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RAPID TRANSIENT MOTION OF A THIN RIGID BODY IN AN ELASTIC MEDIUM*

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Abstract. A two-dimensional transient problem on motion of a thin finite rigid body at sub-Rayleigh speed in an elastic medium is analyzed. When the body moves it leaves a semi-infinite crack-like cavity behind, and a finite crack-like cavity is formed ahead of the body. By means of the Laplace and Fourier transforms the problem maps into a Riemann–Hilbert problem for three pairs of piecewise holomorphic functions with a triangular matrix coefficient. The diagonal elements have plus- or minus-infinite indices, and some nonzero elements are not meromorphic functions. An approximate method based on rational approximation is proposed. It ultimately leads to an infinite system with an exponential rate of convergence of the truncated system solution to the solution of the infinite system. The physical problem is also reformulated as a system of two singular integral equations on a finite and a semi-infinite intervals. The solution is expanded in terms of the Chebyshev polynomials, and the integral system reduces to a second-kind infinite system of linear algebraic equations subjected to an additional transcendental equation. This equation serves for determination of the fore detachment point of the body from the elastic medium. The coefficients of the system originally represented by weakly convergent triple integrals are transformed into fast convergent single integrals possessing Bessel functions. Numerical results for the stress intensity factor, the resistance force, and stresses demonstrate efficiency of the method.

Key words. rapid cavitating motion, dynamic contact of a rigid body and elastic medium, vector Riemann–Hilbert problem, system of singular integral equations

AMS subject classifications. 74H10, 74M15, 74R15, 74S70, 74G70

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1. Introduction.

There are a significant number of modern studies on steady-state motion of a thin two-dimensional rigid body in an elastic medium [6], [7], [13], [14], [15], [3]. The sub-Rayleigh steady-state motion of a semi-infinite rigid body with a finite fore crack-like cavity was treated in [6]. Motion of a thin finite body at super-Rayleigh and transonic speed with a semi-infinite cavity behind and without a cavity ahead of the body was analyzed in [7] and [13], respectively. Subsonic and transonic steady-state regimes for motion of a finite body trailed by a semi-infinite cavity and without a fore cavity when there is Coulomb friction in the contact zone were considered in [14], [15]. Steady-state subsonic frictional motion of a thin rigid body in an elastic medium when there are a semi-infinite rear cavity and a finite cavity ahead of the body was treated in [3] by solving the associated Riemann–Hilbert problem with a piecewise constant coefficient.

Modeling of rapid motion of rigid bodies in an elastic medium is important for penetration mechanics since many phenomena of two-dimensional models are similar to the ones of axisymmetric problems. The similarities between two-dimensional and three-dimensional axisymmetric models used in penetration mechanics were reported and confirmed by computations [9] and experimental tests [11]; in these studies the projectiles were modeled as thin plates and rods. A correspondence between the

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solutions to plane and axisymmetric mixed boundary value problems of contact and fracture mechanics was recently established in [5].

Notice that all existing treatments [6], [7], [13], [14], [15], [3] of thin two-dimensional body motion in an elastic medium concern the steady-state regime. The objective of this work is to develop an efficient method for dealing with the two-dimensional *transient* model of rapid penetration at constant sub-Rayleigh speed of thin finite rigid bodies into an elastic medium. The body is assumed to be self-propelled, and the motion occurs under the impetus of propulsive forces. When it moves, the body leaves a semi-infinite crack-like cavity behind and has a finite cavity ahead.

In section 2, we state the governing equations and the boundary conditions corresponding to the transient problem on motion of a thin body at constant sub-Rayleigh speed. We map the physical problem into a Riemann–Hilbert problem for three pairs of holomorphic functions with a triangular matrix coefficient in section 3. The diagonal elements of the matrix have plus-minus infinite indices, while the matrix elements are not meromorphic functions. Since this problem cannot be solved exactly by the methods available, we modify the method [1], [2] by proposing an approximate method based on rational approximation, poles elimination, and consequent reduction of the problem to an infinite system of linear algebraic equation. Its rate of convergence is exponential. Note that this method is different from the one recently proposed in [12]. In addition, in section 4, we work out a method of singular integral equations. For its solution, the method of orthogonal polynomials is applied. It leads to an infinite system of linear algebraic equations of the second kind supplemented by a transcendental equation that fixes the fore point on the body profile where the body separates from the elastic medium. We determine the stress intensity factor at the fore crack-like cavity tip, the stresses, the normal displacement jump, and the resistance force and report the results of numerical tests.

2. Model problem formulation. Let, at time $t = 0$, a self-propelled rigid slim body of length b and convex profile $\{X_2 = \pm g^\circ(X_1),\ -\infty < X_3 < \infty\}$ start moving at speed V along the plane $X_2 = 0$ in the positive X_1-direction in an elastic medium split at the initial time along the half-plane $X_1 < 0, X_2 = 0$ (Figure 1). The moving rigid body leaves behind a semi-infinite crack-like cavity which, at time t, occupies the domain $-\infty < X_1 < S^-, X_2 = \pm\Omega_1(X_1)$. It is also assumed that ahead of the body a finite crack-like cavity $S^+ < X_1 < S^*, X_2 = \pm\Omega_2(X_1)$ is formed, the cavity length $S^* - S^+$ does not vary with time, and without loss $S^* = 0$. The medium is subjected to plane-strain normal loading symmetric with respect to the plane $X_2 = 0$ that would induce, in the absence of the body and the cavities, the stress field $\sigma_{22}^\circ(X_1, X_2, t)$.

We consider the subsonic regime, $V < c_R$, with c_R being the Rayleigh speed in the elastic medium whose longitudinal and shear waves speeds are c_l and c_s, $c_l = \sqrt{(\lambda + 2\mu)/\rho}$, $c_s = \sqrt{\mu/\rho}$, λ and μ are the Lamé constants, and ρ is the density. On the contact surfaces $S^- < X_1 < S^+, X_2 = \pm g^\circ(X_1)$ between the body and the elastic

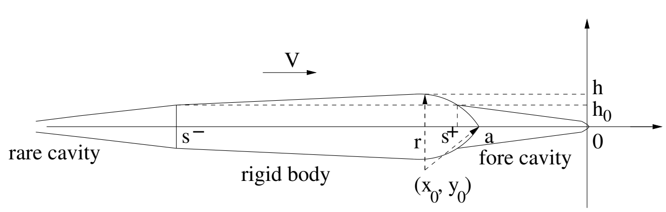

FIG. 1. *Geometry of the model.*

medium, the conditions of the Coulomb friction law are satisfied. These conditions, when linearized and written for the total stress field, read $\sigma_{12} = \pm k(\sigma_{22} + \sigma_{22}^{\circ})$, $X_2 = 0^{\pm}$, where the normal traction component σ_{22} is negative on the upper and lower surfaces of the body, and $k > 0$ is the dynamic friction coefficient.

Due to the problem symmetry it is set in the upper half-plane, and the linearized boundary conditions are written in the line $X_2 = 0$. The plane strain deformation is independent of X_3 and governed by the wave equations for the dilatational and shear potentials [8],

$$\Delta\phi = \frac{\ddot{\phi}}{c_l^2}, \quad \Delta\psi = \frac{\ddot{\psi}}{c_s^2}, \quad |X_1| < \infty, \quad X_2 > 0, \quad t > 0, \tag{2.1}$$

subjected to the boundary conditions in the line $X_2 = 0$,

$$\begin{gathered} \sigma_{22} = -\sigma_{22}^{\circ}, \quad \sigma_{12} = 0, \quad X_1 - Vt \in (-\infty, S^-) \cup (S^+, 0), \\ \sigma_{12} = k(\sigma_{22} + \sigma_{22}^{\circ}), \quad u_{2,1} = g_*(X_1 - Vt), \quad S^- < X_1 - Vt < S^+, \\ \sigma_{12} = 0, \quad u_{2,1} = 0, \quad 0 < X_1 - Vt < \infty, \end{gathered} \tag{2.2}$$

and the initial conditions

$$\phi = \phi_t = 0, \quad \psi = \psi_t = 0, \quad t = 0, \quad |X_1| < \infty, \quad 0 < X_2 < \infty. \tag{2.3}$$

Here, $g_*(X_1) = \frac{d}{dX_1} g^{\circ}(X_1)$, and $f_{,j}$ is the partial derivative of a function f with respect to X_j. The relations [8]

$$\begin{gathered} u_1 = \phi_{,1} + \psi_{,2}, \quad u_2 = \phi_{,2} - \psi_{,1}, \quad \mu^{-1}\sigma_{12} = 2\phi_{,12} - \psi_{,11} + \psi_{,22}, \\ \mu^{-1}\sigma_{11} = c^2\phi_{,11} + \left(c^2 - 2\right)\phi_{,22} + 2\psi_{,12}, \; \mu^{-1}\sigma_{22} = \left(c^2 - 2\right)\phi_{,11} + c^2\phi_{,22} - 2\psi_{,12}, \; c = c_l/c_s, \end{gathered} \tag{2.4}$$

represent the displacements and the stress tensor components in terms of the derivatives of the potentials ϕ and ψ.

It will be convenient to have the governing initial-boundary value problem written in dimensionless coordinates for dimensionless functions. Denote

$$\begin{gathered} X_1 = bx_1, \quad X_2 = bx_2, \quad t = b\tau/c_s, \quad S^{\pm} = bs^{\pm}, \quad V = c_s v, \\ \phi(X_1, X_2, t) = b^2\phi'(x_1, x_2, \tau), \quad \psi(X_1, X_2, t) = b^2\psi'(x_1, x_2, \tau), \\ \mu^{-1}\sigma_{ij}(X_1, X_2, t) = \sigma'_{ij}(x_1, x_2, \tau), \quad u_j(X_1, X_2, t) = bu'_j(x_1, x_2, \tau), \\ \mu^{-1}\sigma_{22}^{\circ}(X_1, 0, t) = \sigma_{22}^{\circ'}(x_1, \tau), \quad g_*(X_1) = g(x_1). \end{gathered} \tag{2.5}$$

This transformation brings us to the following initial-boundary value problem:

$$\begin{gathered} \phi'_{,x_1x_1} + \phi'_{,x_2x_2} = \frac{1}{c^2}\phi'_{,\tau\tau}, \quad \psi'_{,x_1x_1} + \psi'_{,x_2x_2} = \psi'_{,\tau\tau}, \quad |x_1| < \infty, \quad x_2 > 0, \quad \tau > 0, \\ \sigma'_{22} = -\sigma_{22}^{\circ'}(x_1, \tau), \quad \sigma'_{12} = 0, \quad x_1 - v\tau \in (-\infty, s^-) \cup (s^+, 0), \\ \sigma'_{12} = k[(\sigma'_{22} + \sigma_{22}^{\circ'}(x_1, \tau)], \quad u'_{2,1} = g(x_1 - v\tau), \quad s^- < x_1 - v\tau < s^+, \\ \sigma'_{12} = 0, \quad u'_{2,1} = 0, \quad 0 < x_1 - v\tau < \infty, \\ \phi' = \phi'_{\tau} = 0, \quad \psi' = \psi'_{\tau} = 0, \quad \tau = 0, \quad |x_1| < \infty, \quad 0 < x_2 < \infty. \end{gathered} \tag{2.6}$$

To formulate the model problem in the moving frame $x = x_1 - v\tau$, $y = x_2$ fixed in the fore cavity tip, we denote

$$\phi'(x_1, x_2, \tau) = \Phi(x_1 - v\tau, x_2, \tau),\ \ \psi'(x_1, x_2, \tau) = \Psi(x_1 - v\tau, x_2, \tau),$$
$$\sigma^{\circ\prime}_{22}(x_1, \tau) = \sigma(x_1 - v\tau, \tau). \tag{2.7}$$

Then we have the new governing equations

$$\alpha_l^2 \Phi_{,xx} + \Phi_{,yy} + \frac{2v}{c^2}\Phi_{,x\tau} = \frac{1}{c^2}\Phi_{,\tau\tau},$$
$$\alpha_s^2 \Psi_{,xx} + \Psi_{,yy} + 2v\Psi_{,x\tau} = \Psi_{,\tau\tau}, \quad |x| < \infty, \quad y > 0, \quad \tau > 0, \tag{2.8}$$

subjected to the boundary conditions in the line $y = 0$

$$\sigma_{yy} = -\sigma(x, \tau), \quad -\infty < x < s^-, \quad s^+ < x < 0,$$
$$\sigma_{xy} = 0, \quad -\infty < x < s^-, \quad s^+ < x < \infty, \quad \sigma_{xy} = k[\sigma_{yy} + \sigma(x, \tau)], \quad s^- < x < s^+,$$
$$u_{y,x} = g(x), \quad s^- < x < s^+, \quad u_{y,x} = 0, \quad 0 < x < \infty, \tag{2.9}$$

and the initial conditions

$$\Phi = \Phi_{,\tau} = 0, \quad \Psi = \Psi_{,\tau} = 0, \quad \tau = 0. \tag{2.10}$$

Here,

$$\alpha_l = \sqrt{1 - \frac{v^2}{c^2}}, \quad \alpha_s = \sqrt{1 - v^2}, \tag{2.11}$$

σ_{yy}, σ_{xy}, and u_y are the functions σ'_{22}, σ'_{12}, and u'_2 in the body frame. They are expressed through the new potentials Φ and Ψ by formulas (2.4) if ϕ and ψ are replaced first by ϕ' and ψ' and then by Φ and Ψ, respectively.

