamma > 1$, the function $\Psi(z)$ has four inadmissible poles, and, in addition, the contours intersect each other; the solution does not exist.

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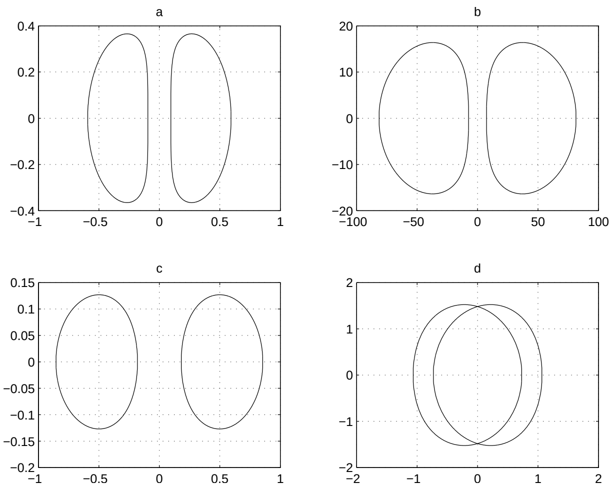

FIG. 6. *Two equal-strength symmetric holes for $\zeta_\infty=\infty$, $c'=1$, $c''=0$, $\tau^\infty=0$, $a=1$.* (a) $k=0.001$, $b=0$, *(*$\gamma=0$*)*; (b) $k=0.01$, $b=-0.5$ *(*$\gamma=0.5$*)*; (c) $k=0.1$, $b=-0.5$ *(*$\gamma=0.5$*)*; (d) $k=0.1$, $b=5$ *(*$\gamma=5$*). In case* (d) *the function $\Psi(z)$ has four inadmissible poles.*

4. Three cavities: ζ_∞ is a finite point. Any triply connected domain D^e can be considered as the image by a conformal map $z=\omega(\zeta)$ of a parametric ζ-plane cut along three segments in the real axis, $l_0=[-1/k,-1]$, $l_1=[k_1,k_2]$, and $l_2=[1,1/k]$, where $0<k<1$, $-1<k_1<k_2<1$. Given the domain D^e, such a map is unique. The point $z=\infty$ is the image of a certain point $\zeta_\infty=\zeta'_\infty+i\zeta''_\infty$, and, in general, the parameters ζ'_∞, ζ''_∞, k, k_1, and k_2 cannot be prescribed and should be recovered from the solution.

Let $\mathcal{R}$ be the hyperelliptic surface of the algebraic function $u^2=p_3(\zeta)$, where

$$p_3(\zeta)=(\zeta^2-1)(\zeta^2-1/k^2)(\zeta-k_1)(\zeta-k_2). \tag{4.1}$$

We fix the single branch $f(\zeta)$ of the function $p_3^{1/2}(\zeta)$ in $\mathbb{C}\setminus l$, $l=l_0\cup l_1\cup l_2$, by the condition $f(\zeta)\sim\zeta^3$, $\zeta\to\infty$. The branch is pure imaginary on the cut sides,

$$f(\zeta)=\pm(-1)^m i|p_3^{1/2}(\xi)|,\quad \zeta=\xi\pm i0,\quad \xi\in l_m,\quad m=0,1,2,$$

$$f(\xi)=|p_3^{1/2}(\xi)|,\quad -1<\xi<k_1,\quad f(\xi)=-|p_3^{1/2}(\xi)|,\quad k_2<\xi<1. \tag{4.2}$$

Similarly to the case $n=2$, on the surface $\mathcal{R}$, we introduce the functions (3.1) which satisfy the Riemann–Hilbert problems (3.3), (3.2). Their solution in the first sheet has the form

$$F_+(\zeta)=\frac{R_+(\zeta)}{f(\zeta)},\quad F_-(\zeta)=\frac{iR_-(\zeta)}{f(\zeta)}, \tag{4.3}$$

where

$$R_\pm(\zeta)=A_1^\pm+A_2^\pm\zeta+A_3^\pm\frac{f(\zeta)+f(\zeta_\infty)}{\zeta-\zeta_\infty}-A_3^\pm\frac{f(\zeta)-f(\bar\zeta_\infty)}{\zeta-\bar\zeta_\infty}+(A_4^\pm+iA_5^\pm)$$
$$\times\frac{f(\zeta)+f(\zeta_\infty)+f'(\zeta_\infty)(\zeta-\zeta_\infty)}{(\zeta-\zeta_\infty)^2}-(A_4^\pm-iA_5^\pm)\frac{f(\zeta)-f(\bar\zeta_\infty)-f'(\bar\zeta_\infty)(\zeta-\bar\zeta_\infty)}{(\zeta-\bar\zeta_\infty)^2}, \tag{4.4}$$

