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Exact solution to the problem of acoustic wave propagation in a semi-infinite waveguide with flexible walls

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HIGHLIGHTS

- Nonstandard integral Laplace transform method is developed.
- Acoustic problems on flexible waveguides reduce to scalar Riemann–Hilbert problems.
- Riemann–Hilbert kernel is a quotient of two cubic polynomials for membrane walls.
- Riemann–Hilbert kernel is a quotient of two degree-5 polynomials for plate walls.
- Exact solution of the problem for a semi-infinite waveguide with flexible walls is found.

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ABSTRACT

Two boundary value problems for the Helmholtz equation in a semi-infinite strip are considered. The main feature of these problems is that, in addition to the function and its normal derivative on the boundary, the functionals of the boundary conditions possess tangential derivatives of the second and fourth orders. Also, the setting of the problems is complemented by certain edge conditions at the two vertices of the semi-strip. The problems model wave propagation in a semi-infinite waveguide with membrane and plate walls. A technique for the exact solution of these fluid–structure interaction problems is proposed. It requires application of two Laplace transforms with respect to both variables. The parameter of the second transform is a certain function of the first Laplace transform parameter. Ultimately, this method yields two scalar Riemann–Hilbert problems with the same coefficient and different right-hand sides. The dependence of the existence and uniqueness results of the physical model problems on the index of the Riemann–Hilbert problem is discussed, and numerical results for the membrane walls model are reported.

1. Introduction

Boundary value problems for the Helmholtz equation with order $n \geq 2$ derivatives in the boundary conditions have been employed in the theory of diffraction since the work [1], where the second order derivatives on the boundary were used to model the surfaces of highly conducted materials. The first order impedance boundary conditions were generalized in [2] by adding second order tangential derivatives on the surface in order to model metal-backed dielectric layers. Order $n \geq 3$ boundary and transition conditions in electromagnetic diffraction theory were systematically studied in [3].

Higher order tangential derivatives in the boundary conditions naturally arise in model problems of aerodynamic noise theory and underwater acoustics when sound waves in fluids interact with flexible surfaces of waveguides [4]. Exact

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solutions and their analysis are available [5] for problems on a compressible fluid bounded by an infinite membrane and an elastic plate fixed along two or more parallel lines when the system is excited by an incident plane wave. These models for membranes and elastic plates are governed by the Helmholtz equation with the third and fifth order derivatives in the boundary conditions, respectively. The Wiener–Hopf method was applied in [6] to study the motion of an infinite plane composed of two half-planes with different elastic constants due to hydroacoustic pressure in the fluid beneath the plate. Edge diffraction of an incident acoustic plane wave by a thin elastic half-plane was treated in [7]. Due to the geometry of the model problem it was also solved by the Wiener–Hopf method. Fifth order boundary conditions were employed in [8] in modeling scattering of sound waves by the edges of a sandwich panel. Analysis of the associated vector Riemann–Hilbert problem was presented in [9,10].

For more complicated domains like a wedge, two joint wedges, and a semi-infinite strip whose boundaries are composed of either membranes or elastic plates, the Wiener–Hopf method is not applicable. The Buchwald method [11] proposed for the solution of the model problem on diffraction of Kelvin waves at a corner was further developed in [12,13] to study diffraction of acoustic waves in a right-angled wedge and a semi-infinite waveguide, respectively, whose surfaces were formed by elastic plates. In this work the authors determined that the number of free constants to be determined from the conditions at the vertices of the structure depends only on the orders of the derivatives in the boundary conditions. They also found an integral representation of the acoustic pressure distribution. Some model problems of sound–structure interaction with high-order boundary conditions by the method of eigenfunction expansions were treated in [14]. These include the problem of propagation of an acoustic wave in a semi-infinite waveguide when the upper boundary is a semi-infinite membrane, while the lower and the finite vertical sides are acoustically rigid walls (in this case, by symmetry, the problem for a half-strip reduces to a problem for an infinite strip).

The Poincaré boundary value problem for the modified Helmholtz operator $\Delta - k^2$ (k is a real number) in a semi-infinite strip was studied in [15] by the method proposed in [16]. It was shown that the problem reduces to an order-2 vector Riemann–Hilbert problem on the real axis whose matrix coefficient, in general, does not admit an explicit factorization by the methods available in the literature. However, in some important cases, including the case of impedance boundary conditions, it allows for an exact factorization. In some other cases the vector problem reduces to a Riemann–Hilbert problem with a triangle matrix coefficient, or can be even decoupled. Notice that in the case of the Helmholtz operator $\Delta + k^2$ ($k \in \mathbb{R}$), the contour of the Riemann–Hilbert problem comprises two semi-infinite rays and two circular arcs [16].

Our goal in this work is to develop an efficient method for the Helmholtz equation in a semi-infinite strip $\{0 < x < \infty, 0 < y < a\}$ with generalized impedance boundary conditions of higher order. The main feature of this technique is that it applies two Laplace transforms in a nonstandard way. The first transform is utilized with respect to x in the classical way, while the second one is applied in the y-direction (the function is extended by zero for $y > a$) with a parameter ζ that is a root of the characteristic polynomial of the ordinary differential operator $d^2/dy^2 + k^2 - \eta^2$, the Laplace image of the original Helmholtz operator. This root is $\zeta = \sqrt{\eta^2 - k^2}$, where k is the wave number, η is the parameter of the first transform, and $\sqrt{\eta^2 - k^2}$ is a fixed branch of the function $\zeta^2 = \eta^2 - k^2$. Such a transform was proposed in [17] for reducing problems on electromagnetic scattering by right-angled wedges to vector Riemann–Hilbert problems and employed in [18] for exact solution of the electromagnetic problem on diffraction of an obliquely incident wave by an impedance right-angled concave wedge. The method to be applied in this paper ultimately yields two symmetric scalar Riemann–Hilbert problems on the real axis equivalent to the original model problem. We emphasize that the contour is the real axis regardless if k is real, imaginary, or a general complex number. The Riemann–Hilbert problems share the same coefficient and have different right-hand sides. Remarkably, the coefficient is a simple rational function, $G(\eta) = q(\eta)/q(-\eta)$, where $q(\eta)$ is a cubic polynomial in the membrane case and a degree-5 polynomial in the elastic plate case, and the degree of $q(\eta)$ coincides with the order of the highest derivative involved in the boundary conditions. We determine the number of free constants in the solution that depends not only on the order of the tangential derivatives in the boundary conditions but also on the position of the zeros of the polynomial $q(\eta)$ and therefore on the parameters of the problem. In the membrane case, we derive explicitly a system of linear algebraic equations for the unknown constants and summarize the results by stating an existence — uniqueness theorem. We also write down representation formulas for the solution by quadratures and, in addition, by series convenient for computational purposes. The theoretical results are illustrated by numerical tests that concern the real and imaginary values for the complex potential in the membrane vertical and horizontal sides.

In Appendix, we analyze the case of the first order impedance boundary conditions and derive an exact solution by the method presented in the paper and also by means of the finite integral transform whose kernel is the solution to the associated Sturm–Liouville problem. We show that the series representations of the solution obtained by these two methods are identical. We note that the classical method of finite integral transforms used in Appendix is not applicable in the case of higher order impedance boundary conditions.

2. Helmholtz equation in a semi-infinite waveguide: membrane walls

2.1. Formulation

Of concern is the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u(x, y) = g(x, y), \quad 0 < x < \infty, \quad 0 < y < a, \tag{2.1}$$

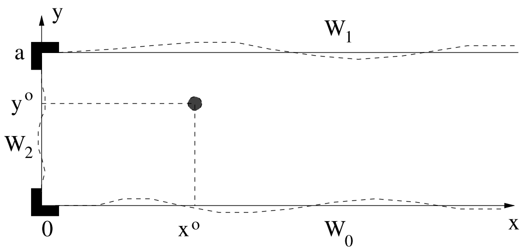

Fig. 1. Geometry of the problem on a semi-infinite waveguide.

with respect to an unknown function $u(x, y)$ subjected to the boundary conditions

$$\left[\left(\frac{\partial^2}{\partial x^2}+\alpha_0^2\right)\frac{\partial}{\partial y}-\mu_0\right]u=g_0(x),\quad (x,y)\in W_0=\{0<x<\infty,\ y=0\},$$
$$\left[\left(\frac{\partial^2}{\partial x^2}+\alpha_1^2\right)\frac{\partial}{\partial y}+\mu_1\right]u=g_1(x),\quad (x,y)\in W_1=\{0<x<\infty,\quad y=a\},$$
$$\left[\left(\frac{\partial^2}{\partial y^2}+\alpha_2^2\right)\frac{\partial}{\partial x}-\mu_2\right]u=g_2(y),\quad (x,y)\in W_2=\{x=0,\quad 0<y<a\}.$$
$$u\to 0,\quad x\to\infty,\quad 0<y<a. \tag{2.2}$$

This boundary value problem governs acoustic wave propagation in a semi-infinite waveguide (Fig. 1). Here, $e^{-i\omega t}g(x, y)$ is the source function, $\mathcal{U}(x, y, t) = \mathrm{Re}[e^{-i\omega t}u(x, y)]$ is the fluid velocity potential, u is a complex potential, ω is the frequency, $\omega = \omega_1+i\omega_2, \omega_j > 0$, t is time, $k = \omega/c$ is the wave number, and c is the sound speed in the fluid. The pressure distribution $p(x, y, t)$ and the surface deflection $w(x, y, t)$ are expressed through the velocity potential as

$$p=-\rho\frac{\partial\mathcal{U}}{\partial t},\quad \frac{\partial w}{\partial t}=\frac{\partial\mathcal{U}}{\partial n}, \tag{2.3}$$

where $\partial/\partial n$ is the external normal derivative and ρ is the mean fluid density. Therefore

$$p=\mathrm{Re}[i\rho\omega e^{-i\omega t}u],\quad w=\mathrm{Re}\left[e^{-i\omega t}\frac{i}{\omega}\frac{\partial u}{\partial n}\right]. \tag{2.4}$$

The equations of motion of the membrane walls have the form [5,14]

$$-T_j w_{xx}+m_j w_{tt}=p,\quad y=y_j,\quad j=0,1,\quad 0<x<\infty,$$

$$-T_2 w_{yy}+m_2 w_{tt}=p,\quad x=0,\quad 0<y<a, \tag{2.5}$$

where $y_0 = 0$, $y_1 = a$, the suffixes x, y, and t denote differentiation with respect to the corresponding variables, m_j is the mass per unit area, and T_j is the surface tension for the membrane W_j. It follows from (2.4) and (2.5) that when

$$g_0(x)=g_1(x)=0,\quad 0<x<\infty;\quad g_2(y)=0,\quad 0<y<a, \tag{2.6}$$

the boundary conditions (2.2) model the deflection of the membrane walls due to pressure loading (we put the functions $g_0(x)$, $g_1(x)$, and $g_2(y)$ into the conditions (2.2) for generality). The parameters involved in (2.2) are

$$\alpha_j=\omega\sqrt{\frac{m_j}{T_j}},\quad \mu_j=\frac{\rho\omega^2}{T_j},\quad j=0,1,2. \tag{2.7}$$

Since $\mathrm{Im}\,\omega > 0$, we have $\mathrm{Im}\,\alpha_j > 0$.