3. Riemann–Hilbert problem for three pairs of functions.

3.1. Derivation of the Riemann–Hilbert problem. To reformulate the transient problem (2.8) to (2.10) as a Riemann–Hilbert problem, we introduce four auxiliary functions in the line $y = 0$,

$$\chi_1(x, \tau) = \begin{cases} \sigma_{yy}(x, 0, \tau), & s^- < x < s^+, \\ 0 & \text{otherwise}, \end{cases} \qquad \chi_2(x, \tau) = \begin{cases} \sigma_{yy}(x, 0, \tau), & 0 < x < \infty, \\ 0 & \text{otherwise}, \end{cases}$$
$$\omega_1(x, \tau) = \begin{cases} u_{y,x}(x, 0, \tau), & -\infty < x < s^-, \\ 0 & \text{otherwise}, \end{cases} \qquad \omega_2(x, \tau) = \begin{cases} u_{y,x}(x, 0, \tau), & s^+ < x < 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

We also extend the definition of the function g for all x assuming that $g(x) = 0$ if $x \notin (s^-, s^+)$ and split the function $\sigma(x, \tau)$ as

$$\sigma_0(x, \tau) = \begin{cases} \sigma(x, \tau), & s^- < x < s^+, \\ 0 & \text{otherwise}, \end{cases}$$
$$\sigma_-(x, \tau) = \begin{cases} \sigma(x, \tau), & -\infty < x < s^-, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma_+(x, \tau) = \begin{cases} \sigma(x, \tau), & s^+ < x < 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Here, the functions $\chi_1(x,\tau)$, $\chi_2(x,\tau)$, $\omega_1(x,\tau)$, and $\omega_2(x,\tau)$ are to be determined, while the functions $\sigma_0(x,\tau)$, $\sigma_-(x,\tau)$, $\sigma_+(x,\tau)$, and $g(x)$ are prescribed. In terms of these new functions, the boundary conditions can be written in the whole real axis,

$$\sigma_{yy} = -\sigma_-(x,\tau) - \sigma_+(x,\tau) + \chi_1(x,\tau) + \chi_2(x,\tau), \quad \sigma_{xy} = k[\chi_1(x,\tau) + \sigma_0(x,\tau)],$$

$$u_{y,x} = g(x) + \omega_1(x,\tau) + \omega_2(x,\tau), \quad -\infty < x < \infty. \tag{3.3}$$

Next we apply the Laplace and Fourier transforms and denote

$$\left(\hat{\Phi}, \hat{\Psi}\right)(x,y,s) = \int_0^\infty (\Phi,\Psi)(x,y,\tau)e^{-s\tau}d\tau, \quad \operatorname{Re} s > 0,$$

$$\left(\tilde{\Phi}, \tilde{\Psi}\right)(p,y,s) = \int_{-\infty}^\infty \left(\hat{\Phi}, \hat{\Psi}\right)(x,y,s)e^{ipx}dx, \quad \operatorname{Re} p = 0. \tag{3.4}$$

On taking into account the initial conditions (2.10) we transform the two wave equations (2.8) to the following ordinary differential equations:

$$\frac{d^2}{dy^2}\tilde{\Phi} - \alpha^2(p,s)\tilde{\Phi} = 0, \quad \frac{d^2}{dy^2}\tilde{\Psi} - \beta^2(p,s)\tilde{\Psi} = 0, \quad 0 < y < \infty, \tag{3.5}$$

where

$$\alpha^2(p,s) = p^2\alpha_l^2 + \frac{2v}{c^2}ips + \frac{s^2}{c^2}, \quad \beta^2(p,s) = p^2\alpha_s^2 + 2vips + s^2. \tag{3.6}$$

Let $\alpha(p,s)$ and $\beta(p,s)$ be the single holomorphic branches of the algebraic functions $w = \alpha^2(p,s)$ and $w = \beta^2(p,s)$,

$$\alpha(p,s) = \alpha_l[p - \alpha_+(s)]^{1/2}[p - \alpha_-(s)]^{1/2}, \quad \beta(p,s) = \alpha_s[p - \beta_+(s)]^{1/2}[p - \beta_-(s)]^{1/2}, \tag{3.7}$$

fixed in the complex p-plane cut along the lines joining the branch points and passing through the infinite point. The branch points of these functions are

$$\alpha_\pm(s) = \pm\frac{is}{c \pm v} \in \mathbb{C}^\pm, \quad \beta_\pm(s) = \pm\frac{is}{1 \pm v} \in \mathbb{C}^\pm, \tag{3.8}$$

where $\mathbb{C}^+ = \{\operatorname{Im} p > 0\}$ and $\mathbb{C}^- = \{\operatorname{Im} p < 0\}$,

$$\arg(p - \alpha_+) \in [\theta_+ - 2\pi, \theta_+], \quad \arg(p - \alpha_-) \in [\theta_-, \theta_- + 2\pi],$$

$$\arg(p - \beta_+) \in [\tau_+ - 2\pi, \tau_+], \quad \arg(p - \beta_-) \in [\tau_-, \tau_- + 2\pi]. \tag{3.9}$$

Here we introduced the notation $\theta_\pm = \arg\alpha_\pm$, $\tau_\pm = \arg\beta_\pm$, and the arguments are chosen as $\theta_+, \tau_+ \in (0,\pi)$ and $\theta_-, \tau_- \in (-\pi, 0)$. The branches $\alpha(p,s)$ and $\beta(p,s)$ share the same property: their real part in the axis $\operatorname{Im} p = 0$ is positive, $\operatorname{Re}\alpha(p,s) > 0$, and $\operatorname{Re}\beta(p,s) > 0$. The general solution of equations (3.5) has the form

$$\tilde{\Phi}(p,y,s) = C_1(p,s)e^{-\alpha(p,s)y}, \quad \tilde{\Psi}(p,y,s) = C_2(p,s)e^{-\beta(p,s)y}. \tag{3.10}$$

On applying the two integral transforms to the first two boundary conditions (3.3) we obtain a system of two algebraic equations for the functions $C_1(p,s)$ and $C_2(p,s)$. Its solution is

$$C_1(p,s) = -\frac{1}{R(p,s)}\left\{[p^2+\beta^2(p,s)][\tilde{\chi}_1(p,s)+\tilde{\chi}_2(p,s)-\tilde{\sigma}_-(p,s)-\tilde{\sigma}_+(p,s)]\right.$$
$$\left.+2ipk\beta(p,s)[\tilde{\chi}_1(p,s)+\tilde{\sigma}_0(p,s)]\right\},$$
$$C_2(p,s) = \frac{1}{R(p,s)}\{2ip\alpha(p,s)[\tilde{\chi}_1(p,s)+\tilde{\chi}_2(p,s)-\tilde{\sigma}_-(p,s)-\tilde{\sigma}_+(p,s)]$$
$$-k[c^2\alpha^2(p,s)-(c^2-2)p^2]\tilde{\chi}_1(p,s)+\tilde{\sigma}_0(p,s)]\}. \tag{3.11}$$

Apply next the Laplace and Fourier transforms to the third equation in (3.3) to obtain

$$\tilde{\omega}_1(p,s)+\tilde{\omega}_2(p,s)+s^{-1}\hat{g}(p) = \frac{p}{R(p,s)}\left\{[i\alpha\left(p^2-\beta^2\right)+kp(2\alpha\beta-p^2-\beta^2)]\tilde{\chi}_1(p,s)\right.$$
$$\left.+i\alpha(p^2-\beta^2)[\tilde{\chi}_2(p,s)-\tilde{\sigma}_-(p,s)-\tilde{\sigma}_+(p,s)]+kp\left(2\alpha\beta-p^2-\beta^2\right)\tilde{\sigma}_0(p,s)\right\}. \tag{3.12}$$

Here and in (3.11), $\tilde{\chi}_j(p,s)$, $\tilde{\omega}_j(p,s)$, $\tilde{\sigma}_\pm(p,s)$, $\tilde{\sigma}_0(p,s)$, and $s^{-1}\hat{g}(p)$ are the Fourier images of the Laplace transforms of the functions $\chi_j(x,\tau)$, $\omega_j(x,\tau)$ $(j=1,2)$, $\sigma_\pm(x,\tau)$, $\sigma_0(x,\tau)$, and $g(x)$, respectively, and $R(p,s)$ is the dimensionless Rayleigh function,

$$R(p,s) = 4p^2\alpha(p,s)\beta(p,s)-[\beta^2(p,s)+p^2]^2. \tag{3.13}$$

Equation (3.12) can be rewritten as the boundary condition of a Riemann–Hilbert problem for three pairs of functions piecewise holomorphic in the p-plane. Denote

$$\varphi_1^+(p,s) = \int_0^{\lambda_1}\hat{\chi}_1(s^-+\xi,s)e^{ip\xi}d\xi, \quad \varphi_1^-(p,s) = \int_{-\lambda_1}^0\hat{\chi}_1(s^++\xi,s)e^{ip\xi}d\xi,$$

$$\varphi_2^+(p,s) = \int_0^{\lambda_2}\hat{\omega}_2(s^++\xi,s)e^{ip\xi}d\xi, \quad \varphi_2^-(p,s) = \int_{-\lambda_2}^0\hat{\omega}_2(\xi,s)e^{ip\xi}d\xi,$$

$$\varphi_3^+(p,s) = \int_0^{\infty}\hat{\chi}_2(\xi,s)e^{ip\xi}d\xi, \quad \varphi_3^-(p,s) = \int_{-\infty}^0\hat{\omega}_1(s^-+\xi,s)e^{ip\xi}d\xi, \tag{3.14}$$

where

$$\lambda_1 = s^+-s^- > 0, \quad \lambda_2 = -s^+ > 0. \tag{3.15}$$

Then we arrive at the following Riemann–Hilbert problem for two order-3 vectors $\varphi^\pm = (\varphi_1^\pm,\varphi_2^\pm,\varphi_3^\pm)^T$:

$$\varphi^+(p,s) = G(p,s)\varphi^-(p,s)+f(p,s), \quad p\in L, \tag{3.16}$$

where L is the contour that consists of two semi-infinite segments $(-\infty,-\varepsilon)$ and $(\varepsilon,+\infty)$ and the lower semicircle $|p|=\varepsilon$, $\operatorname{Im} p\le 0$, $\varepsilon>0$ and small. Denote by $\mathbb{C}_\varepsilon^+$ and $\mathbb{C}_\varepsilon^-$ the domains above and below the contour L, respectively. Then the functions $\varphi_j^\pm(p,s)$ $(j=1,2,3)$ are holomorphic in the domains $\mathbb{C}_\varepsilon^\pm$. The matrix coefficient $G(p,s)$ is a triangular 3×3 matrix, and $f(p,s)$ is an order-3 vector. They are

$$G(p,s) = \begin{pmatrix} e^{ip\lambda_1} & 0 & 0 \\ 0 & e^{ip\lambda_2} & 0 \\ a_1(p,s)e^{-ip\lambda_2} & a(p,s) & a(p,s)e^{-ip(\lambda_1+\lambda_2)} \end{pmatrix}, \quad f(p,s) = \begin{pmatrix} 0 \\ 0 \\ f_3(p,s) \end{pmatrix}, \tag{3.17}$$

where

$$a_1(p,s) = -1+\frac{ikp[2\alpha(p,s)\beta(p,s)-p^2-\beta^2(p,s)]}{\alpha(p,s)[p^2-\beta^2(p,s)]}, \quad a(p,s) = \frac{R(p,s)}{ip\alpha(p,s)[p^2-\beta^2(p,s)]}, \tag{3.18}$$

and

$$f_3(p,s) = a(p,s)e^{-ip\lambda_2}s^{-1}F_0^-(p) + \frac{kpi(2\alpha\beta-p^2-\beta^2)}{\alpha(p^2-\beta^2)}e^{-ip\lambda_2}F_1^-(p,s)$$
$$+\ F_2^-(p,s) + e^{-ip(\lambda_1+\lambda_2)}F_3^-(p,s),$$
$$F_0^-(p) = \int_{-\lambda_1}^0 g(s^+ + \xi)e^{ip\xi}d\xi, \quad F_1^-(p,s) = \int_{-\lambda_1}^0 \hat{\sigma}(s^+ + \xi, s)e^{ip\xi}d\xi,$$

$$F_2^-(p,s) = \int_{-\lambda_2}^0 \hat{\sigma}(\xi,s)e^{ip\xi}d\xi, \quad F_3^-(p,s) = \int_{-\infty}^0 \hat{\sigma}(s^- + \xi, s)e^{ip\xi}d\xi. \tag{3.19}$$

The indices of the first and second diagonal elements are equal to $+\infty$, and the third element has the $-\infty$-index,

$$\varkappa_\pm = \frac{1}{2\pi}[\arg e^{\pm ip\lambda_j}]_{p=-\infty}^{p=+\infty} = \pm\infty. \tag{3.20}$$

The analytical method [1], [2] is not directly applicable in this case since the functions $a_1(p,s)$ and $a(p,s)$ are not meromorphic in the p-plane. In the case of frictionless contact $k=0$, $a_1=-1$, and the matrix becomes simpler. However, the function $a(p,s)$ is independent of k, and since it is not meromorphic, the matrix is still not factorizable in explicit form. In the next section we propose an approximate method for the problem (3.16).