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where $A_j^\pm$ $(j = 1, 2, \dots, 5)$ are free real constants. Analysis of these functions in a neighborhood of the point ζ_∞ yields

$$F_\pm(\zeta) = i^{1/2\mp 1/2}\left(\frac{2(A_4^\pm + iA_5^\pm)}{(\zeta-\zeta_\infty)^2} + \frac{2A_3^\pm}{\zeta-\zeta_\infty}\right) + O(1), \quad \zeta \to \zeta_\infty. \tag{4.5}$$

By virtue of the behavior of the functions $F_\pm(\zeta) = -c_{-1}(b \pm \bar{a})(\zeta-\zeta_\infty)^{-2} + O(1)$, $\zeta \to \zeta_\infty$, we immediately find

$$A_3^\pm = 0, \quad A_4^+ = -\frac{1}{2}(c'\alpha^+ - c''\beta^+), \quad A_4^- = -\frac{1}{2}(c'\beta^- + c''\alpha^-),$$

$$A_5^+ = -\frac{1}{2}(c'\beta^+ + c''\alpha^+), \quad A_5^- = \frac{1}{2}(c'\alpha^- - c''\beta^-). \tag{4.6}$$

The coefficients $A_1^\pm$ and $A_2^\pm$ are still not determined in the expression of the function $\omega'(\zeta)$, that is,

$$\omega'(\zeta) = \frac{1}{2\bar{a}f(\zeta)}\left[A_1 + A_2\zeta + (A_4+iA_5)\frac{f(\zeta)+f(\zeta_\infty)+f'(\zeta_\infty)(\zeta-\zeta_\infty)}{(\zeta-\zeta_\infty)^2}\right.$$

$$\left. -(A_4-iA_5)\frac{f(\zeta)-f(\bar\zeta_\infty)-f'(\bar\zeta_\infty)(\zeta-\bar\zeta_\infty)}{(\zeta-\bar\zeta_\infty)^2}\right], \tag{4.7}$$

where $A_j = A_j^+ - iA_j^-$. To determine the coefficients A_1 and A_2, we need to guarantee that the function $z = \omega(\zeta)$ is a single-valued map, that is, to force the function $\omega'(\zeta)$ to meet the three conditions

$$\int_{l_m} \omega'(\zeta)d\zeta = 0, \quad m = 0, 1, 2, \tag{4.8}$$

or, equivalently,

$$I_{m0}A_1 + I_{m1}A_2 = -J_m, \quad m = 0, 1, 2, \tag{4.9}$$

where

$$I_{mj} = \int_{l_m} \frac{\xi^j d\xi}{f(\xi)}, \quad j = 0, 1,$$

$$J_m = (A_4+iA_5)\int_{l_m} \frac{[f(\zeta_\infty)+f'(\zeta_\infty)(\xi-\zeta_\infty)]d\xi}{f(\xi)(\xi-\zeta_\infty)^2}$$

$$+ (A_4-iA_5)\int_{l_m} \frac{[f(\bar\zeta_\infty)+f'(\bar\zeta_\infty)(\xi-\bar\zeta_\infty)]d\xi}{f(\xi)(\xi-\bar\zeta_\infty)^2}, \quad m = 0, 1, 2. \tag{4.10}$$

The first two equations in (4.9) constitute an inhomogeneous system of two complex equations with respect to complex constants A_1 and A_2. The coefficients of the system, the integrals I_{mj} $(j, m = 0, 1)$, are the A-periods of the abelian integrals

$$\int_{(1/k,0)}^{(\zeta,u(\zeta))} \frac{\xi^j d\xi}{u(\xi)}, \quad j = 0, 1, \tag{4.11}$$

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associated with the genus-2 Riemann surface $\mathcal{R}$ of the algebraic function $u^2(\xi) = p_3(\xi)$. Therefore the 2×2 matrix $\{I_{mj}\}$ $(j, m = 0, 1)$ is not singular, and the unique solution is given by

$$A_1 = \frac{J_1 I_{01} - J_0 I_{11}}{\Delta}, \quad A_2 = \frac{J_0 I_{10} - J_1 I_{00}}{\Delta}, \tag{4.12}$$

where $\Delta = I_{00}I_{11} - I_{01}I_{10}$. The third equation in (4.9) is transformed to the form

$$J_1(I_{01}I_{20} - I_{00}I_{21}) + J_0(I_{10}I_{21} - I_{11}I_{20}) + J_2\Delta = 0 \tag{4.13}$$

and satisfied identically. This is due to the fact that the corresponding abelian integrals in the right hand-side in (4.9) can be represented as a linear combination of the two basis integrals (4.11).