We also need to specify the edge conditions at $x = y = 0$ and $x = 0$, $y = a$. It is assumed that the edges are fixed, and therefore the following four conditions have to be satisfied:

$$\lim_{x\to 0^+}u_y(x,0)=\lim_{x\to 0^+}u_y(x,a)=0,$$
$$\lim_{y\to 0^+}u_x(0,y)=\lim_{y\to a^-}u_x(0,y)=0. \tag{2.8}$$

2.2. Two scalar Riemann–Hilbert problems

Our goal in this section is to present a method that is capable to convert the boundary value problem for the Helmholtz equation (2.1) with higher-order boundary conditions (2.2) in a semi-strip to two scalar Riemann–Hilbert problems. To achieve this, we apply first the Laplace transform with respect to x

$$\tilde{u}(\eta, y) = \int_0^\infty u(x, y)e^{i\eta x}dx \tag{2.9}$$

to Eq. (2.1) and the first and second boundary conditions in (2.2). In conjunction with (2.8) this brings us to the one-dimensional boundary value problem

$$\begin{aligned} &L[\tilde{u}] \equiv \left(\frac{d^2}{dy^2} - \zeta^2\right)\tilde{u}(\eta, y) = f(y), \quad 0 < y < a,\\ &U_0[\tilde{u}] \equiv -\tilde{u}_y(\eta, 0) + \tilde{\mu}_0(\eta)\tilde{u}(\eta, 0) = \tilde{g}^{(0)}(\eta),\\ &U_1[\tilde{u}] \equiv \tilde{u}_y(\eta, a) + \tilde{\mu}_1(\eta)\tilde{u}(\eta, a) = \tilde{g}^{(1)}(\eta), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &\zeta^2 = \eta^2 - k^2, \quad f(y) = u_x(0, y) - i\eta u(0, y) + \tilde{g}(\eta, y),\\ &\tilde{\mu}_j = \frac{\mu_j}{\alpha_j^2 - \eta^2}, \quad \tilde{g}^{(j)}(\eta) = (-1)^{j+1}\frac{\tilde{g}_j(\eta) + c_j}{\alpha_j^2 - \eta^2},\\ &\tilde{g}(\eta, y) = \int_0^\infty g(x, y)e^{i\eta x}dx, \quad \tilde{g}_j(\eta) = \int_0^\infty g_j(x)e^{i\eta x}dx, \quad j = 0, 1, \end{aligned} \tag{2.11}$$

and c_0 and c_1 are unknown constants

$$c_0 = u_{xy}(0^+, 0), \quad c_1 = u_{xy}(0^+, a). \tag{2.12}$$

These constants and the functions $u_x(0, y)$ and $u(0, y)$ in (2.11) are to be determined.

Denote by $\zeta = \sqrt{\eta^2 - k^2}$ the single branch of the two-valued function $\zeta^2 = \eta^2 - k^2$ in the η-plane cut along the straight line joining the branch points $\eta = \pm k$ ($k \in \mathbb{C}^+$) and passing through the infinite point; the branch is fixed by the condition $\zeta = -ik$ as $\eta = 0$. This choice guarantees that $\operatorname{Re}\zeta \geq 0$ whenever $-\infty < \eta < \infty$.

We next employ the fundamental function of the differential operator L

$$\Psi(y, s) = -\frac{e^{-\zeta|y-s|}}{2\zeta} \tag{2.13}$$

and construct the function

$$G(y, s) = \Psi(y, s) - \sum_{j=0}^{1} U_j[\Psi](s)\phi_j(y), \tag{2.14}$$

where $\{\phi_0(y), \phi_1(y)\}$ is the fundamental system of the problem (2.10) formed by the two solutions to the problem

$$\begin{aligned} &L[\phi_j(y)] = 0, \quad 0 < y < a,\\ &U_m[\phi_j] = \delta_{mj}, \quad m, j = 0, 1. \end{aligned} \tag{2.15}$$

The function (2.14) possesses the following properties:

$$\begin{aligned} &L[G] = 0, \quad 0 < y < s < a \quad \text{and} \quad 0 < s < y < a,\\ &\frac{\partial^j G}{\partial y^j}(y, y - 0) - \frac{\partial^j G}{\partial y^j}(y, y + 0) = \delta_{j1}, \quad j = 0, 1,\\ &U_j[G](s) = 0, \quad 0 < s < a, \end{aligned} \tag{2.16}$$

and therefore [19] it is the Green function of the boundary value problem (2.10) with homogeneous boundary conditions. The two solutions to the problem (2.15) can be easily found. They are

$$\phi_0(y) = \frac{\zeta\cosh\zeta(a - y) + \tilde{\mu}_1\sinh\zeta(a - y)}{\tilde{\Delta}(\zeta)}, \quad \phi_1(y) = \frac{\zeta\cosh\zeta y + \tilde{\mu}_0\sinh\zeta y}{\tilde{\Delta}(\zeta)}. \tag{2.17}$$

Here,

$$\tilde{\Delta}(\zeta) = (\tilde{\mu}_0 + \tilde{\mu}_1)\zeta\cosh a\zeta + (\tilde{\mu}_0\tilde{\mu}_1 + \zeta^2)\sinh a\zeta. \tag{2.18}$$

Finally, on substituting the functions (2.17) into (2.14) and evaluating $U_j[\Psi](s)$ we determine the Green function

$$\begin{aligned} G(y, s) = &-\frac{e^{-\zeta|y-s|}}{2\zeta} + \frac{1}{2\zeta\tilde{\Delta}(\zeta)}\left\{(\tilde{\mu}_0 - \zeta)[\zeta\cosh\zeta(a - y) + \tilde{\mu}_1\sinh\zeta(a - y)]e^{-\zeta s}\right.\\ &\left.+(\tilde{\mu}_1 - \zeta)(\zeta\cosh\zeta y + \tilde{\mu}_0\sinh\zeta y)e^{-\zeta(a-s)}\right\}. \end{aligned} \tag{2.19}$$

In terms of the Green function and the functions ϕ_0 and ϕ_1 the solution to the problem (2.10) with the inhomogeneous boundary conditions may be written in the form

$$\tilde{u}(\eta, y) = \int_0^a G(y, s)f(s)ds + \tilde{g}^0(\eta)\phi_0(y) + \tilde{g}^1(\eta)\phi_1(y). \tag{2.20}$$

Now, in this formula, we put $y = 0$ and $y = a$, use the expression for the function $f(y)$ from (2.11) and denote

$$\hat{u}(0, i\zeta) = \int_0^a u(0, y)e^{-\zeta y}dy, \quad \hat{u}_x(0, i\zeta) = \int_0^a u_x(0, y)e^{-\zeta y}dy,$$
$$\hat{\tilde{g}}(\eta, i\zeta) = \int_0^a \tilde{g}(\eta, y)e^{-\zeta y}dy. \tag{2.21}$$

Note that these integrals can be interpreted as the Laplace transforms with the parameter ζ of the corresponding functions whose definition is extended by zero for $y > a$. The two relations for $y = 0$ and $y = a$ derived from (2.20) can be complemented by their counterparts with η being replaced by $-\eta$. The four resulting equations are

$$\begin{aligned}
&\tilde{u}(\pm\eta, 0) + \Lambda_{00}(\zeta)\hat{u}_x(0, i\zeta) \mp i\eta\Lambda_{00}(\zeta)\hat{u}(0, i\zeta) \\
&\quad + \Lambda_{01}(\zeta)\hat{u}_x(0, -i\zeta) \mp i\eta\Lambda_{01}(\zeta)\hat{u}(0, -i\zeta) + h_0(\pm\eta) = 0, \\
&\tilde{u}(\pm\eta, a) + \Lambda_{10}(\zeta)\hat{u}_x(0, i\zeta) \mp i\eta\Lambda_{10}(\zeta)\hat{u}(0, i\zeta) \\
&\quad + \Lambda_{11}(\zeta)\hat{u}_x(0, -i\zeta) \mp i\eta\Lambda_{11}(\zeta)\hat{u}(0, -i\zeta) + h_1(\pm\eta) = 0,
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
&\Lambda_{00}(\zeta) = \frac{1}{2\zeta}\left[1 - \frac{\tilde{\mu}_0 - \zeta}{\tilde{\Delta}(\zeta)}(\zeta \cosh a\zeta + \tilde{\mu}_1 \sinh a\zeta)\right], \\
&\Lambda_{01}(\zeta) = -\frac{\tilde{\mu}_1 - \zeta}{2\tilde{\Delta}(\zeta)}e^{-a\zeta}, \quad \Lambda_{10}(\zeta) = -\frac{\tilde{\mu}_0 - \zeta}{2\tilde{\Delta}(\zeta)}, \\
&\Lambda_{11}(\zeta) = \frac{e^{-a\zeta}}{2\zeta}\left[1 - \frac{\tilde{\mu}_1 - \zeta}{\tilde{\Delta}(\zeta)}(\zeta \cosh a\zeta + \tilde{\mu}_0 \sinh a\zeta)\right].
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&h_0(\eta) = \Lambda_{00}(\zeta)\hat{\tilde{g}}(\eta, i\zeta) + \Lambda_{01}(\zeta)\hat{\tilde{g}}(\eta, -i\zeta) - \frac{(\zeta \cosh a\zeta + \tilde{\mu}_1 \sinh a\zeta)\tilde{g}^{(0)}(\eta)}{\tilde{\Delta}(\zeta)} - \frac{\zeta\tilde{g}^{(1)}(\eta)}{\tilde{\Delta}(\zeta)}, \\
&h_1(\eta) = \Lambda_{10}(\zeta)\hat{\tilde{g}}(\eta, i\zeta) + \Lambda_{11}(\zeta)\hat{\tilde{g}}(\eta, -i\zeta) - \frac{\zeta\tilde{g}^{(0)}(\eta)}{\tilde{\Delta}(\zeta)} - \frac{(\zeta \cosh a\zeta + \tilde{\mu}_0 \sinh a\zeta)\tilde{g}^{(1)}(\eta)}{\tilde{\Delta}(\zeta)}.
\end{aligned} \tag{2.24}$$

We assert that the boundary condition on the vertical side $M_2 = \{x = 0, 0 < y < a\}$ has not been satisfied yet. On extending the function $g_2(y)$ by zero to the interval $y > a$, applying the Laplace transform (2.21) to the third condition in (2.2) with the parameters ζ and $-\zeta$ and utilizing the edge conditions (2.8) we discover

$$-\hat{u}_x(0, \pm i\zeta) + \hat{\mu}_2(\zeta)\hat{u}(0, \pm i\zeta) = \hat{g}^{(2)}(\pm i\zeta), \tag{2.25}$$

where

$$\begin{aligned}
&\hat{\mu}_2(\zeta) = \frac{\mu_2}{\zeta^2 + \alpha_2^2}, \quad \hat{g}^{(2)}(i\zeta) = \frac{-\hat{g}_2(i\zeta) - c_2 + c_3e^{-a\zeta}}{\zeta^2 + \alpha_2^2}, \\
&\hat{g}_2(i\zeta) = \int_0^a g_2(y)e^{-\zeta y}dy,
\end{aligned} \tag{2.26}$$

and c_2 and c_3 are unknown constants,

$$c_2 = u_{xy}(0, 0^+), \quad c_3 = u_{xy}(0, a^-), \tag{2.27}$$

to be fixed.