3.2. Factorization of the functions a_1 and a. First, we rewrite the system with the triangular matrix in the following form:

$$\varphi_3^+ = a_1 e^{-ip\lambda_2}\varphi_1^- + a\varphi_2^- + ae^{-ip(\lambda_1+\lambda_2)}\varphi_3^- + f_3,$$
$$-\ a\varphi_2^+ + e^{ip\lambda_2}\varphi_3^+ = a_1\varphi_1^- + ae^{-ip\lambda_1}\varphi_3^- + e^{ip\lambda_2}f_3,$$
$$-\ a_1\varphi_1^+ - ae^{ip\lambda_1}\varphi_2^+ + e^{ip(\lambda_1+\lambda_2)}\varphi_3^+ = a\varphi_3^- + e^{ip(\lambda_1+\lambda_2)}f_3, \quad p\in L. \tag{3.21}$$

To factorize the elements a_1 and a, we study their properties at the point $p=0$ and as $p\to\pm\infty$. Analysis of the behavior of the functions α, β, and R as $p\to\pm\infty$ yields

$$\alpha \sim \alpha_l|p|, \quad \beta\sim\alpha_s|p|, \quad R\sim R_0p^4, \quad p\to\pm\infty, \tag{3.22}$$

where

$$R_0 = 4\alpha_s\alpha_l - (\alpha_s^2+1)^2. \tag{3.23}$$

If $p\to 0$, then $\alpha\sim s/c$, $\beta\sim s$, and $R\sim -s^4$. This implies

$$a_1 \sim -1-\frac{ikp}{s}(2-c), \quad a\sim\frac{sc}{ip}, \quad p\to 0,$$

$$a_1\sim -1\mp i\gamma, \quad a\sim \pm\frac{R_0}{i\alpha_l(1-\alpha_s^2)}, \quad p\to\pm\infty, \tag{3.24}$$

where

$$\gamma=\frac{k(1+\alpha_s^2-2\alpha_l\alpha_s)}{\alpha_l(1-\alpha_s^2)}\geq 0,\quad k\geq 0. \tag{3.25}$$

Based on these properties we split the functions as

$$a_1=-(1+\tanh\pi p\tanh i\pi\varkappa_0)a_1^\circ,\quad a=\frac{R_0\coth\pi p}{i\alpha_l(1-\alpha_s^2)}a^\circ, \tag{3.26}$$

where

$$\varkappa_0=\frac{1}{\pi}\tan^{-1}\gamma,\quad 0\leq\varkappa_0<\frac{1}{2}. \tag{3.27}$$

The expressions $1+\tanh\pi p\tanh i\pi\varkappa_0$ and $\coth\pi p$ can be factorized in terms of the Γ-functions, and we have

$$\begin{gathered}a_1=-\delta_1\frac{K_1^+(p)}{K_1^-(p)}a_1^\circ(p,s),\quad a=\delta\frac{K^+(p)}{K^-(p)}a^\circ(p,s),\\ K_1^+(p)=\frac{\Gamma(1/2-ip)}{\Gamma(1/2-ip+\varkappa_0)},\quad K_1^-(p)=\frac{\Gamma(1/2+ip-\varkappa_0)}{\Gamma(1/2+ip)},\\ K^+(p)=\frac{\Gamma(1-ip)}{\Gamma(1/2-ip)},\quad K^-(p)=\frac{\Gamma(1/2+ip)}{\Gamma(ip)},\quad \delta_1=\frac{1}{\cos\pi\varkappa_0},\quad \delta=\frac{R_0}{\alpha_l(1-\alpha_s^2)}.\end{gathered} \tag{3.28}$$

The arguments of the functions a_1° and a have a zero increment when p traverses the contour L, and the functions a_1° and a have the following asymptotics:

$$a_1^\circ\sim 1,\quad a^\circ\sim 1,\quad p\to\pm\infty,\qquad a_1^\circ\sim 1,\quad a^\circ\sim\frac{\pi cs}{\delta},\quad p\to 0. \tag{3.29}$$

This implies the factorization of the functions a_1° and a° in terms of the Cauchy integrals

$$a_1^\circ(p,s)=\frac{X_1^+(p,s)}{X_1^-(p,s)},\quad a^\circ(p,s)=\frac{X^+(p,s)}{X^-(p,s)},\quad p\in L, \tag{3.30}$$

where $X_1^\pm$ and $X^\pm$ are the limit values as $p=\operatorname{Re}p\pm i0$ of the functions

$$(X_1,X)(p,s)=\exp\left(\frac{1}{2\pi i}\int_L\frac{\log(a_1^\circ,a^\circ)(\tau,s)d\tau}{\tau-p}\right),\quad p\in\mathbb{C}\setminus L. \tag{3.31}$$

Substitute the functions a_1 and a in (3.21) with their representations through the factors $K_1^\pm$, $K^\pm$, $X_1^\pm$, and $X^\pm$,

$$\begin{gathered}\frac{\varphi_3^+}{K^+X^+}=-\delta_1e^{-ip\lambda_2}\frac{K_1^+X_1^+}{K^+X^+}\frac{\varphi_1^-}{K_1^-X_1^-}+\frac{\delta\varphi_2^-}{K^-X^-}+\delta e^{-ip(\lambda_1+\lambda_2)}\frac{\varphi_3^-}{K^-X^-}+\frac{f_3}{K^+X^+},\\ -\delta\frac{K^+X^+}{K_1^+X_1^+}\varphi_2^++e^{ip\lambda_2}\frac{K^-X^-}{K_1^+X_1^+}\varphi_3^+=-\delta_1\frac{K^-X^-}{K_1^-X_1^-}\varphi_1^-+\delta e^{-ip\lambda_1}\frac{K^+X^+}{K_1^+X_1^+}\varphi_3^-+e^{ip\lambda_2}\frac{K^-X^-}{K_1^+X_1^+}f_3,\\ \delta_1\frac{K_1^+X_1^+}{K^+X^+}\varphi_1^+-\delta e^{ip\lambda_1}\frac{K_1^-X_1^-}{K^-X^-}\varphi_2^++e^{ip(\lambda_1+\lambda_2)}\frac{K_1^-X_1^-}{K^+X^+}\varphi_3^+=\delta\frac{K_1^-X_1^-}{K^-X^-}\varphi_3^-+e^{ip(\lambda_1+\lambda_2)}\frac{K_1^-X_1^-}{K^+X^+}f_3.\end{gathered} \tag{3.32}$$

3.3. Approximate solution of the Riemann–Hilbert problem: Frictionless contact. In what follows, for simplicity, we assume that there is no friction in the contact zone that is $k = 0$. Then $a_1 = -1$, $\delta_1 = 1$, and $K_1^\pm = X_1^\pm \equiv 1$. The method is also applicable when $k \neq 0$, but the level of technicalities increases. The system (3.32) now reads

$$\frac{\varphi_3^+}{K^+X^+} = -\underline{\frac{e^{-ip\lambda_2}}{K^+X^+}}\varphi_1^- + \frac{\delta\varphi_2^-}{K^-X^-} + \delta e^{-ip(\lambda_1+\lambda_2)}\frac{\varphi_3^-}{K^-X^-} + \frac{f_3}{K^+X^+},$$

$$-\delta K^+X^+\varphi_2^+ + e^{ip\lambda_2}\underline{K^-X^-}\varphi_3^+ = -K^-X^-\varphi_1^- + \delta e^{-ip\lambda_1}\underline{K^+X^+}\varphi_3^- + e^{ip\lambda_2}K^-X^-f_3,$$

$$\frac{\varphi_1^+}{K^+X^+} - \delta\underline{\frac{e^{ip\lambda_1}\varphi_2^+}{K^-X^-}} + e^{ip(\lambda_1+\lambda_2)}\frac{\varphi_3^+}{K^+X^+} = \frac{\delta\varphi_3^-}{K^-X^-} + e^{ip(\lambda_1+\lambda_2)}\frac{f_3}{K^+X^+}. \tag{3.33}$$

The ultimate goal of the method is to put all the functions holomorphic in $\mathbb{C}_\varepsilon^+$ into the left-hand side and the functions holomorphic in $\mathbb{C}_\varepsilon^-$ into the right-hand side of the equations. Apart from the known functions having f_3 there are only four terms in (3.33) which do not meet this requirement. These terms are underlined and possess meromorphic functions $[K^+]^{\pm1}$ and $[K^-]^{\pm1}$ and also the functions $[X^+]^{\pm1}$ and $[X^-]^{\pm1}$ which do not admit a meromorphic continuation to the whole complex plane. To proceed further with our solution of the Riemann–Hilbert problem, we approximate the function $a^\circ(p,s)$ by a rational function. Recall that the function a° is neither even nor odd, its argument has zero increment when the point p traverses the whole real axis, $a^\circ(p,s) \sim 1$ as $p \to \pm\infty$, and $a^\circ(p,s) \sim \pi cs/\delta$ as $p \to 0$. Let

$$\tilde{a}^\circ(p,s) = \sum_{j=0}^{2n} \frac{d_j(s)p^j}{e_j(s)p^j}, \tag{3.34}$$

where $d_{2n} = e_{2n} = 1$ and $d_0/e_0 = \pi cs/\delta$. The number of free coefficients is $4n-1$, and they may be determined either from the equations

$$a^\circ(p_j,s) = \tilde{a}^\circ(p_j,s), \quad p_j \in \mathbb{R}_1, \quad p_j \neq 0, \quad j = 1,\ldots,4n-1, \tag{3.35}$$

or by equating not only the functions but also their derivatives,

$$a^\circ(p_j,s) = \tilde{a}^\circ(p_j,s), \quad j = 1,\ldots,2n,$$

$$a^\circ_{,p}(p_j,s) = \tilde{a}^\circ_{,p}(p_j,s), \quad j = 1,\ldots,2n-2, \quad a^\circ_{,p}(0,s) = \tilde{a}^\circ_{,p}(0,s). \tag{3.36}$$

In both cases the collocation points p_j have to be chosen such that the argument of the function $\tilde{a}^\circ$ has a zero gain as p traverses the whole contour L. This is equivalent to the requirement that the number of zeros in $\mathbb{C}_\varepsilon^+$ of the numerator and denominator is the same, n^+. Then, immediately, the number of zeros of the numerator and denominator in the lower half-plane is $n^- = 2n - n^+$. It is also assumed that all the zeros and poles of the rational function $\tilde{a}^\circ_{,p}(p_j,s)$ are simple.