Since the parameters k, k_1, k_2, and $\zeta_\infty = \zeta'_\infty + i\zeta''_\infty$ are free, the derivative $\omega'(\zeta)$ generates a five-parametric family of conformal mappings (we do not count the two free scaling parameters $c_{-1} = c' + ic''$) which transforms the slit domain $\mathbb{C}\setminus l$ into the triple connected domain D^e. By integrating the function (4.7) we find the integral representation of the conformal map

$$\omega(\zeta) = \frac{1}{2\bar{a}}\Bigg\{ -\frac{A_4 + iA_5}{\zeta - \zeta_\infty} + \frac{A_4 - iA_5}{\zeta - \bar{\zeta}_\infty} + \int_{\zeta_0}^{\zeta} \Bigg[A_1 + A_2\xi + (A_4 + iA_5)$$

$$\times \frac{f(\zeta_\infty) + f'(\zeta_\infty)(\xi - \zeta_\infty)}{(\xi - \zeta_\infty)^2} + (A_4 - iA_5)\frac{f(\bar{\zeta}_\infty) + f'(\bar{\zeta}_\infty)(\xi - \bar{\zeta}_\infty)}{(\xi - \bar{\zeta}_\infty)^2}\Bigg] \frac{d\xi}{f(\xi)}\Bigg\}. \tag{4.14}$$

To find the actual profile of the holes L_m, we let ζ run the cuts l_m and obtain

$$z = \mathcal{I}(\zeta) \pm i\mathcal{J}(-1/k, \zeta), \quad \zeta \in l_0^\mp, \quad z \in L_0,$$

$$z = \mathcal{I}(\zeta) + \mathcal{J}(-1, k_1) \mp i\mathcal{J}(k_1, \zeta), \quad \zeta \in l_1^\mp, \quad z \in L_1,$$

$$z = \mathcal{I}(\zeta) + \mathcal{J}(-1, k_1) - \mathcal{J}(k_2, 1) \pm i\mathcal{J}(1, \zeta), \quad \zeta \in l_2^\mp, \quad z \in L_2, \tag{4.15}$$

where we denote

$$\mathcal{I}(\zeta) = \frac{1}{2\bar{a}}\left(-\frac{A_4 + iA_5}{\zeta - \zeta_\infty} + \frac{A_4 - iA_5}{\zeta - \bar{\zeta}_\infty}\right),$$

$$\mathcal{J}(d, \zeta) = \frac{1}{2\bar{a}}\int_d^{\zeta} \Bigg[A_1 + A_2\xi + (A_4 + iA_5)\frac{f(\zeta_\infty) + f'(\zeta_\infty)(\xi - \zeta_\infty)}{(\xi - \zeta_\infty)^2}$$

$$+(A_4 - iA_5)\frac{f(\bar{\zeta}_\infty) + f'(\bar{\zeta}_\infty)(\xi - \bar{\zeta}_\infty)}{(\xi - \bar{\zeta}_\infty)^2}\Bigg] \frac{d\xi}{|f(\xi)|}. \tag{4.16}$$

The function $\omega'(\zeta)$ may not have zeros in the slit domain $\mathbb{C}\setminus l$. Otherwise the function $\psi(\zeta)$ has unacceptable poles. As in the case of the doubly connected domain, we introduce a function $\eta(\zeta)$ which shares the zeros with the function $\omega'(\zeta)$ and is free of singularities of $\omega'(\zeta)$,

$$\omega'(\zeta) = \frac{\eta(\zeta)}{2\bar{a}f(\zeta)(\zeta - \zeta_\infty)^2(\zeta - \bar{\zeta}_\infty)^2}, \tag{4.17}$$

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where

$$\eta(\zeta) = (A_1 + A_2\zeta)(\zeta - \zeta_\infty)^2(\zeta - \bar{\zeta}_\infty)^2 + (A_4 + iA_5)(\zeta - \bar{\zeta}_\infty)^2[f(\zeta) + f(\zeta_\infty)$$

$$+ f'(\zeta_\infty)(\zeta - \zeta_\infty)] - (A_4 - iA_5)(\zeta - \zeta_\infty)^2[f(\zeta) - f(\bar{\zeta}_\infty) - f'(\bar{\zeta}_\infty)(\zeta - \bar{\zeta}_\infty)]. \tag{4.18}$$

The zero counting formula applied yields that the number of zeros, Z, of the function $\eta(\zeta)$ in the slit domain is

$$Z = 5 + \frac{1}{2\pi i}\sum_{m=0}^{2}\int_{l_m}\frac{\eta'(\zeta)d\zeta}{\eta(\zeta)}. \tag{4.19}$$

Here, we used the asymptotics at infinity

$$f(\zeta) \sim \zeta^3, \quad f'(\zeta) \sim 3\zeta^2,$$

$$\eta(\zeta) \sim (A_2 + 2iA_5)\zeta^5, \quad \eta'(\zeta) \sim 5(A_2 + 2iA_5)\zeta^4, \quad \zeta \to \infty, \tag{4.20}$$

and the limit as $R \to \infty$ of the integral over a circle Γ_R of radius R centered at the origin,

$$\lim_{R\to\infty}\int_R \frac{\eta'(\zeta)d\zeta}{\eta(\zeta)} = 5. \tag{4.21}$$

On computing the integrals in (4.19) it is possible to establish that if $\gamma = |b/a| < 1$, then $Z = 2$, and when $\gamma > 1$, the function $\omega'(\zeta)$ has even more zeros, $Z = 8$. In the limiting case $\gamma = 1$, these points lying in the contour l are removable singularities of the function $\psi(\zeta)$ due to the relation $\overline{F_+(\xi) + F_-(\xi)} = -[F_+(\xi) - F_-(\xi)]$, $\xi \in l$. When $\gamma = 1$, the contours L_j are straight segments in the z-plane.