Our intention next is to express the four functions $\hat{u}(0, \pm i\zeta)$ and $\hat{u}_x(0, \pm i\zeta)$ through the functions $\tilde{u}(\pm\eta, 0)$ and $\tilde{u}(\pm\eta, a)$ from the system (2.22) and insert them into the two equations (2.25). We have first

$$\begin{pmatrix} \hat{u}_x(0, i\zeta) \\ \hat{u}(0, i\zeta) \\ \hat{u}_x(0, -i\zeta) \\ \hat{u}(0, -i\zeta) \end{pmatrix} = A(\zeta)\begin{pmatrix} \tilde{u}(\eta, 0) + h_0(\eta) \\ \tilde{u}(-\eta, 0) + h_0(-\eta) \\ \tilde{u}(\eta, a) + h_1(\eta) \\ \tilde{u}(-\eta, a) + h_1(-\eta) \end{pmatrix}, \tag{2.28}$$

where

$$A(\zeta)=-\frac{1}{2\eta}\begin{pmatrix}\eta(\tilde\mu_0+\zeta) & \eta(\tilde\mu_0+\zeta) & \eta(\tilde\mu_1-\zeta)e^{-a\zeta} & \eta(\tilde\mu_1-\zeta)e^{-a\zeta}\\ i(\tilde\mu_0+\zeta) & -i(\tilde\mu_0+\zeta) & i(\tilde\mu_1-\zeta)e^{-a\zeta} & -i(\tilde\mu_1-\zeta)e^{-a\zeta}\\ \eta(\tilde\mu_0-\zeta) & \eta(\tilde\mu_0-\zeta) & \eta(\tilde\mu_1+\zeta)e^{a\zeta} & \eta(\tilde\mu_1+\zeta)e^{a\zeta}\\ i(\tilde\mu_0-\zeta) & -i(\tilde\mu_0-\zeta) & i(\tilde\mu_1+\zeta)e^{a\zeta} & -i(\tilde\mu_1+\zeta)e^{a\zeta}\end{pmatrix}, \tag{2.29}$$

and then, after utilizing equations (2.25), we eventually arrive at the following remarkably simple scalar Riemann–Hilbert problems which share the same coefficient and have different free terms:

$$\Phi_j^+(\eta)=H(\eta)\Phi_j^-(\eta)+f_j(\eta),\quad -\infty<\eta<+\infty, \tag{2.30}$$

subject to the symmetry conditions

$$\Phi_j^+(\eta)=\Phi_j^-(-\eta),\quad \eta\in\mathbb{C},\quad j=0,1, \tag{2.31}$$

where the functions $\Phi_j^+(\eta)$ and $\Phi_j^-(\eta)$ are analytically continued from the contour, the real axis, into the upper and lower half-planes, $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively,

$$\Phi_0^\pm(\eta)=\tilde u(\pm\eta,0),\quad \Phi_1^\pm(\eta)=\tilde u(\pm\eta,a), \tag{2.32}$$

and

$$\begin{aligned}H(\eta)&=-\frac{\eta+i\hat\mu_2(\zeta)}{\eta-i\hat\mu_2(\zeta)}=-\frac{\eta(\eta^2-k^2+\alpha_2^2)+i\mu_2}{\eta(\eta^2-k^2+\alpha_2^2)-i\mu_2},\\ f_j(\eta)&=-h_j(\eta)+H(\eta)h_j(-\eta)+\frac{\eta}{(\eta-i\hat\mu_2)\tilde\Delta(\zeta)}\\ &\quad\times\{[\zeta+(-1)^j\tilde\mu_{1-j}]e^{(1-j)a\zeta}\hat g^{(2)}(i\zeta)+[\zeta-(-1)^j\tilde\mu_{1-j}]e^{(j-1)a\zeta}\hat g^{(2)}(-i\zeta)\},\quad j=0,1.\end{aligned} \tag{2.33}$$

Notice that this procedure can also be applied when the differential operators $\partial^2/\partial x^2+\alpha_j^2$ $(j=0,1)$ and $\partial^2/\partial y^2+\alpha_2^2$ in (2.2) are replaced by $\partial^2/\partial x^2+\beta_j\partial/\partial x+\alpha_j^2$ and $\partial^2/\partial y^2+\beta_2\partial/\partial y+\alpha_2^2$, respectively. However, in this case the Riemann–Hilbert problems are coupled, and the matrix coefficient of the vector Riemann–Hilbert problem does not admit a closed-form factorization by methods available in the literature.

To factorize the rational function $H(\eta)$ we introduce the polynomial

$$q(\eta)=\eta(\eta^2-k^2+\alpha_2^2)+i\mu_2. \tag{2.34}$$

Then $H(\eta)=q(\eta)/q(-\eta)$. Denote the zeros of the cubic polynomial $q(\eta)$ by z_j. For convenience, we introduce two real and positive parameters γ_0 and γ_1 by $\gamma_0=\alpha_2^2/k^2=m_2c^2/T_2$ and $\gamma_1=\mu_2/k^2=\rho c^2/T_2$. Then the polynomial q can be written as

$$q(\eta)=\eta(\eta^2-k^2+k^2\gamma_0)+ik^2\gamma_1. \tag{2.35}$$

We next prove the following statement.

Lemma. *Let $\gamma_0>0$, $\gamma_1>0$, and $k=k_1+ik_2$, $k_j>0$. Then all three zeros of the polynomial $q(\eta)$ are simple and either*
(i) one of the zeros is in the upper half-plane $\mathbb{C}^+$, while the other two are in the lower half-plane $\mathbb{C}^-$, or
(ii) one of the zeros is real, while the other two fall into the opposite half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, or
(iii) one of the zeros is in the lower half-plane $\mathbb{C}^-$, while the other two are in the upper half-plane $\mathbb{C}^+$.

Proof. Let the three zeros be z_0, z_1, and z_2. Then $q(\eta)=(\eta-z_0)(\eta-z_1)(\eta-z_2)$. On comparing this with (2.35) we have $z_2=-(z_0+z_1)$. This eliminates the case when all three zeros fall either in the upper half-plane or in the lower half-plane.

Determine under which conditions one of the zeros say, z_0, is real. Write $k=k_1+ik_2$ and separate the real and imaginary parts of the equation $q(z_0)=0$. This yields two real equations

$$\begin{aligned}&2z_0(\gamma_0-1)k_1k_2+\gamma_1(k_1^2-k_2^2)=0,\\ &z_0^3+z_0(\gamma_0-1)(k_1^2-k_2^2)-2k_1k_2\gamma_1=0.\end{aligned} \tag{2.36}$$

We have two possibilities,

$$\gamma_0=1,\quad k_1=k_2,\quad z_0=(2k_1^2\gamma_1)^{1/3}, \tag{2.37}$$

and

$$\gamma_1=2k_1k_2(k_1^2+k_2^2)\left(\frac{1-\gamma_0}{k_1^2-k_2^2}\right)^{3/2},\quad \frac{1-\gamma_0}{k_1^2-k_2^2}>0,\quad z_0=-\frac{\gamma_1(k_1^2-k_2^2)}{2(\gamma_0-1)k_1k_2}. \tag{2.38}$$

Table 1
The roots z_j $(j=0,1,2)$ of $q(\eta)=\eta(\eta^2-k^2+k^2\gamma_0)+ik^2\gamma_1$ for some values of the parameters γ_0, γ_1, and k.

γ_0	γ_1	k	z_0	z_1	z_2
5	1	$1+0.1i$	$-0.0008424-0.2540i$	$-0.2009+2.115i$	$0.2017-1.861i$
0.5	0.1	$1+0.1i$	$0.7264-0.02424i$	$-0.002135+0.1872i$	$-0.7243-0.1629i$
1	0.1	$1+i$	0.5848	$-0.2924+0.5065i$	$-0.2924-0.5065i$
0.5	0.05	$1+0.1i$	$0.7123+0.02117i$	$-0.0003512+0.09816i$	$-0.7120-0.1193i$
1	0.1	i	$-0.4020+0.2321i$	$0.4020+0.2321i$	$-0.4642i$

Show next that $q(\eta)$ may have at most one real zero. Assume that z_0 and z_1 are two real zeros. Then from (2.35) we have

$$k^2=\frac{z_0^3}{z_0(1-\gamma_0)-i\gamma_1}=\frac{z_1^3}{z_1(1-\gamma_0)-i\gamma_1}. \tag{2.39}$$

This complex equation immediately yields $z_0=z_1$. Therefore, $q'(z_0)=0$, and $k^2=3z_0^2/(1-\gamma_0)$ is a real number that contradicts to the assumption $k_j>0, j=1,2$.

Prove finally that all complex zeros are always simple. Assume $z_0=z_1$. Then z_0 satisfies the system

$$z_0^3+(\gamma_0-1)k^2z_0+i\gamma_1k^2=0,\quad 3z_0^2+(\gamma_0-1)k^2=0. \tag{2.40}$$

From here we deduce

$$k^2=-\frac{27\gamma_1^2}{4(1-\gamma_0)^3},\quad z_0=\frac{3i\gamma_1}{2(1-\gamma_0)}. \tag{2.41}$$

The former equation implies that k^2 is real that again contradicts to the assumption $k_j>0, j=1,2$.

Without loss we consider the following three cases:
(i) $z_0=-\eta_0$, $z_1=\eta_1$, and $z_2=-\eta_2$, $\mathrm{Im}\,\eta_j>0, j=0,1,2$,
(ii) $z_0=\eta_0$, $z_1=\eta_1$, and $z_2=-\eta_2$, $\mathrm{Im}\,\eta_0=0$, $\mathrm{Im}\,\eta_j>0, j=1,2$,
(iii) $z_0=\eta_0$, $z_1=\eta_1$, and $z_2=-\eta_2$, $\mathrm{Im}\,\eta_j>0, j=0,1,2$.