Factorize the rational function $\tilde{a}^\circ$

$$\tilde{a}^\circ(p,s) = \frac{\tilde{X}^+(p,s)}{X^-(p,s)}, \quad p \in L_\varepsilon, \quad \tilde{X}^+(p,s) = \prod_{j=1}^{n^-}\frac{p-\alpha_j^-}{p-\beta_j^-}, \quad \tilde{X}^-(p,s) = \prod_{j=1}^{n^+}\frac{p-\beta_j^+}{p-\alpha_j^+}, \tag{3.37}$$

where $a_j^\pm \in \mathbb{C}_\varepsilon^\pm$, $b_j^\pm \in \mathbb{C}_\varepsilon^\pm$, $j = 1,\dots n^\pm$. Replace now the functions $X^\pm$ in the underlined expressions in (3.33) by the approximate factors $\tilde{X}^\pm$, and to eliminate the inadmissible poles of the functions $K^\pm$, $(K^\pm)^{-1}$, $\tilde{X}^\pm$, and $(\tilde{X}^\pm)^{-1}$, introduce the following functions:

$$\Psi_1^\pm(p,s) = \sum_{m=0}^\infty \frac{A_{1,m}^\pm(s)}{p \pm i(m+1/2)},$$

$$\Psi_2^+(p,s) = \sum_{m=0}^\infty \frac{A_{2,m}^+(s)}{p + i(m+1)}, \quad \Psi_2^-(p,s) = \sum_{m=0}^\infty \frac{A_{2,m}^-(s)}{p - im},$$

$$\Omega_1^\pm(p,s) = \sum_{m=1}^{n^\mp} \frac{B_{1,m}^\pm(s)}{p - \alpha_m^\mp}, \quad \Omega_2^\pm(p,s) = \sum_{m=1}^{n^\mp} \frac{B_{2,m}^\pm(s)}{p - \beta_m^\mp}. \tag{3.38}$$

The coefficients $A_{j,m}^\pm$ $(m = 0, 1, \dots)$ and $B_{j,m}^\pm$ $(m = 1, 2, \dots, n^\mp)$ are to be determined. The only thing left that prevents us from applying the continuity principle and the Liouville theorem is the presence of the terms with f_3. Introduce the functions $H_j^\pm$ as

$$\frac{F_2^- + e^{-ip(\lambda_1+\lambda_2)}F_3^-}{K^+X^+} = H_1^+ - H_1^-,$$

$$\frac{\delta}{s}K^+X^+F_0^- + K^-X^-F_2^+ = H_2^+ - H_2^-,$$

$$\frac{\delta F_0^+}{sK^-X^-} + \frac{F_3^-}{K^+X^+} = H_3^+ - H_3^-, \quad p \in L, \tag{3.39}$$

and define the functions $H_j(p,s)$ $(j = 1,2,3)$ as the Cauchy integrals with the corresponding densities

$$H_1(p,s) = \frac{1}{2\pi i}\int_L \frac{F_2^-(\tau,s) + e^{-i\tau(\lambda_1+\lambda_2)}F_3^-(\tau,s)}{K^+(\tau)X^+(\tau,s)}\frac{d\tau}{\tau - p},$$

$$H_2(p,s) = \frac{1}{2\pi i}\int_L \left[\frac{\delta}{s}K^+(\tau)X^+(\tau,s)F_0^-(\tau) + K^-(\tau)X^-(\tau,s)F_2^+(\tau,s)\right]\frac{d\tau}{\tau - p},$$

$$H_3(p,s) = \frac{1}{2\pi i}\int_L \left[\frac{\delta}{s}\frac{F_0^+(\tau)}{K^-(\tau)X^-(\tau,s)} + \frac{F_3^-(\tau,s)}{K^+(\tau)X^+(\tau,s)}\right]\frac{d\tau}{\tau - p}. \tag{3.40}$$

Rewrite next equations (3.33) as

(3.41)

$$\begin{aligned}
&\frac{\varphi_3^+}{K^+X^+} - \Psi_1^+ - \Omega_1^+ - H_1^+ \\
&\quad = -\frac{e^{-ip\lambda_2}}{K^+\tilde{X}^+}\varphi_1^- + \frac{\delta\varphi_2^-}{K^-X^-} + \delta e^{-ip(\lambda_1+\lambda_2)}\frac{\varphi_3^-}{K^-X^-} - \Psi_1^+ - \Omega_1^+ - H_1^- + \frac{\delta e^{-ip\lambda_2}F_0^-}{sK^-X^-}, \\
&\quad - \delta K^+X^+\varphi_2^+ + e^{ip\lambda_2}K^-\tilde{X}^-\varphi_3^+ - \Psi_1^- - \Omega_1^- - \Psi_2^+ - \Omega_2^+ - H_2^+ \\
&\quad = -K^-X^-\varphi_1^- + \delta e^{-ip\lambda_1}K^+\tilde{X}^+\varphi_3^- - \Psi_1^- - \Omega_1^- - \Psi_2^+ - \Omega_2^+ - H_2^- + K^-X^-e^{-ip\lambda_1}F_3^-, \\
&\quad \frac{\varphi_1^+}{K^+X^+} - \delta\frac{e^{ip\lambda_1}\varphi_2^+}{K^-\tilde{X}^-} + e^{ip(\lambda_1+\lambda_2)}\frac{\varphi_3^+}{K^+X^+} - \Psi_2^- - \Omega_2^- - H_3^+ - \frac{e^{ip\lambda_1}F_2^+}{K^+X^+} \\
&\quad = \frac{\delta\varphi_3^-}{K^-X^-} - \Psi_2^- - \Omega_2^- - H_3^-.
\end{aligned}$$

Notice that although the function $1/K^-(p)$ has a simple pole at the origin, the right-hand side of the first equation in (3.41) is bounded. Indeed, since

$$\int_{-\infty}^{s^-}\omega_1(x,\tau)dx+\int_{s^-}^{s^+}g(x)dx+\int_{s^+}^{0}\omega_2(x,\tau)dx=0,\quad \tau>0, \tag{3.42}$$

we conclude that

$$\lim_{p\to 0}\left(\varphi_2^-+\varphi_3^-+\frac{F_0^-}{s}\right)=0. \tag{3.43}$$

This guarantees the existence of the limit of the right-hand side of the first equation in (3.41) as $p\to 0$. Suppose now that the coefficients $A^{\pm}_{j,m}$ ($m=0,1,\dots$) and $B^{\pm}_{j,m}$ ($m=1,2,\dots,n^{\mp}$) are chosen such that the corresponding unacceptable poles in equations (3.41) become removable singularities. By the continuity principle and the Liouville theorem the left- and right-hand sides in (3.41) are entire functions which are identically equal to zero in the first and third equations and a function $C(s)$ in the second equation. This recovers the solution of the Riemann–Hilbert problem

$$\varphi_1^-=-\frac{C+\Psi_1^-+\Psi_2^++\Omega_1^-+\Omega_2^++H_2^-}{K^-X^-}+e^{-ip\lambda_1}K^+\tilde{X}^+(\Psi_2^-+\Omega_2^-+H_3^-)-e^{-ip\lambda_1}F_3^-,$$

$$\varphi_1^+=-e^{ip\lambda_1}\frac{C+\Psi_1^-+\Psi_2^++\Omega_1^-+\Omega_2^++H_2^+}{K^-\tilde{X}^-}+K^+X^+(\Psi_2^-+\Omega_2^-+H_3^+)+e^{ip\lambda_1}F_2^+,$$

$$\begin{aligned}\varphi_2^-=&-\frac{e^{-ip\lambda_2}}{\delta K^+\tilde{X}^+}(C+\Psi_1^-+\Psi_2^++\Omega_1^-+\Omega_2^++H_2^--K^-X^-e^{-ip\lambda_1}F_3^-)\\&+\frac{K^-X^-}{\delta}(\Psi_1^++\Omega_1^++H_1^-)-e^{-ip\lambda_2}\frac{F_0^-}{s},\end{aligned}$$

$$\varphi_2^+=-\frac{C+\Psi_1^-+\Psi_2^++\Omega_1^-+\Omega_2^++H_2^+}{\delta K^+X^+}+\frac{e^{ip\lambda_2}}{\delta}K^-\tilde{X}^-(\Psi_1^++\Omega_1^++H_1^+),$$

$$\varphi_3^-=\frac{K^-X^-}{\delta}(\Psi_2^-+\Omega_2^-+H_3^-),$$

$$\varphi_3^+=K^+X^+(\Psi_1^++\Omega_1^++H_1^+). \tag{3.44}$$

The arbitrary function $C(s)$ has to be fixed as

$$C(s)=-\Psi_1^-(0,s)-\Psi_2^+(0,s)-\Omega_1^-(0,s)-\Omega_2^+(0,s)-H_2^+(0,s). \tag{3.45}$$

Otherwise, if $C(s)$ is chosen in a different manner, the functions $\varphi_1^{\pm}$ are not bounded as $p\to 0$. On recalling that $\tilde{X}^{\pm}\approx X^{\pm}$ and using the relations (3.39), we can show that $\varphi_1^+=e^{ip\lambda_1}\varphi_1^-$ and $\varphi_2^+=e^{ip\lambda_2}\varphi_2^-$. If the coefficients $A^{\pm}_{j,n}$ and $B^{\pm}_{j,n}$ are arbitrarily chosen, then, in general, the functions $\varphi_1^{\pm}$ and $\varphi_2^{\pm}$ have unacceptable poles. To make them removable singularities, we select them as the solution of the following infinite system of linear algebraic equations for the coefficients $A^{\pm}_{j,n}$:

$$\begin{aligned}A^+_{1,n}+\frac{\mu_{1,n}}{\pi}e^{-(n+\frac12)\lambda_2}\Bigg[&-iC+\sum_{m=0}^{\infty}\left(\frac{A^-_{1,m}}{n+m+1}+\frac{A^+_{2,m}}{n-m-\frac12}\right)+\sum_{m=1}^{n^+}\frac{B^-_{1,m}}{n+\frac12-i\alpha^+_m}\\&+\sum_{m=1}^{n^-}\frac{B^+_{2,m}}{n+\frac12-i\beta^-_m}-iH_2^-\left(-in-\frac{i}{2},s\right)+\frac{X^-\left(-in-\frac{i}{2},s\right)}{\lambda_n e^{(n+\frac12)\lambda_1}}F_3^-\left(-in-\frac{i}{2},s\right)\Bigg]=0,\end{aligned}$$

$$A_{1,n}^- - \frac{\mu_{2,n}}{\pi} e^{-(n+\frac{1}{2})\lambda_2} \left[\sum_{m=0}^{\infty} \frac{A_{1,m}^+}{n+m+1} + \sum_{m=1}^{n^-} \frac{B_{1,m}^+}{n+\frac{1}{2}+i\alpha_m^-} + iH_1^+ \left(in + \frac{i}{2}, s \right) \right] = 0,$$

$$A_{2,n}^+ - \frac{\mu_{3,n}}{\pi} e^{-(n+1)\lambda_1} \left[\sum_{m=0}^{\infty} \frac{A_{2,m}^-}{n+m+1} + \sum_{m=1}^{n^+} \frac{B_{2,m}^-}{n+1-i\beta_m^+} - iH_3^- \left(-in - i, s \right) \right] = 0,$$

$$\begin{aligned} A_{2,n}^- + \frac{\mu_{4,n}}{\pi} e^{-n\lambda_1} \Bigg[iC &+ \sum_{m=0}^{\infty} \left(\frac{A_{1,m}^-}{n-m-\frac{1}{2}} + \frac{A_{2,m}^+}{n+m+1} \right) + \sum_{m=1}^{n^+} \frac{B_{1,m}^-}{n+i\alpha_m^+} \\ &+ \sum_{m=1}^{n^-} \frac{B_{2,m}^+}{n+i\beta_m^-} + iH_2^+ (in, s) \Bigg] = 0, \quad n = 0, 1, \ldots, \end{aligned} \tag{3.46}$$

combined with the finite system for the coefficients $B_{j,n}^{\pm}$

$$\begin{aligned} B_{1,n}^+ - \nu_{1,n}^- e^{-i\alpha_n^- \lambda_2} \Bigg[C &+ \sum_{m=0}^{\infty} \left(\frac{A_{1,m}^-}{\alpha_n^- - im - \frac{i}{2}} + \frac{A_{2,m}^+}{\alpha_n^- + im + i} \right) + \sum_{m=1}^{n^+} \frac{B_{1,m}^-}{\alpha_n^- - \alpha_m^+} \\ &+ \sum_{m=1}^{n^-} \frac{B_{2,m}^+}{\alpha_n^- - \beta_m^-} + H_2^-(\alpha_n^-, s) - e^{-i\alpha_n^- \lambda_1} K^-(\alpha_n^-) X^-(\alpha_n^-, s) F_3^-(\alpha_n^-, s) \Bigg] = 0, \ n=1, \ldots, n^-, \end{aligned}$$

$$B_{1,n}^- - \nu_{1,n}^+ e^{i\alpha_n^+ \lambda_2} \left[\sum_{m=0}^{\infty} \frac{A_{1,m}^+}{\alpha_n^+ + im + \frac{i}{2}} + \sum_{m=1}^{n^-} \frac{B_{1,m}^+}{\alpha_n^+ - \alpha_m^-} + H_1^+(\alpha_n^+, s) \right] = 0, \ n = 1, \ldots, n^+,$$

$$B_{2,n}^+ - \nu_{2,n}^- e^{-i\beta_n^- \lambda_1} \left[\sum_{m=0}^{\infty} \frac{A_{2,m}^-}{\beta_n^- - im} + \sum_{m=1}^{n^+} \frac{B_{2,m}^-}{\beta_n^- - \beta_m^+} + H_3^-(\beta_n^-, s) \right] = 0, \ n = 1, \ldots, n^-,$$

$$\begin{aligned} B_{2,n}^- - \nu_{2,n}^+ e^{i\beta_n^+ \lambda_1} \Bigg[C &+ \sum_{m=0}^{\infty} \left(\frac{A_{1,m}^-}{\beta_n^+ - im - \frac{i}{2}} + \frac{A_{2,m}^+}{\beta_n^+ + im + i} \right) + \sum_{m=1}^{n^+} \frac{B_{1,m}^-}{\beta_n^+ - \alpha_m^+} \\ &+ \sum_{m=1}^{n^-} \frac{B_{2,m}^+}{\beta_n^+ - \beta_m^-} + H_2^+(\beta_n^+, s) \Bigg] = 0, \quad n = 1, \ldots, n^+. \end{aligned} \tag{3.47}$$