Thus, the conformal map which transforms any finite point ζ_∞ of the slit domain into the infinite point $z = \infty$ gives rise to a finite number of poles of the function $\Psi(z)$ regardless of the value of the parameter $\gamma \neq 1$. This means that such a family of maps cannot be employed to identify equal-strength cavities. At the same time, when $\gamma < 1$, the map $z = f(\zeta)$ given by (4.15) generates some contours L_m which do not intersect each other. Samples of such contours when $\gamma < 1$ and the function $\Psi(z)$ has two inadmissible poles in the domain D^e are given in Figures 7(a)–(c). Shown in Figure 7(d), the three loops intersect each other, $\gamma > 1$, and the function $\Psi(z)$ has eight inadmissible poles in the domain D^e.

5. n cavities: Domain D_0^e, $n \geq 3$, and $\zeta_\infty = \infty$. The family of mappings derived in the previous section gives rise to unacceptable poles of the complex potential $\Psi(z)$. All the mappings share the same property: the infinite point $z = \infty$ is the image of a certain finite point ζ_∞ in the slit domain. Since the case $\omega(\infty) = \infty$ cannot be extracted from the solution derived in section 4, we consider this case separately. Also, for generality, we assume that n is not just equal to 3, but any finite integer $n \geq 3$. We confine ourselves to the family of domains $D_0^e \subset D^e$, which are the images of slit domains $\mathcal{D}^e$ such that all the n slits lie in the same line, $\mathcal{D}^e = \mathbb{C} \setminus l$, $l = l_0 \cup \ldots l_{n-1}$, and $l_j = [k_{2j}, k_{2j+1}]$, $j = 0, \ldots, n-1$, $k_{2n-1} = -k_0 = 1$, $-1 < k_1 < \cdots < k_{2n-2} < 1$. The function $\omega(\zeta)$ has a simple pole in the vicinity of the infinite point and for large z can be represented by (2.7). We emphasize that not every triply connected domain

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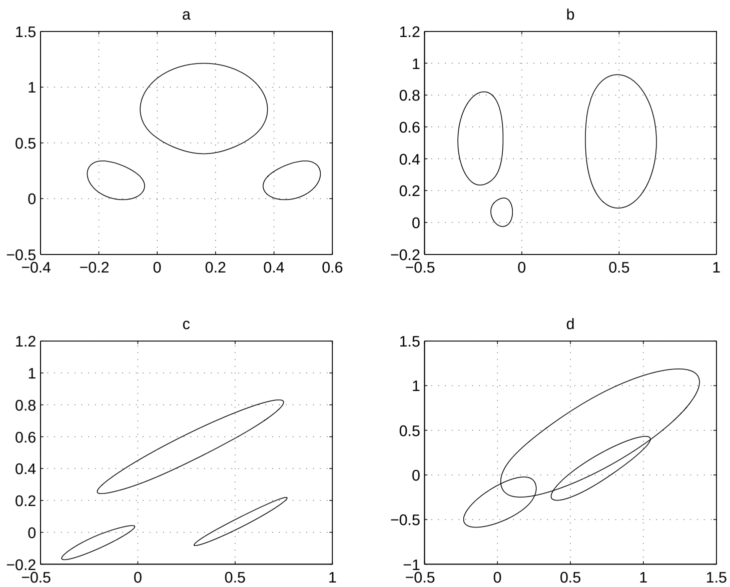

FIG. 7. *The contours L_0, L_1, and L_2 when ζ_∞ is a finite point,* $c' = 1, c'' = 0$, $k_1 = -0.8$, $k = 0.2$, $k_2 = 0.8$, $p = \tau = 0$. (a) $\zeta_\infty = i$, $\sigma_1^\infty = 1$, $\sigma_2^\infty = 2$, $\tau^\infty = 0$ ($\gamma = 1/3$); (b) $\zeta_\infty = 1 + i$, $\sigma_1^\infty = 1$, $\sigma_2^\infty = 2$, $\tau^\infty = 0$ ($\gamma = 1/3$); (c) $\zeta_\infty = i$, $\sigma_1^\infty = 2$, $\sigma_2^\infty = 1$, $\tau^\infty = 1$ ($\gamma = 0.7453560$); (d) $\zeta_\infty = i$, $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = 2$ ($\gamma = 2$). *In cases* (a)–(c) *the function* $\Psi(z)$ *has two inadmissible poles in the domain* D^e, *while in case* (d) *the number of poles is eight.*

D^e is the image of a slit domain $\mathcal{D}^e$ such that $z = \infty$ is the image of $\zeta = \infty$. Needless to say, not every n-connected ($n \geq 4$) domain D^e is the image of the exterior of n slits lying in the same line. On studying the family of mappings $\omega : \mathcal{D}^e \to D_0^e$ when $\omega(\infty) = \infty$, we try to find a set of equal-strength holes such that $\omega'(\zeta)$ does not have zeros in $\mathcal{D}^e$, and therefore the function $\Psi(z)$ is analytic everywhere in the domain D_0^e.