In Table 1 we present the roots of the polynomial $q(\eta)$ in cases (i) (the first two rows), (ii) (the third row), and (iii) (the fourth and fifth rows). Remark that in the last case $k_1=0$.

Consider the first case when the polynomial $q(\alpha)$ has one zero in the upper half-plane and two zeros in the lower half-plane. The location of the zeros enables us to factorize the coefficient $H(\eta)$ as

$$H(\eta)=\frac{H^+(\eta)}{H^-(\eta)},\quad -\infty<\eta<+\infty, \tag{2.42}$$

where

$$H^+(\eta)=\frac{(\eta+\eta_0)(\eta+\eta_2)}{\eta+\eta_1},\quad H^-(\eta)=-\frac{(\eta-\eta_0)(\eta-\eta_2)}{\eta-\eta_1}, \tag{2.43}$$

and $\pm\eta_j\in\mathbb{C}^\pm$, $j=0,1,2$. The index (the winding number) of the function $H(\eta)$

$$\mathrm{ind}\,H(\eta)=\frac{1}{2\pi}\,[\arg H(\eta)]_{-\infty}^{\infty}=-1. \tag{2.44}$$

Therefore, in the class of functions having a simple zero at the infinite point, one expects the Riemann–Hilbert problems being solvable if and only if a certain condition is fulfilled. However, we assert that in our case this condition is automatically satisfied, and the solution always exists. This solution is unique provided the functions $f_j(\tau)$ are uniquely defined. Indeed, introduce the Cauchy integrals

$$\Psi_j^\pm(\eta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_j(\tau)d\tau}{H^+(\tau)(\tau-\eta)},\quad \eta\in\mathbb{C}^\pm,\quad j=0,1, \tag{2.45}$$

with the densities $f_j(\tau)/H^+(\tau)$ vanishing at the infinite point: $|f_j(\tau)/H^+(\tau)|\le c|\tau|^{-4}$, $\tau\to\pm\infty$, c is a nonzero constant. The standard application of the continuity principle and the Liouville theorem brings us to the following representation formulas for the solution of the Riemann–Hilbert problems (2.33):

$$\Phi_j^\pm(\eta)=H^\pm(\eta)\Psi_j^\pm(\eta),\quad \eta\in\mathbb{C}^\pm,\quad j=0,1. \tag{2.46}$$

It is directly verified that

$$\frac{f_j(\tau)}{H^+(\tau)}=-\frac{f_j(-\tau)}{H^+(-\tau)}, \tag{2.47}$$

and therefore the condition (2.31) is satisfied, and (2.45) implies

$$\Psi_j^\pm(\eta) = \frac{1}{\pi i}\int_0^\infty \frac{f_j(\tau)}{H^+(\tau)}\frac{\tau d\tau}{\tau^2-\eta^2} = O\left(\frac{1}{\eta^2}\right), \quad \eta \in \mathbb{C}^\pm, \quad \eta \to \infty. \tag{2.48}$$

Clearly, the functions $\Phi_0^\pm(\eta)$ and $\Phi_1^\pm(\eta)$ have a simple zero at the infinite point as it is required. Due to the presence of the functions $f_j(\eta)$ they have four unknown constants c_m $(m = 0, 1, 2, 3)$.

Consider now case (ii) when one of the zeros of the polynomial $q(\eta)$, $z_0 = \eta_0$, is real, and the other two, $z_1 = \eta_1$ and $z_2 = -\eta_2$, lie in the half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Owing to this, we select the Wiener–Hopf factors in (2.42) as

$$H^+(\eta) = \frac{\eta+\eta_2}{(\eta+\eta_0)(\eta+\eta_1)}, \quad H^-(\eta) = -\frac{\eta-\eta_2}{(\eta-\eta_0)(\eta-\eta_1)}, \tag{2.49}$$

Upon representing the function $f_j(\eta)/H^+(\eta)$ as the difference $\Psi_j^+(\eta) - \Psi_j^-(\eta)$ of the boundary values on the real axis of the Cauchy integral (2.45) with the function $H^+(\tau)$ being the one in (2.49) and using the asymptotics of $H^\pm(\eta) = O(\eta^{-1})$, $\eta \to \infty$, we deduce

$$\Phi_j^\pm(\eta) = H^\pm(\eta)[b_j + \Psi_j^\pm(\eta)], \quad \eta \in \mathbb{C}^\pm, \quad j = 0, 1. \tag{2.50}$$

The functions $\Phi_j^\pm(\eta)$ satisfy the symmetry condition (2.31), and b_0 and b_1 are arbitrary constants. Because of the simple poles of the factors $H^\pm(\eta)$ at $\mp\eta_0$ in the real axis, the functions $\Phi_j^\pm(\eta)$ are not continuous at these points. Due to the symmetry, to remove these singularities, it is necessary and sufficient to put

$$b_j = -\Psi_j^+(-\eta_0), \quad j = 0, 1. \tag{2.51}$$

By the Sokhotski–Plemelj formulas and since $f_j(\eta)/H^+(\eta) = 0$ at $\eta = -\eta_0$, the condition (2.51) is equivalent to

$$b_j = -\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f_j(\tau)(\tau+\eta_1)d\tau}{\tau+\eta_2}, \quad j = 0, 1. \tag{2.52}$$

Thus, as in case (i), the solution of the problem has only four arbitrary constants of the functions $f_j(\eta)$, $c_0, \dots, c_3$, and $\Phi_j^\pm(\eta) = O(\eta^{-1})$, $\eta \to \infty$.

In case (iii), two zeros of the polynomial $q(\eta)$ lie in the upper half-plane, $z_0 = \eta_0$ and $z_1 = \eta_1$, and the third one lies in the lower half-plane, $z_2 = -\eta_2$. The index of the function $H(\eta)$ is now equal to 1. We employ the same factorization as in the previous case. However, the factors $H^+(\eta)$ and $H^-(\eta)$ given by (2.49) do not have poles on the real axis. What is common with case (ii) is the asymptotics of the factors at the infinite point, $H^\pm(\eta) = O(\eta^{-1})$, $\eta \to \infty$. Therefore the solution of each Riemann–Hilbert problems (2.30), (2.31) has an arbitrary constant and has the form

$$\Phi_j^\pm(\eta) = H^\pm(\eta)[b_j + \Psi_j^\pm(\eta)], \quad \eta \in \mathbb{C}^\pm, \quad b_j = \text{const}, \quad j = 0, 1. \tag{2.53}$$

On the contrary to the previous case, there is no additional condition (2.52), and the constants b_0 and b_1 remain undetermined.

2.3. Determination of the unknown constants c_j $(j = 1, 2, 3, 4)$

To determine the free constants c_j, for simplicity, we assume that the three functions g_j $(j = 0, 1, 2)$ in the boundary conditions (2.2) vanish, $g_0(x) = g_1(x) = 0$, $0 < x < \infty$, and $g_2(y) = 0$, $0 < y < a$, and select the function $g(x, y)$ in (2.1) as $g(x, y) = -\delta(x - x^\circ)\delta(y - y^\circ)$, $0 < x^\circ < \infty$ and $0 < y^\circ < a$, where $\delta(\cdot)$ is the Dirac delta function. Then

$$\hat{\tilde{g}}(\eta, i\zeta) = -e^{ix^\circ\eta - y^\circ\zeta}, \quad \tilde{g}^{(0)}(\eta) = \frac{c_0}{\eta^2-\alpha_0^2},$$
$$\tilde{g}^{(1)}(\eta) = -\frac{c_1}{\eta^2-\alpha_1^2}, \quad \hat{g}^{(2)}(i\zeta) = \frac{-c_2 + c_3 e^{-a\zeta}}{\zeta^2+\alpha_2^2}, \tag{2.54}$$

and the functions $f_j(\eta)$ in the Riemann–Hilbert problems (2.30) can be represented as

$$f_j(\eta) = -\frac{1}{q(-\eta)\Delta(\eta)}\left[\sum_{m=0}^{3} c_m f_j^m(\eta) + f_j^4(\eta)\right], \quad j = 0, 1, \tag{2.55}$$

where

$$f_j^j(\eta) = 2(-1)^{j+1}\eta[\mu_{1-j}\sinh a\zeta + (\alpha_{1-j}^2 - \eta^2)\zeta\cosh a\zeta](\alpha_2^2 + \zeta^2),$$
$$f_j^{1-j}(\eta) = 2(-1)^j\eta\zeta(\alpha_j^2 - \eta^2)(\alpha_2^2 + \zeta^2),$$
$$f_j^{j+2}(\eta) = 2(-1)^{j+1}\eta(\alpha_j^2 - \eta^2)[\mu_{1-j}\sinh a\zeta + (\alpha_{1-j}^2 - \eta^2)\zeta\cosh a\zeta],$$
$$f_j^{3-j}(\eta) = 2(-1)^j\eta\zeta(\alpha_0^2 - \eta^2)(\alpha_1^2 - \eta^2),$$

$$f_j^4(\eta) = [\eta(\alpha_2^2 + \zeta^2)\cos\eta x^\circ + \mu_2 \sin\eta x^\circ] f_j^*(\eta), \quad j = 0, 1,$$
$$f_0^*(\eta) = -2(\alpha_0^2 - \eta^2)[\mu_1 \sinh(y^\circ - a)\zeta - \zeta(\alpha_1^2 - \eta^2)\cosh(y^\circ - a)\zeta],$$
$$f_1^*(\eta) = 2(\alpha_1^2 - \eta^2)[\mu_0 \sinh y^\circ\zeta + \zeta(\alpha_0^2 - \eta^2)\cosh y^\circ\zeta], \tag{2.56}$$

and

$$\Delta(\eta) = [\alpha_1^2\mu_0 + \alpha_0^2\mu_1 - (\mu_0 + \mu_1)\eta^2]\zeta \cosh a\zeta + [\mu_0\mu_1 + (\alpha_0^2 - \eta^2)(\alpha_1^2 - \eta^2)\zeta^2]\sinh a\zeta. \tag{2.57}$$