Here, the numbers $\mu_{j,n}$ and $\nu_{j,n}^{\pm}$ are given by

$$\mu_{1,n} = \frac{\rho_n^2}{X^-(-in - \frac{i}{2}) \tilde{X}^+(-in - \frac{i}{2})}, \quad \mu_{2,n} = \left(n + \frac{1}{2} \right)^2 \rho_n^2 \tilde{X}^- \left(in + \frac{i}{2} \right) X^+ \left(in + \frac{i}{2} \right),$$

$$\mu_{3,n} = \left(n + \frac{1}{2} \right)^2 \rho_n^2 \tilde{X}^+ (-in - i) X^- (-in - i), \quad \mu_{4,n} = \frac{\rho_n^2}{\tilde{X}^-(in) X^+(in)},$$

$$\rho_n = \frac{\Gamma(n+1/2)}{n!},$$

$$\nu_{1,n}^{\pm} = [K^-(\alpha_n^{\pm})K^+(\alpha_n^{\pm})X^{\pm}(\alpha_n^{\pm})]^{\pm 1}(\alpha_n^{\pm} - \beta_n^{\pm}) \prod_{k=1, k\neq n}^{n^{\pm}} \frac{\alpha_n^{\pm} - \beta_k^{\pm}}{\alpha_n^{\pm} - \alpha_k^{\pm}},$$

$$\nu_{2,n}^{\pm} = [K^-(\beta_n^{\pm})K^+(\beta_n^{\pm})X^{\pm}(\beta_n^{\pm})]^{\mp 1}(\beta_n^{\pm} - \alpha_n^{\pm}) \prod_{k=1, k\neq n}^{n^{\pm}} \frac{\beta_n^{\pm} - \alpha_k^{\pm}}{\beta_n^{\pm} - \beta_k^{\pm}}. \tag{3.48}$$

The presence of the factors $e^{-n\lambda_1}$ and $e^{-n\lambda_2}$ implies an exponential decay of the coefficients $A_{1,n}^{\pm}$ and $A_{2,n}^{\pm}$. Due to the factors $e^{\pm i\alpha_n^{\pm}}$ and $e^{\pm i\beta_n^{\pm}}$ and since $\pm \operatorname{Im} \alpha_n^{\pm} > 0$ and $\pm \operatorname{Im} \beta_n^{\pm} > 0$, the same is true for the coefficients $B_{1,n}^{\mp}$ and $B_{2,n}^{\mp}$. It is directly verified that if the coefficients $A_{j,n}^{\pm}$ and $B_{j,m}^{\pm}$ solve the systems (3.46) and (3.47), and $C(s)$ is determined by (3.45), then the functions (3.44) approximately solve the vector Riemann–Hilbert problem (3.21).

4. System of two singular integral equations.

4.1. Derivation of the governing system of integral equations in the general case. In this section we intend to propose another method to be used for numerical tests in section 5. It also reduces the problem to an infinite system of linear algebraic equations. However, it does not hinge on the rational approximation $\tilde{a}^\circ(p,s)$ of the function $a^\circ(p,s)$ and determination of the zeros and poles of the function $\tilde{a}^\circ(p,s)$. Instead, it maps the original vector Riemann–Hilbert problem to a system of singular integral equations and solves it approximately by the method of orthogonal polynomials.

The Fourier image of the Laplace transform, $\tilde{u}_{y,x}$, of the function $u_{y,x}$ in the line $y=0$ admits the representation

$$\tilde{u}_{y,x} = [q_0(p,s)+q_1(p,s)]\tilde{\chi}_1(p,s)+q_0(p,s)[\tilde{\chi}_2(p,s)-\tilde{\sigma}_-(p,s)-\tilde{\sigma}_+(p,s)]+q_1(p,s)\tilde{\sigma}_0(p,s), \tag{4.1}$$

where

$$q_0(p,s) = \frac{ip\alpha}{R}(p^2-\beta^2), \quad q_1(p,s) = \frac{kp^2}{R}(2\alpha\beta - p^2 - \beta^2). \tag{4.2}$$

On inverting the Fourier transform we discover

$$2\pi\hat{u}_{y,x}(x,0^+,s) = \int_{s^-}^{s^+} \hat{\chi}_1(\xi,s)d\xi \int_{-\infty}^{\infty} e^{ip(\xi-x)}[q_0(p,s)+q_1(p,s)]dp + \int_0^{\infty} \hat{\chi}_2(\xi,s)d\xi$$
$$\times \int_{-\infty}^{\infty} e^{ip(\xi-x)} q_0(p,s)dp - \left(\int_{-\infty}^{s^+} \hat{\sigma}_-(\xi,s) + \int_{s^+}^{0} \hat{\sigma}_+(\xi,s)\right) d\xi \int_{-\infty}^{\infty} e^{ip(\xi-x)} q_0(p,s)dp$$
$$+ \int_{s^-}^{s^+} \hat{\sigma}_0(\xi,s)d\xi \int_{-\infty}^{\infty} e^{ip(\xi-x)} q_1(p,s)dp. \tag{4.3}$$

The relations (3.22) imply the following asymptotics of the functions $q_0(p,s)$ and $q_1(p,s)$:

$$q_0(p,s) \sim i\operatorname{sgn} p \frac{\alpha_l}{R_0}(1-\alpha_s^2), \quad q_1(p,s) \sim \frac{k}{R_0}(2\alpha_l\alpha_s - 1 - \alpha_s^2), \quad p \to \pm\infty. \tag{4.4}$$

Now, by making use of the relations

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty} e^{-|p|y}e^{ip(\xi-x)} i\,\mathrm{sgn}\,p\,dp = -\frac{2}{\xi-x},$$

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty} e^{-|p|y}e^{ip(\xi-x)} dp = 2\pi\delta(\xi-x) \tag{4.5}$$

and the boundary conditions

$$\hat{u}_{y,x}(x,0^+,s) = \frac{1}{s}g(x), \quad s^- < x < s^+, \quad \hat{u}_{y,x}(x,0^+,s) = 0, \quad 0 < x < \infty, \tag{4.6}$$

we arrive at a system of two singular integral equations with respect to the Laplace transforms of the stresses σ_{yy} in the contact zone (s^-, s^+) and ahead of the fore cavity, in the interval $(0,\infty)$,

$$\hat{\chi}_1(x,s) = \int_{s^-}^{s^+} \chi_1(x,\tau)e^{-s\tau}d\tau, \quad \hat{\chi}_2(x,s) = \int_0^{\infty} \chi_2(x,\tau)e^{-s\tau}d\tau. \tag{4.7}$$

The system has the form

$$\begin{aligned}
&\int_{s^-}^{s^+} \left[-\frac{\gamma_1}{\xi-x} + K_{11}(\xi-x,s)\right]\hat{\chi}_1(\xi,s)d\xi + \gamma_2\hat{\chi}_1(x,s) \\
&\quad + \int_0^{\infty} K_{12}(\xi-x,s)\hat{\chi}_2(\xi,s)d\xi = \frac{2\pi}{s}g(x) - \gamma_2\hat{\sigma}_0(x,s) + F(x,s), \quad s^- < x < s^+, \\
&\int_{s^-}^{s^+} K_{21}(\xi-x,s)\hat{\chi}_1(\xi,s)d\xi \\
&\quad + \int_0^{\infty}\left[-\frac{\gamma_1}{\xi-x} + K_{22}(\xi-x,s)\right]\hat{\chi}_2(\xi,s)d\xi = F(x,s), \quad 0 < x < \infty.
\end{aligned} \tag{4.8}$$

Here,

$$\gamma_1 = \frac{2\alpha_l(1-\alpha_s^2)}{R_0}, \quad \gamma_2 = \frac{2k\pi}{R_0}(2\alpha_l\alpha_s - 1 - \alpha_s^2),$$

$$K_{11}(x,s) = K_0(x,s) + K_*(x,s), \quad K_{12}(x,s) = -\frac{\gamma_1}{x} + K_0(x,s),$$

$$K_{21}(x,s) = -\frac{\gamma_1}{x} + K_0(x,s) + K_*(x,s), \quad K_{22}(x,s) = K_0(x,s),$$

$$K_0(x,s) = \int_{-\infty}^{\infty}\left[q_0(p,s) - \frac{i\gamma_1}{2}\,\mathrm{sgn}\,p\right]e^{ipx}dp, \;\; K_*(x,s) = \int_{-\infty}^{\infty}\left[q_1(p,s) - \frac{\gamma_2}{2\pi}\right]e^{ipx}dp,$$

$$\begin{aligned}
F(x,s) = &-\int_{s^-}^{s^+} K_*(\xi-x,s)\hat{\sigma}_0(\xi,s)d\xi \\
&+ \int_{-\infty}^{s^-} K_{12}(\xi-x)\hat{\sigma}_-(\xi,s)d\xi + \int_{s^+}^{0} K_{12}(\xi-x)\hat{\sigma}_+(\xi,s)d\xi.
\end{aligned} \tag{4.9}$$

Note that the kernels $K_{jj}(x,s)$ have the logarithmic singularity, $K_{jj}(x,s) \sim c_0 \log|x|$ as $x \to 0$ ($c_0 = \mathrm{const}$, $j = 1,2$). In the representations of the kernels $K_{12}(x,s)$ and $K_{21}(x,s)$, x never equals 0. The integral representations of the functions $K_0(x,s)$ and $K_*(x,s)$ in (4.9) are convergent since

$$q_0(p,s) - \frac{i\gamma_1}{2}\,\mathrm{sgn}\,p = O(p^{-1}), \quad q_1(p,s) - \frac{\gamma_2}{2\pi} = O(p^{-1}), \quad p \to \pm\infty. \tag{4.10}$$

4.2. Solution of the system of integral equations: Frictionless contact. As in section 3.3, we assume that the thin rigid body is moving without friction between the body surface and the elastic medium, that is, $k = 0$. For the solution of the system (4.8) one might use the orthogonal Chebyshev and Laguerre polynomials and the orthogonal G-functions introduced in [4] that deals with the system of integral equations in the semi-infinite interval and avoids its trancation. Here, we present an alternative simple approach efficient for numerical purposes. First, we truncate the semi-infinite interval by a finite segment $[0, B]$, $B >> 1$ and assume that $\hat{\chi}(x, s) = 0$ as $x \geq B$. By mapping the system (4.8) into the interval $(-1, 1)$ we obtain

$$\int_{-1}^{1} \left[-\frac{\gamma_1}{\xi - \eta} + \nu_0 K_0(\nu_0\xi - \nu_0\eta, s) \right] \chi_1^0(\xi, s) d\xi$$
$$+ \int_{-1}^{1} \left[-\frac{\gamma_1\mu_0}{\mu_0\xi - \nu_0\eta + d} + \mu_0 K_0(\mu_0\xi - \nu_0\eta + d, s) \right] \chi_2^0(\xi, s) d\xi = f_1(\eta, s), \ -1 < \eta < 1,$$
$$\int_{-1}^{1} \left[-\frac{\gamma_1}{\xi - \eta} + \mu_0 K_0(\mu_0\xi - \mu_0\eta, s) \right] \chi_2^0(\xi, s) d\xi$$
$$+ \int_{-1}^{1} \left[-\frac{\gamma_1\nu_0}{\nu_0\xi - \mu_0\eta - d} + \nu_0 K_0(\nu_0\xi - \mu_0\eta - d, s) \right] \chi_1^0(\xi, s) d\xi = f_2(\eta, s), \ -1 < \eta < 1,$$
(4.11)

where

$$\nu_0 = \frac{s^+ - s^-}{2}, \quad \nu_1 = \frac{s^+ + s^-}{2}, \quad \mu_0 = \frac{B}{2}, \quad d = \mu_0 - \nu_1,$$
$$\hat{\chi}_1(\nu_0\xi + \nu_1, s) = \chi_1^0(\xi, s), \quad \hat{\chi}_2(\mu_0\xi + \mu_0, s) = \chi_2^0(\xi, s),$$

$$f_1(\eta, s) = F(\nu_0\eta + \nu_1, s) + \frac{2\pi}{s} g(\nu_0\eta + \nu_1), \quad f_2(\eta, s) = F(\mu_0\eta + \mu_0, s). \tag{4.12}$$