First we fix the branch $f(\zeta)$ of the function $p_n^{1/2}(\zeta)$,

$$p_n(\zeta) = \prod_{j=0}^{2n-1} (\zeta - k_j), \tag{5.1}$$

in the domain $\mathcal{D}^e$ by the condition $f(\zeta) \sim \zeta^n$. The functions $F_\pm(\zeta)$ are bounded at infinity, and their counterparts defined in the Riemann surface have simple poles at the branch points of the surface $\mathcal{R}$. We have

$$F_+(\zeta) = \frac{1}{f(\zeta)} \sum_{j=1}^{n+1} A_j^+ \zeta^{j-1} + iA_0^+, \quad F_-(\zeta) = \frac{i}{f(\zeta)} \sum_{j=1}^{n+1} A_j^- \zeta^{j-1} + A_0^-, \tag{5.2}$$

where $A_j^\pm$ are arbitrary real constants. By expanding these functions for large z and comparing these expansions with the asymptotics of $F_\pm(\zeta)$ in (2.12), we find

$$A_n^\pm = -kA_{n+1}^\pm, \quad A_{n+1}^- = \alpha^- c'' + \beta^- c', \quad A_{n+1}^+ = \alpha^+ c' - \beta^+ c'',$$

$$A_0^- = \alpha^- c' - \beta^- c'', \quad A_0^+ = \alpha^+ c'' + \beta^+ c', \tag{5.3}$$

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where $k = \frac{1}{2}(k_1 + k_2 + \cdots + k_{2n-2})$. The derivative of the conformal map $\omega'(\zeta)$,

$$\omega'(\zeta) = \frac{1}{2\bar{a}}\left[iA_0^+ - A_0^- + \frac{1}{f(\zeta)}\sum_{j=1}^{n+1}(A_j^+ - iA_j^-)\zeta^{j-1}\right], \tag{5.4}$$

has to generate a one-to-one map. This is guaranteed by the following n complex conditions:

$$\int_{l_m} \omega'(\zeta)d\zeta = 0, \quad m = 0, 1, \ldots, n-1. \tag{5.5}$$

These conditions can be rewritten as

$$\sum_{j=0}^{n} a_{mj}(A_{j+1}^+ - iA_{j+1}^-) = 0, \quad m = 0, 1, \ldots, n-1. \tag{5.6}$$

Here,

$$a_{mj} = \int_{l_m} \frac{\zeta^j d\zeta}{f(\zeta)}, \quad m = 0, 1, \ldots, n-1, \quad j = 0, 1, \ldots, n. \tag{5.7}$$

The integrals a_{mj} $(m, j = 0, 1, \ldots, n-2)$ are the A-periods of the abelian integrals

$$\int_{(1,0)}^{(\zeta, u(\zeta))} \frac{\xi^j d\xi}{u(\xi)}, \quad j = 0, 1, \ldots, n-2, \tag{5.8}$$

associated with the genus-$(n-1)$ Riemann surface $\mathcal{R}$ of the algebraic function $u^2(\xi) = p_n(\xi)$. Therefore the matrix a_{mj} $(m, j = 0, 1, \ldots, n-2)$ is not singular [16]. Denote

$$I_{mj} = \int_{l_m^+} \frac{\xi^j d\xi}{|f(\xi)|}, \qquad m = 0, 1, \ldots, n-1, \quad j = 0, 1, \ldots, n. \tag{5.9}$$

The coefficients $A_j^\pm$ are uniquely determined through the known coefficients $A_{n+1}^\pm$ from the nonsingular system

$$\sum_{j=0}^{n-2} I_{mj}A_{j+1}^\pm = -(I_{mn} - kI_{mn-1})A_{n+1}^\pm, \quad m = 0, 1, \ldots, n-2. \tag{5.10}$$

The last equation in (5.6) is satisfied automatically because the basis of the abelian integrals (5.8) has dimension $n-1$, and the $n \times n$ matrix

$$\begin{pmatrix} I_{00} & \ldots & I_{0n-2} & I_{0n} - kI_{0n-1} \\ \ldots & \ldots & \ldots & \ldots \\ I_{n-10} & \ldots & I_{n-1n-2} & I_{n-1n} - kI_{n-1n-1} \end{pmatrix} \tag{5.11}$$

is singular.