The first two conditions for the constants c_j ($j = 0, \ldots, 3$) come from the boundary conditions (2.10) which, in the case of consideration, may be written as

$$(-1)^{j+1}\tilde{u}_y(\eta, y_j) + \frac{\mu_j \tilde{u}(\eta, y_j)}{\alpha_j^2 - \eta^2} = \frac{(-1)^{j+1}c_j}{\alpha_j^2 - \eta^2}, \quad j = 0, 1, \tag{2.58}$$

where, as in (2.5), $y_0 = 0$ and $y_1 = a$. We invert the Laplace transforms in (2.58) and assert that $\tilde{u}(\eta, y_j) = \Phi_j^+(\eta), j = 0, 1$. Due to the first two boundary conditions in (2.8)

$$\lim_{x\to 0^+} \int_{-\infty}^{\infty} \tilde{u}_y(\eta, y_j)e^{-i\eta x}d\eta = 0, \tag{2.59}$$

and one deduces from (2.58)

$$\int_{-\infty}^{\infty} \frac{[\mu_j \Phi_j^+(\eta) + (-1)^j c_j]d\eta}{\alpha_j^2 - \eta^2} = 0, \quad j = 0, 1, \tag{2.60}$$

or, equivalently, by the theory of residues,

$$\mu_j \Phi_j^+(\alpha_j) + (-1)^j c_j = 0, \quad j = 0, 1. \tag{2.61}$$

In view of formulas (2.46), (2.50), (2.53) and the representation (2.55) we can write down the following two equations for the constants:

$$(-1)^j c_j + \mu_j H^+(\alpha_j)\left[\sum_{m=0}^{3} c_m \psi_j^m(\alpha_j) + \psi_j^4(\alpha_j) + b_j\right] = 0, \quad j = 0, 1, \tag{2.62}$$

where $b_j = 0$ in case (i),

$$b_j = -\sum_{m=0}^{3} c_m \psi_j^m(-\eta_0) - \psi_j^4(-\eta_0) \tag{2.63}$$

in case (ii), and b_j are free constants in case (iii). Here,

$$\psi_j^m(\eta) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f_j^m(\tau)d\tau}{q(-\tau)\Delta(\tau)H^+(\tau)(\tau - \eta)}, \quad j = 0, 1, \quad m = 0, \ldots, 4. \tag{2.64}$$

For numerical purposes, formula (2.64) can be written as

$$\psi_j^m(\eta) = \frac{1}{\pi i}\int_0^{\infty} \frac{f_j^m(\tau)\tau d\tau}{\Delta(\tau)(\tau^2 - \eta_0^2)(\tau^2 - \eta_2^2)(\tau^2 - \eta^2)}, \quad j = 0, 1, \quad m = 0, \ldots, 4, \tag{2.65}$$

in case (i) and as

$$\psi_j^m(\eta) = \frac{1}{\pi i}\int_0^{\infty} \frac{f_j^m(\tau)\tau d\tau}{\Delta(\tau)(\tau^2 - \eta_2^2)(\tau^2 - \eta^2)}, \quad j = 0, 1, \quad m = 0, \ldots, 4, \tag{2.66}$$

in cases (ii) and (iii). Analysis of the functions (2.56) and (2.57) shows that the integrands, $F_j^m(\tau)$, in the integrals (2.65) and (2.66) can be estimated as $\tau \to \infty$ by

$$|F_j^m(\tau)| \le A\tau^{-5+l}, \quad m = j, j+2, \quad |F_j^m(\tau)| \le A\tau^{-5+l}e^{-a\eta}, \quad m = 1-j, 3-j,$$

$$|F_j^4(\tau)| \le A\eta^{2+l}e^{-y^\circ\eta}, \quad |F_j^4(\tau)| \le A|\eta|^{2+l}e^{(y^\circ - a)\eta}, \tag{2.67}$$

where $A = \text{const}$, $l = 0$ in case (i) and $l = 2$ in cases (ii) and (iii). Since $\operatorname{Im}\alpha_j > 0$, the integrals (2.65) and (2.66) in (2.62) are not singular and can be evaluated by simple numerical methods.

Alternatively, if $m \ne 4$, then series expansions of the integrals (2.64) can by derived by the Cauchy residue theorem. For this approach, in case (i), we invoke the representation

$$\frac{f_j(\eta)}{H^+(\eta)} = \frac{1}{(\eta^2 - \eta_0^2)(\eta^2 - \eta_2^2)\Delta(\eta)}\left[\sum_{m=0}^{3} c_m f_j^m(\eta) + f_j^4(\eta)\right], \quad j = 0, 1, \tag{2.68}$$

that follows from (2.34), (2.43) and (2.55). Since $f_j^m(\eta)/\Delta(\eta)$ is a meromorphic function of η, for $\eta\in\mathbb{C}^+$ this enables us to write

$$\psi_j^m(\eta)=\frac{1}{2(\eta_0^2-\eta_2^2)}\left(-\frac{f_j^m(-\eta_0)}{\eta_0\Delta(-\eta_0)(\eta_0+\eta)}+\frac{f_j^m(-\eta_2)}{\eta_2\Delta(-\eta_2)(\eta_2+\eta)}\right)+\sum_{s=0}^{\infty}\frac{f_j^m(-\tau_s)}{(\tau_s^2-\eta_0^2)(\tau_s^2-\eta_2^2)\Delta'(-\tau_s)(\tau_s+\eta)},\quad m=0,1,2,3,\tag{2.69}$$

and τ_s ($s=0,1,\ldots,$) are the zeros (they are all simple) of the functions $\Delta(\eta)/\zeta$ in the upper half-plane. The corresponding series representation for $\eta\in\mathbb{C}^+$ in cases (ii) and (iii) has the form

$$\psi_j^m(\eta)=-\frac{f_j^m(-\eta_2)}{2\eta_2\Delta(-\eta_2)(\eta_2+\eta)}+\sum_{s=0}^{\infty}\frac{f_j^m(-\tau_s)}{(\tau_s^2-\eta_2^2)\Delta'(-\tau_s)(\tau_s+\eta)}.\tag{2.70}$$

We turn now to the inverse Laplace transform of the conditions (2.25) as $y\to0^+$ and $y\to a^-$ and derive two more equations for the constants c_j ($j=0,\ldots,3$). The boundary conditions on the horizontal walls have the form

$$\hat{u}_x(0,i\zeta)+\frac{-\mu_2\hat{u}(0,i\zeta)-c_2+c_3e^{-a\zeta}}{\zeta^2+\alpha_2^2}=0.\tag{2.71}$$

Because of (2.8)

$$u_x(0,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\hat{u}_x(0,i\zeta)e^{\zeta y}d\zeta\to0,\quad y\to0^+,\quad y\to a^-.\tag{2.72}$$

This brings us two more equations for the constants c_j

$$-\mu_2\mathcal{J}(y_j)+\frac{c_3e^{i\alpha_2(a-y_j)}-c_2e^{i\alpha_2y_j}}{2i\alpha_2}=0,\quad j=0,1,\tag{2.73}$$

where

$$\mathcal{J}(y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\hat{u}(0,i\zeta)e^{\zeta y}d\zeta}{\zeta^2+\alpha_2^2}.\tag{2.74}$$

According to (2.28) and (2.29) and since

$$\hat{\hat{g}}(\eta,i\zeta)=-e^{ix^\circ\eta-y^\circ\zeta}\tag{2.75}$$

and

$$\frac{1}{2i\eta}\{(\tilde{\mu}_0+\zeta)[h_0(\eta)-h_0(-\eta)]+(\tilde{\mu}_1-\zeta)[h_1(\eta)-h_1(-\eta)]\}=-\frac{\sin x^\circ\eta}{\eta}e^{-y^\circ\zeta},\tag{2.76}$$

the function $\hat{u}(0,i\zeta)$ is given by

$$\hat{u}(0,i\zeta)=-\frac{\sin x^\circ\eta}{\eta}e^{-y^\circ\zeta}-\frac{\Phi_0^+(\eta)-\Phi_0^-(\eta)}{2i\eta(\eta^2-\alpha_0^2)}[\mu_0-(\eta^2-\alpha_0^2)\zeta]-\frac{\Phi_1^+(\eta)-\Phi_1^-(\eta)}{2i\eta(\eta^2-\alpha_1^2)}e^{-a\zeta}[\mu_1+(\eta^2-\alpha_1^2)\zeta].\tag{2.77}$$

The main difficulty in computing the integral in (2.74) is the presence of the two-valued function $\eta^2=\zeta^2+k^2$ in $\hat{u}(0,i\zeta)$. We make the substitution $\zeta=-i\xi$ and fix a branch of the function $\eta^2=k^2-\xi^2$ in the ξ-plane cut along the line joining the branch points $\xi=\pm k$ and passing through the infinite point $\xi=\infty$. Notice that due to the symmetry condition (2.31) the functions

$$\frac{\Phi_j^+(\eta)-\Phi_j^-(\eta)}{\eta}=\frac{\Phi_j^+(\sqrt{k^2-\xi^2})-\Phi_j^-(\sqrt{k^2-\xi^2})}{\sqrt{k^2-\xi^2}},\quad j=0,1,\tag{2.78}$$

are meromorphic in the ξ-plane and independent of the branch choice. On using the boundary condition of the Riemann–Hilbert problem

$$\Phi_j^-(\eta)=\frac{\Phi_j^+(\eta)-f_j(\eta)}{H(\eta)},\quad j=0,1,\tag{2.79}$$

one may continue analytically the functions $\Phi_j^-(\eta)$ into the upper η-half-plane $\mathbb{C}_\eta^+$ cut along the semi-infinite line $\{|\eta|>k,\arg\eta=\alpha\}$, $\alpha=\arg k\in(0,\pi/2)$. In a similar manner we continue the functions $\Phi_j^+(\eta)$ from the half-plane

$\mathbb{C}_\eta^+$ into the lower η-half-plane $\mathbb{C}_\eta^-$ cut along the ray $\{|\eta|>k, \arg\eta=\alpha+\pi\}$. Because of the meromorphicity of the functions (2.78), the function $(\zeta^2+\alpha_2^2)^{-1}\hat{u}(0,i\zeta)$ is meromorphic everywhere in the ξ-plane, and it can be equivalently represented as

$$\frac{\hat{u}(0,i\zeta)}{\zeta^2+\alpha_2^2}=\frac{i}{\eta(\alpha_2^2+\zeta^2)+i\mu_2}\left\{\frac{\mu_0-(\eta^2-\alpha_0^2)\zeta}{\eta^2-\alpha_0^2}\Phi_0^+(\eta)+\frac{\mu_1+(\eta^2-\alpha_1^2)\zeta}{\eta^2-\alpha_1^2}e^{-a\zeta}\Phi_1^+(\eta)\right.$$
$$\left.+e^{i\eta x^\circ-y^\circ\zeta}-\left(\frac{c_0}{\alpha_0^2-\eta^2}+\frac{c_2}{\zeta^2+\alpha_2^2}\right)+\left(\frac{c_1}{\alpha_1^2-\eta^2}+\frac{c_3}{\zeta^2+\alpha_2^2}\right)e^{-a\zeta}\right\},\quad \zeta=-i\xi, \tag{2.80}$$

for all ξ such that $\eta\in\mathbb{C}_\eta^+$. Here we employed the identities

$$1-\frac{1}{H(\eta)}=\frac{2\eta(\zeta^2+\alpha_2^2)}{\eta(\zeta^2+\alpha_2^2)+i\mu_2} \tag{2.81}$$

and

$$-\frac{\sin x^\circ\eta}{\eta}e^{-y^\circ\zeta}+\frac{1}{2i\eta q(-\eta)\Delta(\eta)H(\eta)}\left[\frac{\mu_0+(\alpha_0^2-\eta^2)\zeta}{\eta^2-\alpha_0^2}f_0^4(\eta)\right.$$
$$\left.+\frac{\mu_0-(\alpha_1^2-\eta^2)\zeta}{\eta^2-\alpha_1^2}e^{-a\zeta}f_1^4(\eta)\right]=\frac{i(\alpha_2^2+\zeta^2)}{(\alpha_2^2+\zeta^2)\eta+i\mu_2}e^{ix^\circ\eta-y^\circ\zeta}. \tag{2.82}$$