The function $\hat{\chi}_1(x, s)$ is sought in the class of functions vanishing at the points s^- and s^+, while the second function $\hat{\chi}_2(x, s)$ admits the square-root singularity at the point $x = 0$ and has to vanish at the point $x = B$. One of possible representations for the unknown functions is the expansions of them in terms of the Chebyshev polynomials of the first kind

$$\chi_j^0(\xi, s) = \frac{1}{\sqrt{1 - \xi^2}} \sum_{m=0}^{\infty} b_m^{(j)}(s) T_m(\xi), \quad j = 1, 2, \tag{4.13}$$

subjected to the additional conditions

$$\sum_{m=0}^{\infty} (-1)^m b_m^{(1)}(s) = 0, \quad \sum_{m=0}^{\infty} b_m^{(2)}(s) = 0, \tag{4.14}$$

which guarantee that $\chi_1^0(\xi, s) = O(\sqrt{1 + \xi})$, $\xi \to -1$, and $\chi_2^0(\xi, s) = O(\sqrt{1 - \xi})$, $\xi \to 1$, and also the condition

$$\sum_{m=0}^{\infty} b_m^{(1)}(s) = 0. \tag{4.15}$$

This condition implies that the function $\chi_1^0(\xi, s)$ vanishes at the point $\xi = 1$, $\chi_1^0(\xi, s) = O(\sqrt{1 - \xi})$, $\xi \to 1$.

On substituting the series (4.13) into the system (4.11) and using the integral relation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_m(\xi)}{\sqrt{1-\xi^2}}\frac{d\xi}{\xi-\eta}=\begin{cases}U_{m-1}(\eta), & m=1,2,\dots,\\ 0, & m=0,\end{cases}\qquad -1<\eta<1, \tag{4.16}$$

and the orthogonality of the Chebyshev polynomials we eventually arrive at the following infinite system of linear algebraic equations of the second kind:

$$b_{n+1}^{(j)}(s)+\sum_{m=0}^{\infty}\left\{c_{n,m}^{(j,j)}(s)b_m^{(j)}(s)+[c_{n,m}^{(j,3-j)}(s)+d_{n,m}^{(j)}]b_m^{(3-j)}(s)\right\}=f_n^{(j)}(s),$$
$$n=0,1,\dots,\quad j=1,2. \tag{4.17}$$

These equations need to be supplemented by the two equations (4.14). The coefficients of the system and the right-hand side are given by

$$c_{n,m}^{(l,j)}(s)=-\frac{2}{\pi^2\gamma_1}\int_{-1}^{1}\int_{-1}^{1}k_{l,j}(\xi,\eta,s)\frac{T_m(\xi)}{\sqrt{1-\xi^2}}\sqrt{1-\eta^2}U_n(\eta)d\xi d\eta,$$
$$d_{n,m}^{(j)}=-\frac{2}{\pi^2\gamma_1}\int_{-1}^{1}\int_{-1}^{1}d_j(\xi,\eta)\frac{T_m(\xi)}{\sqrt{1-\xi^2}}\sqrt{1-\eta^2}U_n(\eta)d\xi d\eta,$$
$$f_n^{(l)}(s)=-\frac{2}{\pi^2\gamma_1}\int_{-1}^{1}f_l(\eta,s)\sqrt{1-\eta^2}U_n(\eta)d\eta,\quad l=1,2,\quad j=1,2, \tag{4.18}$$

where

$$k_{11}(\xi,\eta,s)=\nu_0K_0(\nu_0[\xi-\eta],s),\quad k_{12}(\xi,\eta,s)=\mu_0K_0(\mu_0\xi-\nu_0\eta+d,s),$$
$$k_{21}(\xi,\eta,s)=\nu_0K_0(\nu_0\xi-\mu_0\eta-d,s),\quad k_{22}(\xi,\eta,s)=\mu_0K_0(\mu_0[\xi-\eta],s),$$
$$d_1(\xi,\eta)=-\frac{\gamma_1\mu_0}{\mu_0\xi-\nu_0\eta+d},\quad d_2(\xi,\eta)=-\frac{\gamma_1\nu_0}{\nu_0\xi-\mu_0\eta-d}. \tag{4.19}$$

The double integrals $d_{n,m}^{(j)}$ and the right-hand-side integrals $f_n^{(j)}$ $(j=1,2)$ are evaluated by the corresponding $\mathcal{N}$-points Gauss quadrature formulas. In the case under consideration, they can be written as

$$\int_{-1}^{1}\frac{f(\xi)T_m(\xi)d\xi}{\sqrt{1-\xi^2}}=\frac{\pi}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}f(\cos\xi_i)\cos m\xi_i,\quad \xi_i=\frac{(2i-1)\pi}{2\mathcal{N}},$$
$$\int_{-1}^{1}f(\xi)\sqrt{1-\xi^2}U_m(\xi)d\xi=\frac{\pi}{\mathcal{N}+1}\sum_{i=1}^{\mathcal{N}}f(\cos\zeta_i)\sin\zeta_i\sin(n+1)\zeta_i,\quad \zeta_i=\frac{i\pi}{\mathcal{N}+1}. \tag{4.20}$$

Since the function K_0 is the inverse Fourier integral (4.9), the coefficients $c_{n,m}^{(lj)}(s)$ are triple integrals. We next aim to transform these integrals into single integrals. Substituting the integral representation (4.9) of the kernel K_0 into the formulas for $c_{nm}^{(lj)}(s)$ in (4.18) and changing the order of integration we derive

$$c_{n,m}^{(11)}(s)=-\frac{2\nu_0}{\pi^2\gamma_1}[(-1)^n-(-1)^m]\int_0^{\infty}\hat{q}_0(p,s)t_m(p\nu_0)u_n(p\nu_0)dp,$$

$$c^{(12)}_{n,m}(s) = -\frac{2\mu_0}{\pi^2\gamma_1}\int_0^\infty \hat{q}_0(p,s)[(-1)^n e^{ipd} - (-1)^m e^{-ipd}]t_m(p\mu_0)u_n(p\nu_0)dp,$$

$$c^{(21)}_{n,m}(s) = -\frac{2\nu_0}{\pi^2\gamma_1}\int_0^\infty \hat{q}_0(p,s)[(-1)^n e^{-ipd} - (-1)^m e^{ipd}]t_m(p\nu_0)u_n(p\mu_0)dp,$$

$$c^{(22)}_{n,m}(s) = -\frac{2\mu_0}{\pi^2\gamma_1}[(-1)^n - (-1)^m]\int_0^\infty \hat{q}_0(p,s)t_m(p\mu_0)u_n(p\mu_0)dp, \tag{4.21}$$

where

$$\hat{q}_0(p,s) = q_0(p,s) - \frac{i\gamma_1}{2}\operatorname{sgn} p,$$

$$t_n(p) = \int_{-1}^1 e^{ip\eta}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta, \quad u_n(p) = \int_{-1}^1 e^{ip\eta}\sqrt{1-\eta^2}U_n(\eta)d\eta. \tag{4.22}$$

On using formulas 7.355(1,2) and 7.393(1,2) [10] we express the integrals $t_n(p)$ and $u_n(p)$ through the Bessel functions

$$t_{2l}(p) = \pi(-1)^l J_{2l}(p), \quad t_{2l+1}(p) = \pi i(-1)^l J_{2l+1}(p),$$

$$u_{2l}(p) = \frac{\pi(-1)^l(2l+1)}{p}J_{2l+1}(p), \; u_{2l+1}(p) = \frac{2\pi i(-1)^l(l+1)}{p}J_{2l+2}(p), \quad l = 0,1,\ldots. \tag{4.23}$$

These formulas can be easily verified by the series expansions of the function $e^{ip\eta}$, changing the order of summation and integration, evaluating the integrals in terms of the Γ-function, and reducing the resulting expressions to the Bessel functions series representations. Notice that, due to the presence of the Bessel functions, the integrands in (4.21) oscillate and are bounded by Ap^{-3} as $p \to \infty$, $A > 0$, and a good convergence of the integrals (4.21) is guaranteed.

Since the system of integral equations (4.11) has a unique solution in the class of functions represented in the form (4.13) and satisfying the two conditions (4.14), the associated infinite system (4.17) subject to the conditions (4.14) also has a unique solution. The infinite system is solved approximately by its truncation. Introduce two $(N+1)$-dimensional vectors $b^{(l)} = (b^{(l)}_0, b^{(l)}_1, \ldots, b^{(l)}_N)$, $l = 1,2$. Due to the additional condition (4.14) the coefficients $\tilde{b}^{(l)}_0$ are expressed through the other coefficients $\tilde{b}^{(l)}_j$ by

$$b^{(1)}_0(s) = -\sum_{m=1}^N (-1)^m b^{(1)}_m(s), \quad b^{(2)}_0(s) = -\sum_{m=1}^N b^{(2)}_m(s). \tag{4.24}$$

To eliminate the coefficients $b^{(1)}_0$ and $b^{(2)}_0$ from the system, we represent the unknowns as

$$b^{(l)}_m(s) = \beta^{(l)}_{m,0}(s) + b^{(1)}_0(s)\beta^{(l)}_{m,1}(s) + b^{(2)}_0(s)\beta^{(l)}_{m,2}(s), \quad m = 1,\ldots,N, \quad l = 1,2. \tag{4.25}$$

This brings us to the following three linear systems of $2N$ equations:

$$\beta_{n,l}^{(1)}(s) + \sum_{m=1}^{N} \left\{ c_{n-1,m}^{(1,1)}(s)\beta_{m,l}^{(1)}(s) + [c_{n-1,m}^{(1,2)}(s) + d_{n-1,m}^{(1)}]\beta_{m,l}^{(2)}(s) \right\} = f_{n,l}^{(1)}(s),$$

$$\beta_{n,l}^{(2)}(s) + \sum_{m=1}^{N} \left\{ [c_{n-1,m}^{(2,1)}(s) + d_{n-1,m}^{(2)}]\beta_{m,l}^{(1)}(s) + c_{n-1,m}^{(2,2)}(s)\beta_{m,l}^{(2)}(s) \right\} = f_{n,l}^{(2)}(s),$$

$$n = 1, 2, \ldots, N, \quad l = 0, 1, 2. \tag{4.26}$$

These systems share the same matrix and have different right-hand sides,

$$f_{n,0}^{(1)} = f_{n-1}^{(1)}, \quad f_{n,0}^{(2)} = f_{n-1}^{(2)},$$

$$f_{n,1}^{(1)} = -c_{n-1,0}^{(1,1)}, \quad f_{n,1}^{(2)} = -c_{n-1,0}^{(2,1)} - d_{n-1,0}^{(2)},$$

$$f_{n,2}^{(1)} = -c_{n-1,0}^{(1,2)} - d_{n-1,0}^{(1)}, \quad f_{n,2}^{(2)} = -c_{n-1,0}^{(2,2)}. \tag{4.27}$$

The three systems (4.26) are free of the unknowns $b_0^{(1)}(s)$ and $b_0^{(2)}(s)$. After they are solved these two unknowns can be easily expressed through the solution to the system (4.26). They are

$$b_0^{(1)} = \frac{\beta_{*0}^{(2)}\beta_{*2}^{(1)} - \beta_{*0}^{(1)}(1 + \beta_{*2}^{(2)})}{\Delta}, \quad b_0^{(2)} = \frac{\beta_{*0}^{(1)}\beta_{*1}^{(2)} - \beta_{*0}^{(2)}(1 + \beta_{*1}^{(1)})}{\Delta}, \tag{4.28}$$

where

$$\beta_{*l}^{(1)} = \sum_{m=1}^{N} (-1)^m \beta_{m,l}^{(1)}, \quad \beta_{*l}^{(2)} = \sum_{m=1}^{N} \beta_{m,l}^{(2)}, \quad l = 0, 1, 2,$$

$$\Delta = (1 + \beta_{*1}^{(1)})(1 + \beta_{*2}^{(2)}) - \beta_{*1}^{(2)}\beta_{*2}^{(1)}. \tag{4.29}$$

