Evaluation of Line Integrals by Green's Theorem

Using Green's theorem, evaluate the line integral \( \int_C \mathbf{F}(r) \cdot dr \) counterclockwise around the boundary \( C \) of the region \( R \), where:

1. \( \mathbf{F} = [x^2 e^y, y^2 e^x] \), \( C \) the rectangle with vertices (0, 0), (2, 0), (2, 3), (0, 3)
2. \( \mathbf{F} = [3y^2, x - y^4] \), \( R \) the square with vertices (1, 1), (−1, 1), (−1, −1), (1, −1)
3. \( \mathbf{F} = [y, -x] \), \( C \) the circle \( x^2 + y^2 = 1/4 \)
4. \( \mathbf{F} = [2xy^3, 3x^2 y^2] \), \( C \) of \( x^4 + y^4 = 1 \) (sketch it)
5. \( \mathbf{F} = \text{grad} (\sin x \cos y) \), \( C \) the ellipse \( 25x^2 + 9y^2 = 225 \)
6. \( \mathbf{F} = [\sin y, \cos x] \), \( R \) the triangle with vertices (0, 0), (π, 0), (π, 1)
7. \( \mathbf{F} = [\tan 0.2x, x^2 y] \), \( R \): \( x^2 + y^2 \leq 25, y \geq 0 \)
8. \( \mathbf{F} = [\cosh y, -\sinh x] \), \( R \): \( 1 \leq x \leq 3, x \leq y \leq 3x \)
9. \( \mathbf{F} = [e^x /x, e^x \ln (x + 2x)] \), \( R \): \( 1 + x^4 \leq y \leq 2 \)
10. \( \mathbf{F} = [x \cosh 2y, 2x^3 \sinh 2y] \), \( R \): \( x^2 \leq y \leq x \)

Further Applications of Green's Theorem

Area. Find the area of the following regions.

11. The region in the first quadrant within the cardioid (see Example 3)
12. The region under one arch of the cycloid \( r = a(t - \sin t)i + a(1 - \cos t)j \), \( 0 \leq t \leq 2\pi \).

   [Sketch it. Use (4).]
13. The region in the first quadrant under the arc of the limaçon (snail of Pascal) \( r = 1 + 2 \cos \theta \), \( 0 \leq \theta \leq \pi/2 \).

   [Use (5).]

Integral of the normal derivative. In Probs. 14–18, using (9), evaluate \( \int_C \frac{\partial w}{\partial n} \, ds \) counterclockwise over the boundary curve \( C \) of the region \( R \). (Show the details of your work.)

14. \( w = e^x + e^y \), \( R \) the rectangle \( 0 \leq x \leq 2, 0 \leq y \leq 1 \)
15. \( w = \cosh x \), \( R \) the triangle with vertices (0, 0), (4, 2), (0, 2)
16. \( w = e^x \sin y \), \( R \) as in Prob. 15

17. \( w = 3x^3 y - y^3 + y^5 \), \( C \): \( 25x^2 + y^2 = 25 \)
18. \( w = x^3 y + xy^5 \), \( R \): \( x^2 + y^2 \leq 1, x \geq 0, y \geq 0 \)

19. (Laplace's equation) Show that for a solution \( w(x, y) \) of Laplace's equation \( \nabla^2 w = 0 \) in a region \( R \) with boundary curve \( C \) and outer unit normal vector \( n \),

\[
\int_R \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \, dx \, dy = \int_C w \cdot \frac{\partial w}{\partial n} \, ds.
\]

20. PROJECT. Other Forms of Green's Theorem in the Plane. Let \( R \) and \( C \) be as in Green's theorem, \( r' \) a unit tangent vector, and \( n \) an outer unit normal vector of \( C \) (Fig. 220 in Example 4). Show that (1) may be written

\[
\int_R \text{div} \mathbf{F} \, dx \, dy = \int_C \mathbf{F} \cdot n \, ds
\]

or

\[
\int_R (\text{curl} \mathbf{F}) \cdot \mathbf{k} \, dx \, dy = \int_C \mathbf{F} \cdot r' \, ds
\]

where \( \mathbf{k} \) is a unit vector perpendicular to the \( xy \)-plane. Verify (11) and (12) for \( \mathbf{F} = [7x, -3y] \) and \( C \) the circle \( x^2 + y^2 = 4 \) as well as for an example of your own choice.