Similarly, by analytic continuation into the lower half-plane we derive

$$\frac{\hat{u}(0,i\zeta)}{\zeta^2+\alpha_2^2}=\frac{i}{\eta(\alpha_2^2+\zeta^2)-i\mu_2}\left\{-\frac{\mu_0-(\eta^2-\alpha_0^2)\zeta}{\eta^2-\alpha_0^2}\Phi_0^-(\eta)-\frac{\mu_1+(\eta^2-\alpha_1^2)\zeta}{\eta^2-\alpha_1^2}e^{-a\zeta}\Phi_1^-(\eta)\right.$$
$$\left.-e^{-i\eta x^\circ-y^\circ\zeta}+\left(\frac{c_0}{\alpha_0^2-\eta^2}+\frac{c_2}{\zeta^2+\alpha_2^2}\right)-\left(\frac{c_1}{\alpha_1^2-\eta^2}+\frac{c_3}{\zeta^2+\alpha_2^2}\right)e^{-a\zeta}\right\},\quad \zeta=-i\xi, \tag{2.83}$$

when η lies in the lower η-half-plane $\mathbb{C}_\eta^-$. This formula can also be obtained upon replacing η by $-\eta$ in (2.80) and using the symmetry relation (2.31). On account of (2.80) the integral (2.74) is

$$\mathcal{J}(y)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[e^{i\xi y^\circ+i\eta x^\circ}-\left(\frac{c_0}{\alpha_0^2-k^2+\xi^2}+\frac{c_2}{\alpha_2^2-\xi^2}\right)\right.$$
$$+\left(\frac{c_1}{\alpha_1^2-k^2+\xi^2}+\frac{c_3}{\alpha_2^2-\xi^2}\right)e^{ia\xi}-\frac{\mu_0-i\xi(\alpha_0^2-k^2+\xi^2)}{\alpha_0^2-k^2+\xi^2}\Phi_0^+(\eta)$$
$$\left.-\frac{\mu_1+i\xi(\alpha_1^2-k^2+\xi^2)}{\alpha_1^2-k^2+\xi^2}e^{ia\xi}\Phi_1^+(\eta)\right]\frac{e^{-i\xi y}d\xi}{(\alpha_2^2-\xi^2)\eta+i\mu_2}. \tag{2.84}$$

where $\operatorname{Im}\eta\ge 0$, and η is defined by the equation $\eta^2=k^2-\xi^2$. In the half-plane $\operatorname{Im}\eta\ge 0$ the cubic equation $q(\eta)=0$ may have either one root η_1 (case i), or one complex root η_1 and one real root η_0, or two complex roots η_0 and η_1. Therefore, for $\operatorname{Im}\eta\ge 0$ the function $h(\xi)=(\alpha_2^2-\xi^2)\eta+i\mu_2$ has either two zeros $\pm\xi_1=\sqrt{k^2-\eta_1^2}\in\mathbb{C}_\xi^\pm$ (case i), or four zeros $\pm\xi_j=\sqrt{k^2-\eta_j^2}\in\mathbb{C}_\xi^\pm$, $j=0,1$ (cases ii and iii). Notice that for certain combinations of the problem parameters these zeros may fall in the real axis of the ξ-plane. In this singular case the formulas to be derived need to be adjusted accordingly. Assume that $\operatorname{Im}\xi_j\ne 0$ and denote by ξ_j the root of $\xi_j^2=k^2-\eta_j^2$ lying in the upper ξ-plane, $\xi_j>0$. By using the theory of residues we compute the integral and represent it in the form

$$\mathcal{J}(y)=\sum_{m=s}^{1}t_m\left[e^{i\xi_m|y-y^\circ|+i\eta_m x^\circ}-e^{i\xi_m y}\frac{c_0+[\mu_0+i\xi_m(\alpha_0^2-\eta_m^2)]\Phi_0^+(\eta_m)}{\alpha_0^2-\eta_m^2}\right.$$
$$\left.+e^{i\xi_m(a-y)}\frac{c_1-[\mu_1+i\xi_m(\alpha_1^2-\eta_m^2)]\Phi_1^+(\eta_m)}{\alpha_1^2-\eta_m^2}+\frac{c_2e^{i\xi_m y}-c_3e^{i(a-y)\xi_m}}{\xi_m^2-\alpha_2^2}\right]$$
$$-\frac{c_2e^{i\alpha_2 y}-c_3e^{i(a-y)\alpha_2}}{2i\mu_2\alpha_2}. \tag{2.85}$$

Here, $s=1$ in case (i) and $s=0$ in cases (ii) and (iii). We denoted

$$t_m=\frac{\eta_m}{\xi_m(\alpha_2^2-\xi_m^2+2\eta_m^2)} \tag{2.86}$$

and used the fact that if $\xi=\pm\sqrt{k^2-\alpha_j^2}$, then $\eta=\alpha_j\in\mathbb{C}_\eta^+$, $j=0,1$, and

$$c_0+\mu_0\Phi_0^+(\alpha_0)=0,\quad c_1-\mu_1\Phi_1^+(\alpha_1)=0. \tag{2.87}$$

Owing to formula (2.85) we transform the relations (2.73) at $y = y_0 = 0$ and $y = y_1 = a$ to read

$$\sum_{m=s}^{1} t_m \left[e^{i\xi_m|y_j - y^\circ| + i\eta_m x^\circ} - e^{i\xi_m y_j} \frac{c_0 + [\mu_0 + i\xi_m(\alpha_0^2 - \eta_m^2)]\Phi_0^+(\eta_m)}{\alpha_0^2 - \eta_m^2} \right. \\ \left. + e^{i\xi_m(a-y_j)} \frac{c_1 - [\mu_1 + i\xi_m(\alpha_1^2 - \eta_m^2)]\Phi_1^+(\eta_m)}{\alpha_1^2 - \eta_m^2} + \frac{c_2 e^{i\xi_m y_j} - c_3 e^{i\xi_m(a-y_j)}}{\xi_m^2 - \alpha_2^2} \right] = 0, \quad j = 0, 1. \tag{2.88}$$

Now, upon plugging (2.46), (2.50) and (2.53) into the relations (2.88) and since

$$\Phi_j^+(\eta_m) = H^+(\eta_m) \left[\sum_{n=0}^{3} c_n \psi_j^n(\eta_m) + \psi_j^4(\eta_m) + b_j \right], \quad j = 0, 1, \tag{2.89}$$

we deduce the following two equations for the constants c_j $(j = 0, \ldots, 3)$:

$$D_{j0}c_0 + D_{j1}c_1 + D_{j2}c_2 + D_{j3}c_3 + D_{j4}b_0 + D_{j5}b_1 = E_j, \quad j = 0, 1, \tag{2.90}$$

where

$$D_{jn} = \sum_{m=s}^{1} (d_{jnm}^0 + d_{jnm}^1),$$
$$d_{j0m}^0 = \frac{t_m e^{i\xi_m y_j}}{\alpha_0^2 - \eta_m^2}, \quad d_{j1m}^0 = -\frac{t_m e^{i\xi_m(a-y_j)}}{\alpha_1^2 - \eta_m^2},$$
$$d_{j2m}^0 = \frac{t_m e^{i\xi_m y_j}}{\alpha_2^2 - \xi_m^2}, \quad d_{j3m}^0 = -\frac{t_m e^{i\xi_m(a-y_j)}}{\alpha_2^2 - \xi_m^2},$$
$$d_{jnm}^1 = t_m H^+(\eta_m)[r_{m0}\psi_0^n(\eta_m)e^{i\xi_m y_j} + r_{m1}\psi_1^n(\eta_m)e^{i\xi_m(a-y_j)}], \quad n = 0, 1, 2, 3,$$
$$D_{j4} = \sum_{m=0}^{1} t_m H^+(\eta_m) r_{m0} e^{i\xi_m y_j}, \quad D_{j5} = \sum_{m=0}^{1} t_m H^+(\eta_m) r_{m1} e^{i\xi_m(a-y_j)},$$
$$E_j = \sum_{m=s}^{1} t_m \left\{ e^{i\xi_m|y_j - y^\circ| + i\eta_m x^\circ} - H^+(\eta_m) \right. \\ \left. \times \left[r_{m0}\psi_0^4(\eta_m)e^{i\xi_m y_j} + r_{m1}\psi_1^4(\eta_m)e^{i\xi_m(a-y_j)} \right] \right\},$$
$$r_{mj} = \frac{\mu_j + i\xi_m(\alpha_j^2 - \eta_m^2)}{\alpha_j^2 - \eta_m^2}, \quad j = 0, 1, \quad m = s, 1. \tag{2.91}$$

and $\psi_j^m(\eta_1)$ are determined by the quadrature (2.64) and by the series (2.69), (2.70). Eqs. (2.62) and (2.90) comprise a system of four equations with respect to the four constants c_j $(j = 0, \ldots, 3)$. In case (iii), the constants b_j are still not determined, while in the other two cases, they are fixed: $b_j = 0$ in case (i) and b_j are given by (2.63) in case (ii).

2.4. Analysis of the solution

To write down the function $u(x, y)$ for any internal point in the semi-strip, one needs to know the function and its normal derivative in the vertical wall $W_2 = \{x = 0, 0 < y < a\}$ and in both horizontal sides $W_j = \{0 < x < \infty, y = y_j\}$, $j = 0, 1$. Then the solution can be constructed in the standard manner by the Green formula for the Helmholtz operator. We start with the function $u(0, y)$ for $0 < y < a$. By inverting the Laplace transform we have

$$u(0, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \hat{u}(0, i\zeta) e^{\zeta y} d\zeta. \tag{2.92}$$

On employing the theory of residues, similarly to the previous section, one deduces

$$u(0, y) = -\sum_{m=s}^{1} t_m(\alpha_2^2 - \xi_m^2) \left[\left(r_{m0}\Phi_0^+(\eta_m) + \frac{c_0}{\alpha_0^2 - \eta_m^2} + \frac{c_2}{\alpha_2^2 - \xi_m^2} \right) e^{i\xi_m y} \right. \\ \left. + \left(r_{m1}\Phi_1^+(\eta_m) - \frac{c_1}{\alpha_1^2 - \eta_m^2} - \frac{c_3}{\alpha_2^2 - \xi_m^2} \right) e^{i\xi_m(a-y)} - e^{i\xi_m|y-y^\circ| + i\eta_m x^\circ} \right], \quad 0 < y < a. \tag{2.93}$$

Here, $s = 1$ in case (i) and $s = 0$ in cases (ii) and (iii), the functions $\Phi_j^+(\eta)$ are given by (2.89), and the functions $\psi_j^n(\eta)$ are determined by (2.65) in case (i) and (2.66) in cases (ii) and (iii).