5. Separation point, the crack-like cavity length, and the resistance force. The Laplace transform of the normal traction $\hat{\chi}_1(x, s) = \hat{\sigma}_y(x, 0, s)$ found in the previous section has the square root singularity at the separation point $x = s^+$. It has to be removed to guarantee smooth separation of the body from the elastic medium. This can be achieved by satisfying the condition (4.15). Denote by $B_m^{(l)}(t)$ the inverse Laplace transform of the coefficient $b_m^{(l)}(s)$, $l = 1, 2$, $m = 0, 1, \ldots, N$. On applying the inverse Laplace transform to the truncated series in (4.15) we obtain

$$\sum_{m=0}^{N} B_m^{(1)}(\tau) = 0. \tag{5.1}$$

This condition is a transcendental equation with respect to the parameter $s^+ < 0$. Its solution determines the separation point in terms of the distance of the body tip from the origin. Let the x-coordinate of the body tip be $x = a < 0$. The parameter a can be determined by the dimensionless dynamic Griffith criterion [8], [3]

$$K_I(\tau) = \frac{2}{v}\sqrt{\frac{TR_0}{\alpha_l}}, \tag{5.2}$$

where T is the dimensionless Griffith constant and $K_I(\tau)$ is the stress intensity factor at the fore tip of the crack-like cavity. Equations (5.1) and (5.2) constitute a system of two nonlinear equations with respect to the parameters s^+ and a. If the distance $|a|$ from the tip of the body to the fore tip of the cavity ahead of the body is prescribed, then (5.2) specifies the constant T,

$$T = \frac{K_I^2(\tau)v^2\alpha_l}{4R_0}. \tag{5.3}$$

The stress intensity factor

$$K_I(\tau) = \lim_{x\to 0^+} \sqrt{2\pi x}\sigma_y(x,0,\tau) \tag{5.4}$$

is determined from formula (4.13) by

$$K_I(\tau) = \sqrt{\pi\mu_0}\sum_{m=0}^{N}(-1)^m B_m^{(2)}(\tau). \tag{5.5}$$

For numerical tests we select the Poisson ratio $\nu = 0.33$ and the dimensionless speed $v = \frac{1}{2}c_R/c_s$ (for $\nu = 0.33$, $c_R/c_s = 0.9320$). Normalize the geometric parameters by b or, equivalently, choose $b = 1$. Let the dimensionless profile of the body be characterized by the function $h(x)$ (Figure 1),

$$h(x) = \begin{cases} y_0 + \sqrt{r^2-(x-x_0)^2}, & a-l \le x \le s^+ \le a, \\ e_0x + e_1, & s^- \le x \le a-l. \end{cases} \tag{5.6}$$

Here,

$$l = \sqrt{r^2-(r-h_0)^2}, \quad e_0 = \frac{h_0-h_1}{a-s^- -l}, \quad e_1 = \frac{(a-l)h_1 - s^-h_0}{a-l-s^-}, \tag{5.7}$$

and it is assumed that the dimensionless cavity length $|a|$ is independent of time. The derivative of $h(x)$, the function $g(x)$, has the form

$$g(x) = \begin{cases} -(x-x_0)[r^2-(x-x_0)^2]^{-1/2}, & a-l \le x \le s^+ \le a, \\ e_0, & s^- \le x \le a-l. \end{cases} \tag{5.8}$$

Select the Griffith constant T such that $a = -1/3$. Let $h_0 = 1/20$, $h_1 = h_0/2$, $r = 5h_0$. In Figure 2, we present the normal traction σ_{yy} for $s^- < x < s^+$ for $\tau = 1$,

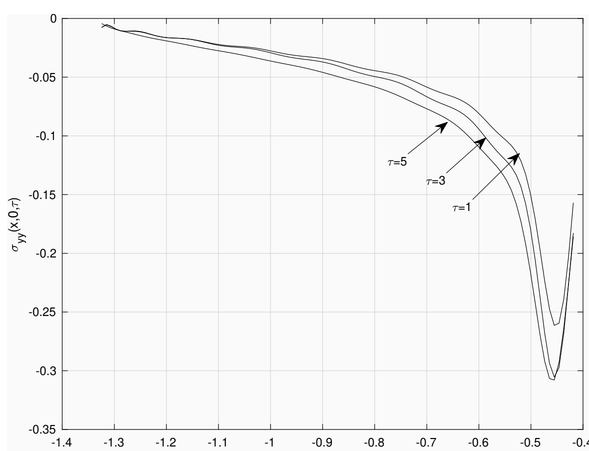

FIG. 2. *The dimensionless normal contact stress σ_{yy} in the dimensionless contact zone s^-s^+ for some values of the dimensionless time parameter $\tau = c_st/b$.*

$\tau = 3$, and $\tau = 5$. The normal traction is bounded and equal to zero at the separation point s^+. For the parameters chosen and when $\tau = 1$, $\tau = 3$, and $\tau = 5$, the point s^+ is given by $s^+ = x_0 + lu$ with the parameter $u \in [0, 1]$ being equal to 0.406, 0.472, and 0.484, respectively. Notice that the magnitude of the normal traction attains a local maximum at the point $x = x_0 = a - l$ of the projectile profile at which the head joins the rest of the body.

The normal stress σ_{yy} in the line $y = 0$ ahead of the fore cavity is given in Figure 3. The normal stress tends to infinity as $x \to 0^+$. The stress intensity factor versus the speed parameter $v \in (0, c_R/c_s)$ when $\tau = 1$, $\tau = 3$, and $\tau = 5$ is shown in Figure 4. As in the case of the transient model of a semi-infinite Mode-I crack propagating at sub-Rayleigh speed in an elastic medium [8], as $v \to c_R/c_s$, the stress intensity factor at the fore tip of the cavity monotonically decreases to 0.

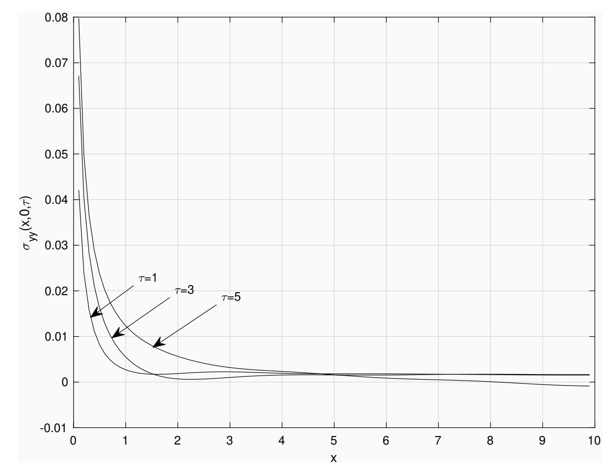

FIG. 3. *The dimensionless normal contact stress in the line $y = 0$ ahead of the cavity, $x > 0$, for some values of the parameter τ.*

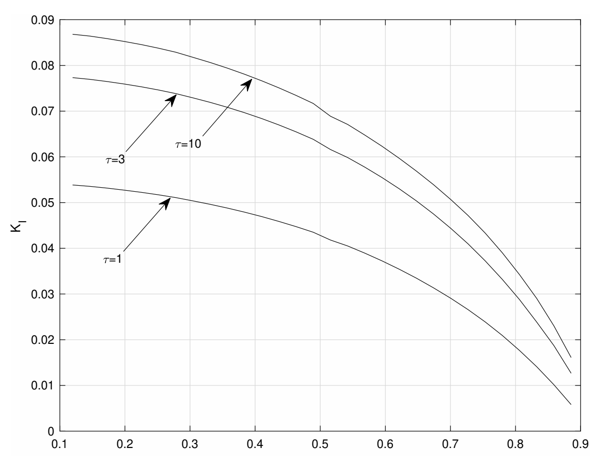

FIG. 4. *The dimensionless stress intensity factor K_I at the fore tip of the finite cavity versus the dimensionless speed parameter v for some values of the parameter τ.*

We wish now to evaluate the magnitude of the dimensionless resistance force due to the motion of the body

$$F = 2\int_\Gamma \{[-\sigma_n]_x + [\sigma_{nt}]_x\} dl. \tag{5.9}$$

Approximately, this formula reduces [1] to

$$F = 2\int_{s^-}^0 [g(x)\sigma_{xx} - \sigma_{xy}] dx. \tag{5.10}$$

Here, $\Gamma = [s^- s^+] \cup [s^+ 0]$, $[s^- s^+]$ is the upper contact zone, $[s^+ 0]$ is the upper boundary of the fore cavity, dl is the elementary arc length, σ_n and σ_{nt} are the normal and tangential components of the traction with the unit normal n to the contact surface directed inside the rigid body, while $[\sigma_n]_x$ and $[\sigma_{nt}]_x$ are their projections on the x-axis. Note that the resistance force derived in [1] for the steady-state case neglected the stress σ_{xx} in the cavity boundary. Since in our case the contact is frictionless, $\sigma_{xy} = 0$ in the contact zone, and the resistance force depends only on the stress σ_{xx} and the tangential derivative of the normal displacement $\omega_2(x,\tau)$,

$$F(\tau) = 2\int_{s^-}^{s^+} g(x)\sigma_{xx}(x,\tau) dx + 2\int_{s^+}^0 \omega_2(x,\tau)\sigma_{xx}(x,\tau) dx. \tag{5.11}$$

To determine the dimensionless stress σ_{xx}, we use formula (2.4) and the notation (2.6). On applying the Laplace and Fourier transform we have

$$\tilde{\sigma}_{xx} = -c^2p^2\tilde{\phi} + (c^2-2)\frac{d^2}{dy^2}\tilde{\phi} - 2ip\frac{d}{dy}\tilde{\psi}. \tag{5.12}$$

For simplicity, we suppose that the external field vanishes, that is, $\sigma(x,\tau) = 0$ for $-\infty < x < \infty$. For $y = 0^+$, due to (3.10) and (3.11) we deduce

$$\tilde{\sigma}_{xx}(p,0^+,s) = -\left\{1 + \frac{2[\beta^2(p,s)+p^2][\beta^2(p,s)-\alpha^2(p,s)]}{R(p,s)}\right\}[\tilde{\chi}_1(p,s) + \tilde{\chi}_2(p,s)]. \tag{5.13}$$

On analyzing the behavior of the integrand as $p \to \pm\infty$ and inverting the Fourier transform we obtain the following expression for the Laplace transform of the stress σ_{xx} as $x \in (s^-, s^+)$ and $y \to 0^+$:

$$\begin{aligned}\hat{\sigma}_{xx}(x,0^+,s) &= -(1-\gamma_0)[\hat{\chi}_1(x,s) + \hat{\chi}_2(x,s)] \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2[\beta^2(p,s)+p^2][\beta^2(p,s)-\alpha^2(p,s)]}{R(p,s)} + \gamma_0\right\}[\tilde{\chi}_1(p,s)+\tilde{\chi}_2(p,s)]e^{-ipx} dp,\end{aligned} \tag{5.14}$$

where

$$\gamma_0 = \frac{2v^2(2-v^2)(1-1/c^2)}{R_0} > 0, \tag{5.15}$$

and the expression in the curly brackets decays as $\text{const}\, p^{-1}$ when $p \to \pm\infty$. This formula may be simplified. Since

$$\hat{\chi}_1(x,s) = \chi_1^0\left(\frac{x-\nu_1}{\nu_0}, s\right), \quad \hat{\chi}_2(x,s) = \chi_2^0\left(\frac{x-\mu_0}{\mu_0}, s\right), \tag{5.16}$$

due to formulas (4.13) and (4.23) we find

$$\int_{s_-}^{s^+} \hat{\chi}_1(x,s)e^{ipx}dx = \pi\nu_0 e^{ip\nu_1}\sum_{l=0}^{\infty}(-1)^l[b_{2l}^{(1)}(s)J_{2l}(p\nu_0) + ib_{2l+1}^{(1)}(s)J_{2l+1}(p\nu_0)],$$

$$\int_0^B \hat{\chi}_2(x,s)e^{ipx}dx = \pi\mu_0 e^{ip\mu_0}\sum_{l=0}^{\infty}(-1)^l[b_{2l}^{(2)}(s)J_{2l}(p\mu_0) + ib_{2l+1}^{(2)}(s)J_{2l+1}(p\mu_0)], \tag{5.17}$$