To determine the number of zeros of the function $\omega'(\zeta)$, we introduce the function

$$\eta(\zeta) = \sum_{j=1}^{n+1}(A_j^+ - iA_j^-)\zeta^{j-1} + (iA_0^+ - A_0^-)f(\zeta). \tag{5.12}$$

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The functions $\eta(\zeta)$ and $\omega'(\zeta)$ share their zeros. The number of zeros of the function $\eta(\zeta)$ coincides with the number of inadmissible poles of the function $\psi(\zeta)$ and is given by

$$Z = \frac{1}{2\pi i}\left(\sum_{j=0}^{n-1}\int_{l_j} + \lim_{R\to\infty}\int_{\Gamma_R}\right)\frac{\eta'(\zeta)d\zeta}{\eta(\zeta)} = n + \frac{1}{2\pi i}\sum_{j=0}^{n-1}\int_{l_j}\frac{\eta'(\zeta)d\zeta}{\eta(\zeta)}. \tag{5.13}$$

Our numerical tests implemented for the case $n = 3$ reveal that if $\gamma = |b/a| < 1$, then $Z = 0$, and the function $\Psi(z)$ is analytic in D_0^e and continuous up to the boundary. If $\gamma > 1$, then $Z = 6$, and the function $\Psi(z)$ has six poles in the domain D_0^e. When $|a| = |b|$, the domain D_0^e is a set of straight segments, and the function $\Psi(z)$ has removable singularities in the boundary of D_0^e. As in the simply and doubly connected cases, if $|a| < |b|$, then the problem does not have solutions. When $|a| > |b|$, the solution exists, and the conformal map $z = \omega(\zeta)$ is defined up to seven arbitrary constants, an additive constant, two scaling parameters c' and c'', and the four parameters k_j $(j = 1, \dots, 4)$. Integrating (5.4) and employing (5.5), we have

$$z = \frac{1}{2\bar{a}}[(iA_0^+ - A_0^-)\zeta \mp iJ(-1,\zeta)] + B, \quad \zeta \in l_0^\pm, \quad z \in L_0,$$

$$z = \frac{1}{2\bar{a}}[(iA_0^+ - A_0^-)\zeta + J(k_1,k_2) \pm iJ(k_2,\zeta)] + B, \quad \zeta \in l_1^\pm, \quad z \in L_1,$$

$$z = \frac{1}{2\bar{a}}[(iA_0^+ - A_0^-)\zeta + J(k_1,k_2) - J(k_3,k_4) \mp iJ(k_4,\zeta)] + B, \quad \zeta \in l_2^\pm, \quad z \in L_2. \tag{5.14}$$

Here, B is an additive constant and without loss can be taken as zero, and J is the real integral

$$J(\alpha,\beta) = \int_\alpha^\beta \frac{1}{|f(\xi)|}\sum_{j=1}^{4}(A_j^+ - iA_j^-)\xi^{j-1}d\xi. \tag{5.15}$$

Figures 8(a)–(c) show how the change of the loading parameter γ and the conformal mapping parameters affects the profiles of equal-strength cavities in the case $n = 3$ and when $\zeta_\infty = \infty$. Figure 8(d) gives a sample of the contours L_j when $\gamma > 1$. Although the contours L_j do not have common points, the function $\Psi(z)$ has six poles in the domain D^e, and the solution does not exist.

Conclusions. We have analyzed the inverse plane problem of constructing n equal-strength cavities in an unbounded elastic body when constant loading is applied at infinity and to the cavities' boundaries. By advancing the method of conformal mappings employed in [3] for $n = 1$ and $n = 2$ (two symmetric holes) to general doubly and triply connected domains, we have found by quadratures a four- and a seven-parametric family of mappings and therefore a four- and a seven-parametric family of two and three equal-strength cavities, respectively. In both cases two out of four and two out of seven free parameters, respectively, are scaling parameters. For the doubly connected problem, the map ω transforms a slit domain $\mathcal{D}^e$, the exterior of two slits $[-1/k, -1]$ and $[1, 1/k]$ $(0 < k < 1)$, into the elastic domain D^e, the exterior of the holes, and $\omega(\zeta_\infty) = \infty$, $\zeta_\infty \in (-1, 1)$. For the triply connected problem, we analyzed two cases of the preimage of the infinite point: ζ_∞ is a finite point and $\zeta_\infty = \infty$. In the former case, $\mathcal{D}^e$ is the exterior of three slits $[-1/k, -1]$,

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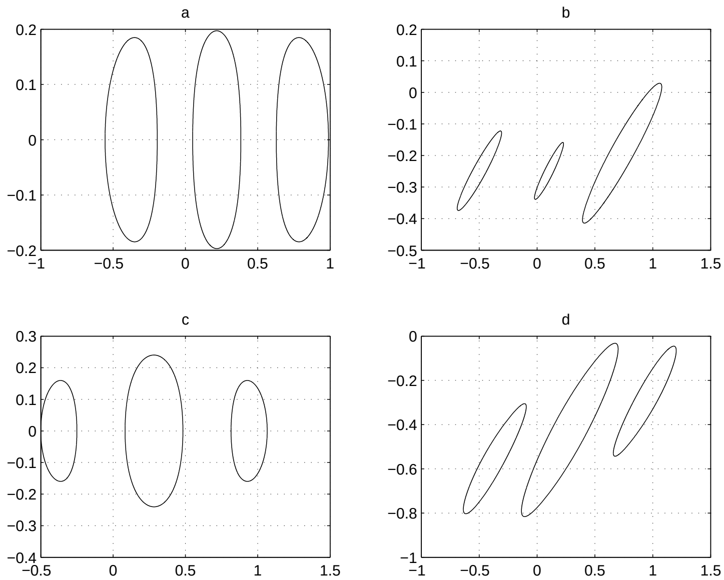