Next we wish to derive expressions for the function u on the two horizontal boundaries of the half-strip,

$$u(x, y_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_j^+(\eta)e^{-i\eta x}d\eta, \quad 0 < x < \infty. \tag{2.94}$$

On continuing analytically the functions $\Phi_j^+(\eta)$ into the lower half-plane by making use of the Riemann–Hilbert boundary conditions (2.30) we rewrite the representation (2.94) as

$$u(x, y_j) = \mathcal{I}_{1j}(x) + \mathcal{I}_{2j}(x), \tag{2.95}$$

where

$$\mathcal{I}_{1j}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^+(\eta)[\Psi_j^-(\eta) + b_j]e^{-i\eta x}d\eta, \quad \mathcal{I}_{2j}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_j(\eta)e^{-i\eta x}d\eta. \tag{2.96}$$

The integral $\mathcal{I}_1(x)$ is evaluated by the theory of residues. In case (i), $b_j = 0$, $-\eta_1$ is the only one pole of the function $H^+(\eta)$ in the lower half-plane, and

$$\mathcal{I}_{1j}(x) = -i(\eta_0 - \eta_1)(\eta_2 - \eta_1)e^{i\eta_1 x}\Psi_j^-(-\eta_1). \tag{2.97}$$

In cases (ii) and (iii),

$$\mathcal{I}_{1j}(x) = -\frac{i(\eta_2 - \eta_1)}{\eta_0 - \eta_1}e^{i\eta_1 x}[\Psi_j^-(-\eta_1) + b_j] + \epsilon\frac{i(\eta_2 - \eta_0)}{\eta_0 - \eta_1}e^{i\eta_0 x}[\Psi_j^-(-\eta_0) + b_j], \tag{2.98}$$

where $\epsilon = \frac{1}{2}$ in case (ii) and $\epsilon = 1$ in case (iii). Notice that $\Psi_j^-(-\eta_m) = \Psi_j^+(\eta_m)$ and

$$\Psi^-(-\eta_m) = \sum_{n=0}^{3} c_n\psi_j^n(\eta_m) + \psi_j^4(\eta_m), \quad j = 0, 1, \quad m = s, 1. \tag{2.99}$$

The second integral $\mathcal{I}_2(x)$ can be evaluated either directly numerically or on applying the theory of residues and representing it as a series. The second approach requires however to determine zeros of the function $\Delta(\eta)$. To proceed with the former method, it is convenient to rewrite the integral $\mathcal{I}_2(x)$ as

$$\mathcal{I}_{2j}(x) = \frac{1}{2\pi}\int_0^{\infty}\left[\sum_{m=0}^{3} c_m f_j^m(\eta) + f_j^4(\eta)\right]\left[\frac{e^{i\eta x}}{q(\eta)} - \frac{e^{-i\eta x}}{q(-\eta)}\right]\frac{d\eta}{\Delta(\eta)}. \tag{2.100}$$

In general, the functions $u(0, y)$ and $u(x, y_j)$ derived do not satisfy the compatibility conditions

$$\lim_{x\to 0^+} u(x, 0) = \lim_{y\to 0^+} u(0, y), \quad \lim_{x\to 0^+} u(x, a) = \lim_{y\to a^-} u(0, y), \tag{2.101}$$

which guarantee the continuity of the function $u(x, y)$ and therefore, due to (2.4), the continuity of the pressure distribution $p(x, y)$ at the corners of the semi-strip. In cases (i) and (ii), after the constants c_j ($j = 0, \ldots, 3$) have been fixed by solving the system of four equations (2.62) and (2.90), there is no way to satisfy the compatibility conditions (2.101), and in general, both functions, $u(x, y)$ and $p(x, y, t)$, are discontinuous at the corners. The situation is different in case (iii). We still have two free constants b_0 and b_1.

Furnished with the expressions (2.93), (2.95), (2.98), (2.99), and (2.100) of the function $u(x, y)$ on the boundary we are able to satisfy the conditions (2.101) and fix the remaining constants b_0 and b_1. The two new equations have the form

$$\beta_{j0}b_0 + \beta_{j1}b_1 + \beta b_j + \sum_{n=0}^{3}(\sigma_{jn} + \lambda_{jn})c_n = \nu_j, \quad j = 0, 1. \tag{2.102}$$

Here,

$$\beta_{j0} = \sum_{m=0}^{1} t_m(\alpha_2^2 - \xi_m^2)r_{m0}H^+(\eta_m)e^{i\xi_m y_j}, \quad \beta_{j1} = \sum_{m=0}^{1} t_m(\alpha_2^2 - \xi_m^2)r_{m1}H^+(\eta_m)e^{i\xi_m(a-y_j)},$$

$$\beta = -\frac{i}{\eta_1 - \eta_0}\sum_{m=0}^{1}(-1)^m(\eta_2 - \eta_m), \quad \lambda_{j0} = \sum_{m=0}^{1}\frac{t_m(\alpha_2^2 - \xi_m^2)e^{i\xi_m y_j}}{\alpha_0^2 - \eta_m^2},$$

$$\lambda_{j1} = -\sum_{m=0}^{1}\frac{t_m(\alpha_2^2 - \xi_m^2)e^{i\xi_m(a-y_j)}}{\alpha_1^2 - \eta_m^2}, \quad \lambda_{j2} = \sum_{m=0}^{1} t_m e^{i\xi_m y_j}, \quad \lambda_{j3} = -\sum_{m=0}^{1} t_m e^{i\xi_m(a-y_j)},$$

$$\sigma_{jn} = -\frac{i}{\eta_1 - \eta_0}\sum_{m=0}^{1}(-1)^m(\eta_2 - \eta_m)\psi_j^n(-\eta_m) + M_j^n(0)$$

$$+\sum_{m=0}^{1} t_m(\alpha_2^2-\xi_m^2)H^+(\eta_m)\left[r_{m0}\psi_0^n(\eta_m)e^{i\xi_m y_j}+r_{m1}\psi_1^n(\eta_m)e^{i\xi_m(a-y_j)}\right],$$

$$\nu_j = -M_j^4(0)+\frac{i}{\eta_1-\eta_0}\sum_{m=0}^{1}(-1)^m(\eta_2-\eta_m)\psi_j^4(-\eta_m)+\sum_{m=0}^{1} t_m(\alpha_2^2-\xi_m^2)$$
$$\times\left\{e^{i\xi_m|y_j-y^\circ|+i\eta_m x^\circ}-H^+(\eta_m)\left[r_{m0}\psi_0^4(\eta_m)e^{i\xi_m y_j}+r_{m1}\psi_1^4(\eta_m)e^{i\xi_m(a-y_j)}\right]\right\}, \quad (2.103)$$

and

$$M_j^n(x)=\frac{1}{2\pi}\int_0^\infty\left[\frac{e^{i\eta x}}{q(\eta)}-\frac{e^{-i\eta x}}{q(-\eta)}\right]\frac{f_j^n(\eta)d\eta}{\Delta(\eta)},\quad j=0,1,\quad n=0,\dots,4. \quad (2.104)$$

These equations combined with (2.62) and (2.90) form a system of six equations for the six constants $c_0,\dots,c_3$, b_0, and b_1. Therefore we may conclude (in general the matrix of the system is not singular) that in case (iii) the solution of the boundary value problem (2.1), (2.2), (2.8) exists, it is unique and satisfies the compatibility conditions (2.101).

The results obtained are collected in the theorem below.

Theorem. *Let $g(x,y)\in L_1(\mathbb{R}^2)$, $g_j(x)\in L_1(0,\infty)$, $j=0,1$, $g_2(y)\in L_1(0,a)$ and let these functions satisfy the Dirichlet conditions that is be piecewise monotonic and have a finite number of discontinuities. Suppose $k=\omega/c$, $\alpha_j=\omega/\varepsilon_j$, $\mu_j=\omega^2/\delta_j$, where c, ε_j and δ_j $(j=0,1,2)$ are positive constants, and $\omega=\omega_1+i\omega_2$, $\omega_1>0$, $\omega_2>0$. Denote $y_0=0$ and $y_1=a$.*

Consider the boundary value problem

$$(\Delta+k^2)u(x,y)=g(x,y),\quad 0<x<\infty,\quad 0<y<a,$$
$$u_{xxy}+\alpha_j^2u_y-(-1)^j\mu_j u=g_j(x),\quad 0<x<\infty,\quad y=y_j,\quad j=0,1,$$
$$u_{xyy}+\alpha_2^2u_x-\mu_2 u=g_2(y),\quad x=0,\quad 0<y<a, \quad (2.105)$$

whose solution satisfies the four conditions

$$u_y(0^+,0)=u_y(0^+,a)=0,\qquad u_x(0,0^+)=u_x(0,a^-)=0. \quad (2.106)$$

Let the three zeros of the polynomial $q(\eta)=\eta(\eta^2-k^2+\alpha_2^2)+i\mu_2$ be z_0, z_1 and z_2. Then two zeros say, z_1 and z_2, lie in the opposite half-planes, $\operatorname{Im} z_1>0$ and $\operatorname{Im} z_2<0$, while the third zero z_0 either (i) falls in the lower half-plane, or (ii) is in the real axis, or (iii) $\operatorname{Im} z_0>0$.

In all cases (i) to (iii) the solution of the problem (2.105) exists, and the Dirichlet data on the two horizontal sides of the semi-strip, $u(x,0)$ and $u(x,a)$, are expressed by (2.94) through the solution of the two symmetric scalar Riemann–Hilbert problems (2.30), (2.31), $\Phi_0^+(\eta)$ and $\Phi_1^+(\eta)$. In the first two cases these solutions given by (2.46) and *(2.50), (2.52), respectively, have four arbitrary constants c_j $(j=0,\dots,3)$. In case (iii), the functions $\Phi_0^+(\eta)$ and $\Phi_1^+(\eta)$ have the form (2.53) and possess six arbitrary constants c_j $(j=0,\dots,3)$, b_0, and b_1.*

In particular, if $g_0(x)=g_1(x)=0$ $(0<x<\infty)$, $g_2(y)=0$ $(0<y<a)$, and $g(x,y)=-\delta(x-x^\circ)\delta(y-y^\circ)$, $x^\circ\in(0,\infty)$, $y^\circ\in(0,a)$, then the edge conditions (2.106) are equivalent to the system of four linear algebraic equations (2.62), (2.90), where $b_j=0$ in case (i), b_j are given by (2.52) in case (ii), and remain free in case (iii). In general, in cases (i) and (ii), the function $u(x,y)$ is discontinuous at the edges $x=y=0$ and $x=0, y=a$. In case (iii), however, on fixing the constants b_1 and b_2 by solving the two equations (2.102), it is possible to satisfy the compatibility conditions (2.101) and find the unique solution of the problem (2.105), (2.106) continuous up to the boundary including the corners of the semi-strip.