Here, $b_n^{(j)} = 0$ if $n \geq N$, $j = 1,2$. Finally, we introduce the function

$$Y(p,s) = \pi\nu_0\sum_{l=0}^{N/2}(-1)^l[b_{2l}^{(1)}(s)\cos p(\nu_1 - x)J_{2l}(p\nu_0) - b_{2l+1}^{(1)}(s)\sin p(\nu_1 - x)J_{2l+1}(p\nu_0)]$$

$$+\pi\mu_0\sum_{l=0}^{N/2}(-1)^l[b_{2l}^{(2)}(s)\cos p(\mu_0 - x)J_{2l}(p\mu_0) - b_{2l+1}^{(2)}(s)\sin p(\mu_0 - x)J_{2l+1}(p\mu_0)] \tag{5.18}$$

with $b_{N+1}^{(1)}(s) = b_{N+1}^{(2)}(s) = 0$ and derive the following expression for the Laplace transform of the stress $\sigma_{xx}(x,0^+,\tau)$:

$$\begin{aligned}\hat{\sigma}_{xx}(x,0^+,s) = &-(1-\gamma_0)[\hat{\chi}_1(x,s) + \hat{\chi}_2(x,s)]\\ &-\frac{1}{\pi}\int_0^\infty \left\{\frac{2[\beta^2(p,s)+p^2][\beta^2(p,s)-\alpha^2(p,s)]}{R(p,s)} + \gamma_0\right\}Y(p,s)dp.\end{aligned} \tag{5.19}$$

Determine next the displacement derivative $u_{y,x} = \omega_2(x,\tau)$ needed for the resistance force formula (5.11). From the general formula (4.3) in the case of the frictionless contact when there is no external loading, and the body is self-propelled we find

$$\begin{aligned}\hat{\omega}_2(x,s) = &-\frac{\gamma_1}{2\pi}\left(\int_{s^-}^{s^+}\frac{\hat{\chi}_1(\xi,s)d\xi}{\xi - x} + \int_0^\infty \frac{\hat{\chi}_2(\xi,s)d\xi}{\xi - x}\right)\\ &+\frac{1}{\pi}\int_0^\infty\left[\frac{p\alpha}{R}(p^2-\beta^2) - \frac{\gamma_1}{2}\right]Z(p,s)dp, \quad s^+ < x < 0,\end{aligned} \tag{5.20}$$

where

(5.21)

$$\begin{aligned}Z(p,s) = &-\pi\nu_0\sum_{l=0}^{N/2}(-1)^l[b_{2l}^{(1)}(s)\sin p(\nu_1 - x)J_{2l}(p\nu_0) + b_{2l+1}^{(1)}(s)\cos p(\nu_1 - x)J_{2l+1}(p\nu_0)]\\ &-\pi\mu_0\sum_{l=0}^{N/2}(-1)^l[b_{2l}^{(2)}(s)\sin p(\mu_0 - x)J_{2l}(p\mu_0) - b_{2l+1}^{(2)}(s)\cos p(\mu_0 - x)J_{2l+1}(p\mu_0)].\end{aligned}$$

On inverting the Laplace transform and substituting the representations (4.12), (4.13) of the functions $\chi_1(x,\tau)$ and $\chi_2(x,\tau)$ into (5.20) we obtain

$$\begin{aligned}\omega_2(x,\tau) = &\frac{\gamma_1}{4\pi}\sum_{m=0}^{N}\left[B_m^{(1)}(\tau)\mathcal{J}_m\left(\frac{x-s^+}{2\nu_0}\right) - (-1)^m B_m^{(2)}(\tau)\mathcal{J}_m\left(-\frac{x}{2\mu_0}\right)\right]\\ &+\frac{1}{\pi}\mathcal{L}_\tau^{-1}\left\{\int_0^\infty\left[\frac{p\alpha}{R}(p^2-\beta^2) - \frac{\gamma_1}{2}\right]Z(p,s)dp\right\}.\end{aligned} \tag{5.22}$$

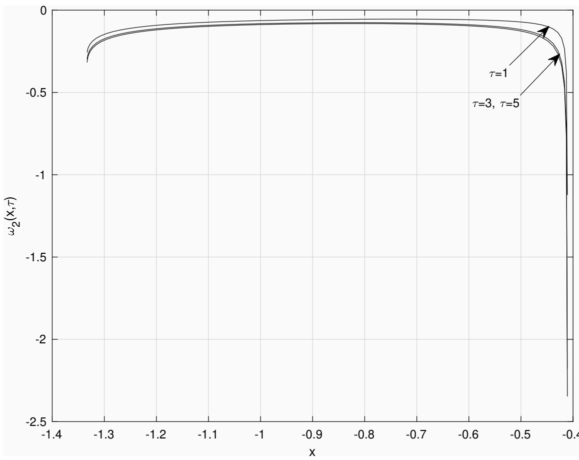

FIG. 5. *The derivative $\omega_2(x,\tau)$ of the upper boundary of the fore cavity for some values of the parameter $\tau = c_s t/b$.*

Here, $B_m^{(j)}(\tau) = \mathcal{L}^{-1}\{b_m^{(j)}(s)\}$, $j = 1, 2$, $\mathcal{L}_\tau^{-1}\{f(s)\}$ is the inverse Laplace transform of a function $f(s)$, and

$$\mathcal{J}_m(y) = \int_0^1 \frac{T_m(1-2\eta)}{\sqrt{\eta(1-\eta)}} \frac{d\eta}{\eta + y}. \tag{5.23}$$

On writing the integral (5.23) as the Mellin convolution integral, using the convolution theorem and the theory of residues we express $\mathcal{J}_m(y)$ through the hypergeometric function F as

$$\mathcal{J}_m(y) = -2\pi m F\left(1-m, 1+m; \frac{3}{2}; -y\right) + \frac{\pi}{\sqrt{y}} F\left(\frac{1}{2}-m, \frac{1}{2}+m; \frac{1}{2}; -y\right). \tag{5.24}$$

For computations, we use the function $g(x)$ given by (5.8) and the parameters employed in Figures 2 and 3. The tangential derivative of the normal displacement, the function $\omega_2(x)$, for $s^+ < x < 0$ is shown in Figure 5. It is bounded at the point $x = s^+$ and tends to $-\infty$ as $x \to 0^-$. For the resistance force, we need the stress σ_{xx}. We show the variation of the stress σ_{xx} with x for some values of τ in the contact zone and in the cavity boundary in Figures 6 and 7, respectively. As the magnitude of the normal traction σ_{yy}, the function $|\sigma_{xx}|$ attains a local maximum at the point $x = x_0 = a - l$ of the projectile profile. It is observed from Figures 6 and 7, that the magnitude of the stress σ_{xx} is small in a neighborhood of the cavity profile. However, the product $\omega_2(x)\sigma_{xx}$ in the interval $s^+ < x < 0$ is not small, and the second integral in formula (5.11), F_2, is comparable to the first integral, F_1, given by

$$F_1(\tau) = 2\int_{s^-}^{s^+} g(x)\sigma_{xx}(x,\tau)dx, \quad F_2(\tau) = 2\int_{s^+}^{0} \omega_2(x,\tau)\sigma_{xx}(x,\tau)dx. \tag{5.25}$$

Our numerical tests (Table 1) show that if the integral F_2 is neglected and the dimensionless resistance force includes only the integral over the body surface, then the resistance force may have a noticeable error. Finally, we emphasize that although the dimensionless force F is small, the actual value of the resistance force, say, $\mathcal{F}$, due to the frictionless motion of the body is not small for the presence of the Young

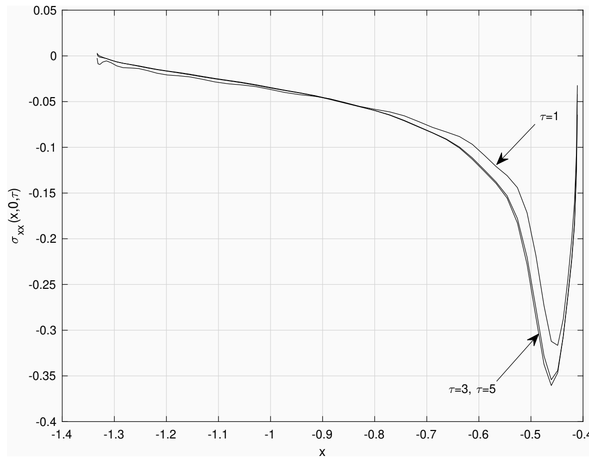

FIG. 6. *The dimensionless normal contact stress* σ_{xx} *for* $s^- < x < s^+$, $y = g(x)$ *for some values of the parameter* $\tau = c_s t/b$.

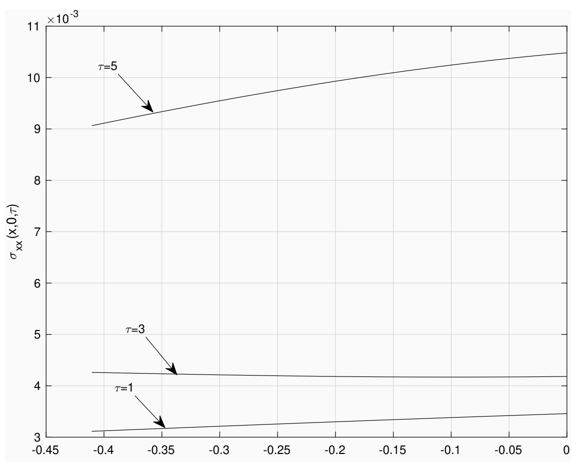

FIG. 7. *The dimensionless normal contact stress* σ_{xx} *for* $s^+ < x < 0$, $y = g(x)$ *for some values of the parameter* $\tau = c_s t/b$.

TABLE 1

The values of the resistance force F *and the integrals* F_1 *and* F_2 *for some values of the parameter* τ.

τ	F_1	F_2	F
1	3.94810^{-4}	-2.17910^{-4}	1.77010^{-4}
3	2.29610^{-3}	-3.80310^{-4}	1.91610^{-3}
5	2.60510^{-3}	-9.49910^{-4}	1.65510^{-3}

modulus μ of the elastic medium in the formula $\mathcal{F} = b\mu F(\tau)$. Here, b is the body length, $\tau = c_s t/b$, c_s is the shear wave speed in the medium, and t is time.

6. Conclusions. We have analyzed a transient two-dimensional problem on motion of a thin finite rigid body at constant sub-Rayleigh speed in an elastic medium. The motion of the body results in creating two cavities, a semi-infinite rear cavity and a finite cavity ahead of the body. By applying the Laplace transform with respect to time and the Fourier transform along the direction of motion we managed to formulate the model problem as a vector Riemann–Hilbert problem with a triangular 3×3 matrix coefficient. The winding numbers (indices) of the first two diagonal elements equal to $+\infty$, while the third element index equals $-\infty$. We proposed an approximate method of solution that develops further the procedure [1], [2] worked out for triangular matrices with meromorphic coefficients and $\pm\infty$-indices of the diagonal elements to the case when some elements of the matrix to be factorized are not meromorphic. Eventually, the vector Riemann–Hilbert problem was converted into an infinite system of linear algebraic equations. The main feature of the procedure is that the rate of convergence of an approximate solution to the exact one is exponential.

In addition to this method we have proposed another procedure that recasts the vector Riemann–Hilbert problem as a system of two singular integral equations. In the case of frictionless contact of the body and the elastic medium the system was solved by the method of orthogonal polynomials. The method reduces the integral equations to an infinite system of linear algebraic equations whose coefficients are some triple integrals. We managed to transform them into single rapidly convergent integrals of the Bessel functions. We determined the point at which the body separates from the elastic medium by solving an associated transcendental equation that guarantees boundedness of the normal traction and smoothness of the cavity profile at the separation point. The length of the fore cavity was derived by employing the dynamic Griffith criterion by neglecting the variation of the cavity length with time. We analyzed the dependence of the stress intensity factor at the fore tip of the finite cavity on speed. A formula for the dimensionless resistance force was derived. It counts on the stress σ_{xx} not only in the interface between the curvilinear body surface but also between the cavity and the elastic medium.

One of the assumptions of the model analyzed was the presence of a Mode-I crack-like cavity ahead of the body. When the body penetrates the elastic medium at sub-Rayleigh speed, the energy release rate at the crack tip is positive, and the solution to the model exists. In the transonic regime, when $v/c_s > 1$, the energy release rate vanishes, and a Mode-I cavity may not move ahead of the body. On the other hand, a Mode-II crack is not formed due to the symmetry with respect to the plane of motion. In the super-Rayleigh regime, when $c_s/c_R < v < 1$, the model that admits the existence of a fore Mode-I crack-like cavity is incorrect since in this case the energy release rate is negative. At the same time the transient model of motion of a finite body trailed by a semi-infinite cavity and without a fore cavity with or without Coulomb friction in the contact zone seems to be correct and is worth analyzing.

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