FIG. 8. *Three equal-strength holes when* $\zeta_\infty = \infty$, $c' = 1$, *and* $c'' = 0$, $\tau = 0$. (a) $\sigma_1^\infty = 0$, $\sigma_2^\infty = 1$, $\tau^\infty = 0$, $p = 5$ $(\gamma = 1/9)$, $k_1 = -k_4 = -0.35$, $k_2 = -k_3 = -0.3$; (b) $\sigma_1^\infty = 2$, $\sigma_2^\infty = 1$, $\tau^\infty = 1$, $p = 0$ $(\gamma = 0.745360)$, $k_1 = -0.5$, $k_2 = -0.3$, $k_3 = 0$, $k_4 = 0.1$; (c) $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = 0$, $p = 5$ $(\gamma = 0)$, $k_1 = -k_4 = -0.5$, $k_2 = -k_3 = -0.4$; (d) $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = 1.5$, $p = 0$ $(\gamma = 1.5)$, $k_1 = -k_4 = -0.5$, $k_2 = -k_3 = -0.4$. *In cases* (a)–(c) *the function* $\Psi(z)$ *is analytic, while in case* (d) *the function* $\Psi(z)$ *has six inadmissible poles.*

$[k_1, k_2]$, and $[1, 1/k]$, while in the second case the slits are $[-1, k_1]$, $[k_2, k_3]$, and $[k_4, 1]$. The conformal mappings are derived in terms of elliptic integrals for $n = 2$ and hyperelliptic integrals for $n = 3$. We have also analyzed the zeros of the derivative $\omega'(\zeta)$ of the conformal mapping and shown that these zeros, if they exist, generate inadmissible poles of the solution. If $\gamma = |b/a| < 1$ (a and b are complex loading parameters), then the Kolosov–Muskhelishvili potential $\Psi(z)$ is free of poles when $n = 1$ and $n = 2$. In the triply connected case, if the conformal map is chosen such that $\omega(\infty) = \infty$, then the centers of the cavities are located in the same line, and the condition $\gamma < 1$ is necessary and sufficient for the solution to exist. If $\gamma > 1$, then the function $\Psi(\zeta)$ has two, four, and six inadmissible poles in the cases $n = 1$, $n = 2$, and $n = 3$ ($\omega(\infty) = \infty$), respectively. If $\gamma = 1$, then the equal-strength cavities are straight segments, and the potential $\Psi(z)$ has removable singularities in the boundary. It has also been discovered that when $\gamma > 1$ and big enough, then the contours intersect each other. If $n = 3$ and ζ_∞ is a finite point, then $\Psi(\zeta)$ has two and eight poles for the cases $\gamma < 1$ and $\gamma > 1$, respectively; that is, there does not exist a set of three equal-strength cavities whose centers do not lie in the same line. By using the Riemann–Hilbert problem on a genus-$(n-1)$ surface and the theory of abelian integrals, we have also derived an integral representation in terms of hyperelliptic integrals for a $2n$-parametric family (two of them are scaling parameters) of conformal mappings for the case $n \geq 4$ and $\omega(\infty) = \infty$, where the slits lie in the same line. We conjecture that (i) if the centers of equal-strength cavities lie in the same line and $\omega(\infty) = \infty$, then the function $\Psi(z)$ is free of poles and the solution exists when $\gamma < 1$ and the function $\Psi(z)$ has $2n$ poles in the domain D^e when $\gamma > 1$, and (ii) regardless of the value of the parameter $\gamma \neq 1$, if $|\zeta_\infty| < \infty$, there does not

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exist a set of $n \geq 4$ equal-strength cavities whose centers are not in the same line.

Appendix A. One cavity. If $n = 1$, then with no loss of generality, l_0 and the point ζ_∞ may be selected as $[-1, 1]$ and ∞, respectively. Such a map is defined up to one real parameter, and we assume that $\operatorname{Im} c_{-1} = 0$. Denote $p_1(\zeta) = \zeta^2 - 1$. We fix the branch $f(\zeta)$ of $p^{1/2}(\zeta)$ in the ζ-plane cut along l_0 by the condition $p_1(\zeta) \sim \zeta$, $\zeta \to \infty$. This branch is pure imaginary on the sides of the cut, $f(\xi \pm i0) = \pm i|f(\xi)|$, $-1 < \xi < 1$, and is real in the real axis outside the cut. We have

$$F_+(\zeta) = \frac{A_1^+ + A_2^+ \zeta}{f(\zeta)} + iA_0^+, \quad F_-(\zeta) = \frac{i(A_1^- + A_2^- \zeta)}{f(\zeta)} + A_0^-, \tag{A.1}$$

where $A_j^\pm$ $(j = 0, 1, 2)$ are arbitrary real constants. Due to the asymptotics (2.12) of the functions $F_\pm(\zeta)$ we have

$$A_1^\pm = 0, \quad A_2^+ = c_{-1} \operatorname{Re}(b + \bar{a}), \quad A_2^- = c_{-1} \operatorname{Im}(b - \bar{a}),$$

$$A_0^+ = c_{-1} \operatorname{Im}(b + \bar{a}), \quad A_0^- = c_{-1} \operatorname{Re}(b - \bar{a}). \tag{A.2}$$

Substituting these coefficients into (A.1) and then into (2.14), we derive

$$\omega'(\zeta) = \frac{c_{-1}}{2}\left[m_- + \frac{m_+ \zeta}{f(\zeta)}\right], \quad m_\pm = 1 \pm \frac{\bar{b}}{\bar{a}}, \quad \psi(\zeta) = \bar{a}\frac{(a+b)\zeta - (a-b)f(\zeta)}{(\bar{a}+\bar{b})\zeta + (\bar{a}-\bar{b})f(\zeta)}. \tag{A.3}$$

Notice that

$$\int_{l_0} \omega'(\zeta) d\zeta = 0, \tag{A.4}$$

and the map is one-to-one. The conformal map $z = \omega(\zeta)$ is defined up to an additive constant B, has the form [3]

$$\omega(\zeta) = \frac{c_{-1}}{2}[m_- \zeta + m_+(\zeta^2 - 1)^{1/2}] + B, \tag{A.5}$$

and $z = \omega(\zeta)$, $\zeta \in l_0$, is a parametric equation of a family of ellipses (c_{-1} is an arbitrary nonzero real parameter). Denote $a_1 + ib_1 = \frac{1}{2}c_{-1}m_-$, $a_2 + ib_2 = \frac{1}{2}c_{-1}m_+$ and put $B = 0$. Then from (A.5) we may write

$$x = a_1\xi \mp b_2\sqrt{1 - \xi^2}, \quad y = b_1\xi \pm a_2\sqrt{1 - \xi^2}. \tag{A.6}$$

On excluding $\sqrt{1 - \xi^2}$ we express ξ through x and y as

$$\xi = \frac{a_2 x + b_2 y}{a_1 a_2 + b_1 b_2}. \tag{A.7}$$

We next square x and y in (A.6) and employ formula (A.7). After simple algebra we obtain a quadratic equation in x and y:

$$(a_2^2 + b_1^2)x^2 + (a_1^2 + b_2^2)y^2 - 2(a_1 b_1 - a_2 b_2)xy = (a_1 a_2 + b_1 b_2)^2. \tag{A.8}$$

Since its discriminant $-4\Delta = -4(a_1a_2 + b_1b_2)^2$ is negative, (A.8) represents an ellipse.

Now, the function $\psi(\zeta)$ has to be analytic everywhere in $\mathbb{C} \setminus l_0$. Possible singularities of this function coincide with the zeros of the derivative of the map or,

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equivalently, with the zeros of the function $\eta(\zeta) = m_-\sqrt{\zeta^2-1} + m_+\zeta$. The number of zeros of $\eta(\zeta)$ inside the contour Γ is given by

$$Z = \frac{1}{2\pi i}\int_\Gamma \frac{\eta'(\zeta)d\zeta}{\eta(\zeta)}. \tag{A.9}$$

Computing the integral over the contour Γ_R, letting $R \to \infty$, and transforming the integral over the contour l_0, we obtain

$$Z = 1 - \frac{m_+m_-}{\pi(m_+^2 - m_-^2)}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}(\xi^2+\mu^2)}, \quad \mu^2 = \frac{m_-^2}{m_+^2 - m_-^2}. \tag{A.10}$$

This integral can be computed by making the subsequent substitutions $\xi = \cos\phi$ and $e^{i\theta} = w$ and applying the theory of residues in the w-plane. Eventually we derive that $Z = 0$ if $|b/a| < 1$ and $Z = 2$ if $\gamma = |b/a| > 1$. These poles can be easily determined:

$$\zeta_{1,2} = \pm\frac{i}{2}\left(\sqrt{\frac{\bar{b}}{\bar{a}}} - \sqrt{\frac{\bar{a}}{\bar{b}}}\right). \tag{A.11}$$

In the case $\gamma = 1$, the ellipse becomes a segment, while the poles (A.11) of the function $\omega'(\zeta)$, $\zeta_{1,2} = \pm\sin\frac{1}{2}(\arg a - \arg b)$, lie in the segment l_0. Analysis of the second formula in (A.3) shows that they are removable singularities of the function $\psi(\zeta)$.

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