2.5. Case (iii): the homogeneous problem

In this section we consider case (iii) when the solution to the Riemann–Hilbert problems (2.30), (2.31) have the free constants b_j. We split the general solution as

$$\Phi_j(\eta)=b_j\Phi_0(\eta)+\tilde\Phi_j(\eta), \quad (2.107)$$

where

$$\Phi_0^\pm(\eta)=H^\pm(\eta),\quad \tilde\Phi_j^\pm(\eta)=H^\pm(\eta)\Psi^\pm(\eta),\quad j=1,2, \quad (2.108)$$

and focus on the solution u associated with the function $\Phi_0(\eta)$. It can be interpreted as the nontrivial solution to the homogeneous boundary value problem (2.1), (2.2) when the conditions at the edges $x=y=0$ and $x=0$, $y=a$ are ignored. We wish to determine the values of this nontrivial solution in the walls of the semi-strip. By inverting the Fourier transform we find the function u in the semi-infinite walls

$$u(x,y_j)=\frac{b_j}{2\pi}\int_{-\infty}^{\infty}H^+(\eta)e^{-i\eta x}d\eta$$

$$= -\frac{ib_j}{\eta_1 - \eta_0}[(\eta_2 - \eta_0)e^{i\eta_0 x} - (\eta_2 - \eta_1)e^{i\eta_1 x}], \quad 0 < x < \infty, \quad y_0 = 0, \quad y_1 = a. \tag{2.109}$$

Since $\operatorname{Im}\eta_j > 0$, the solution to the homogeneous problem decays exponentially in the walls $y = 0$ and $y = a$ as $x \to \infty$.

Next, on employing the relations (2.28), (2.29), (2.108), and (2.49) we write down the expressions for the transforms $\hat{u}(0, i\zeta)$ and $\hat{u}_x(0, i\zeta)$

$$\hat{u}(0, i\zeta) = -\frac{i(\eta^2 + \eta_0\eta_1 - \eta_0\eta_2 - \eta_1\eta_2)}{(\eta^2 - \eta_0^2)(\eta^2 - \eta_1^2)}[(\tilde{\mu}_0 + \zeta)b_0 + (\tilde{\mu}_1 - \zeta)e^{-a\zeta}b_1],$$

$$\hat{u}_x(0, i\zeta) = \frac{\eta^2(\eta_0 + \eta_1 - \eta_2) - \eta_0\eta_1\eta_2}{(\eta^2 - \eta_0^2)(\eta^2 - \eta_1^2)}[(\tilde{\mu}_0 + \zeta)b_0 + (\tilde{\mu}_1 - \zeta)e^{-a\zeta}b_1], \tag{2.110}$$

where $\eta^2 = \zeta^2 + k^2$. It is seen that both functions, $\hat{u}(0, i\zeta)$ and $\hat{u}_x(0, i\zeta)$, are meromorphic functions of ζ, and the inverse Laplace transforms of them can be easily evaluated by the theory of residues. For the function $u(0, y)$ we have

$$u(0, y) = b_0\mathcal{J}_0(y) + b_1\mathcal{J}_1(y), \quad 0 < y < a, \tag{2.111}$$

where

$$\mathcal{J}_0(y) = \rho_0\left(\frac{\mu_0}{\alpha_0^2 - \zeta_0^2 - k^2} - \zeta_0\right)e^{-\zeta_0 y} + \rho_1\left(\frac{\mu_0}{\alpha_0^2 - \zeta_1^2 - k^2} - \zeta_1\right)e^{-\zeta_1 y} - \hat{\rho}_0 e^{-\hat{\zeta}_0 y},$$
$$\mathcal{J}_1(y) = \rho_0\left(\frac{\mu_1}{\alpha_1^2 - \zeta_0^2 - k^2} - \zeta_0\right)e^{\zeta_0(y-a)} + \rho_1\left(\frac{\mu_1}{\alpha_1^2 - \zeta_1^2 - k^2} - \zeta_1\right)e^{\zeta_1(y-a)} - \hat{\rho}_1 e^{\hat{\zeta}_1(y-a)}, \tag{2.112}$$

$\zeta_j = (\eta_j^2 - k^2)^{1/2}$, $\hat{\zeta}_j = (\alpha_j^2 - k^2)^{1/2}$, and the branches are chosen such that $\operatorname{Re}\zeta_j > 0$ and $\operatorname{Re}\hat{\zeta}_j > 0$, $j = 0, 1$,

$$\rho_j = \frac{i(\zeta_j^2 + k^2 + \eta_0\eta_1 - \eta_0\eta_2 - \eta_1\eta_2)}{2\zeta_j(\zeta_j^2 + k^2 - \eta_{1-j}^2)},$$
$$\hat{\rho}_j = \frac{i\mu_j(\hat{\zeta}_j^2 + k^2 + \eta_0\eta_1 - \eta_0\eta_2 - \eta_1\eta_2)}{2\hat{\zeta}_j(\hat{\zeta}_j^2 + k^2 - \eta_0^2)(\hat{\zeta}_j^2 + k^2 - \eta_1^2)}, \quad j = 0, 1. \tag{2.113}$$

2.6. Numerical results

To test the numerical efficiency of the method proposed, we evaluate the complex potential $u(x, y)$ in the vertical and horizontal walls. For the test we take the following values of the problem parameters: $a = 1$, $k = 1 + 0.1i$, $x^\circ = a$, and $y^\circ = 0.5a$. In Fig. 2, we plot the real and imaginary parts of the complex potential $u(x, y)$ in the vertical wall $x = 0$ when $\gamma_0 = 5$, $\gamma_1 = 1$. Recall that the parameters α_2 μ_2 are determined through γ_0 and γ_1 by $\alpha_2 = \sqrt{\gamma_0}k$ and $\mu_2 = \gamma_1 k^2$. Our numerical tests show that although the solutions to the Riemann–Hilbert problems and the constants c_j ($j = 0, 1, 2, 3$) depend on the other parameters α_0, α_1, μ_0, and μ_1, the expressions

$$r_{m0}\Phi_0^+(\eta_m) + \frac{c_0}{\alpha_0^2 - \eta_m^2} + \frac{c_2}{\alpha_2^2 - \xi_m^2} \tag{2.114}$$

and

$$r_{m1}\Phi_1^+(\eta_m) - \frac{c_1}{\alpha_1^2 - \eta_m^2} - \frac{c_3}{\alpha_2^2 - \xi_m^2} \tag{2.115}$$

are invariant with respect to these parameters. This results in the fact that the solution u in the vertical wall $x = 0$, $0 < y < a$ is independent of the parameters α_0, α_1, μ_0, and μ_1.

The real and imaginary parts of the function $u(x, y)$ in the semi-infinite horizontal walls $y = 0$ and $y = a$ when $0 < x < 30a$ are shown in Fig. 3. These values depend not only on α_2 and μ_2 but also on α_0, α_1, μ_0, and μ_1. In Fig. 3, we select $\alpha_0 = 100\alpha_2$, $\alpha_1 = \alpha_2$, $\mu_0 = 100\mu_2$, $\mu_1 = \mu_2$. It is seen from Figs. 2 and 3 that the function $u(x, y)$ is discontinuous at the edges $x = 0$, $y = 0$ and $x = 0$, $y = a$.

In Figs. 4 and 5, we plot sample curves in the case (iii) when the solution possesses the constants b_0 and b_1 and is continuous at the edges $x = y = 0$ and $x = 0$, $y = a$. The parameters a, k, x°, and y° used for the numerical test are the same as in Figs. 2 and 3. The other parameters are $\gamma_0 = 0.5$, $\gamma_1 = 0.05$, $\alpha_0 = 0.5\alpha_2$, $\alpha_1 = 2\alpha_2$, $\alpha_2 = \sqrt{\gamma_0}k$, $\mu_0 = 0.5\mu_2$, $\mu_1 = 2\mu_2$, and $\mu_2 = \gamma_1 k^2$. In this case two zeros of the polynomial $q(\eta)$ are in the upper half-plane, and the third one is in the lower half-plane (see Table 1), that is we have case (iii). The function u on both vertical and horizontal walls depends on all the parameters of the problem. The functions $u(x, 0)$ and $u(x, a)$, $0 < x < 30a$, are plotted in Fig. 6 for the same parameters as in Fig. 5 except for μ_0, $\mu_0 = 5\mu_2$.

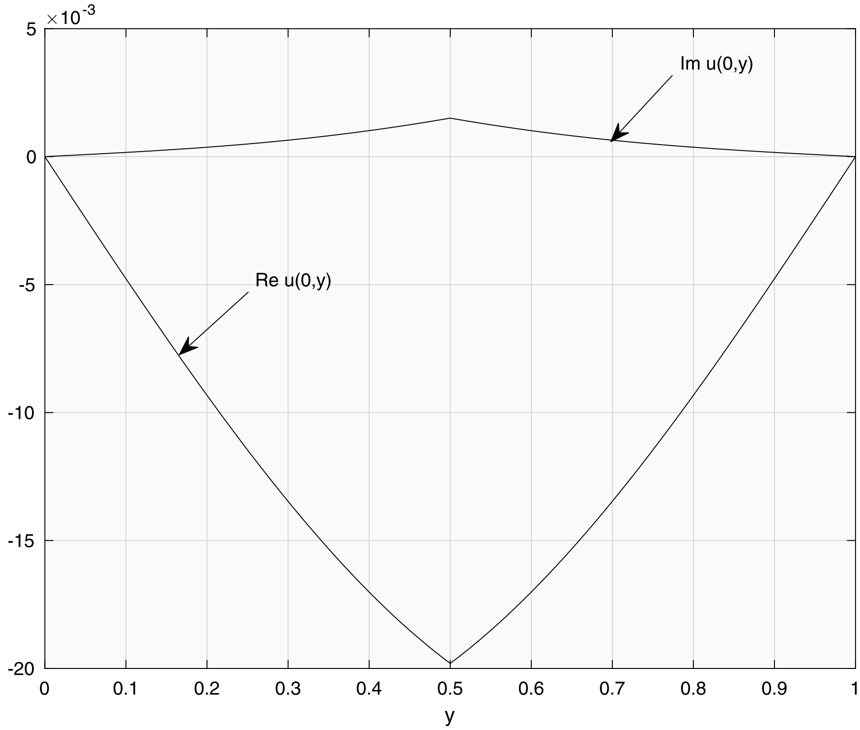

Fig. 2. The real and imaginary parts of the complex potential $u(0, y)$, $0 \le y \le a$, $a = 1$, in the vertical wall in case (i).

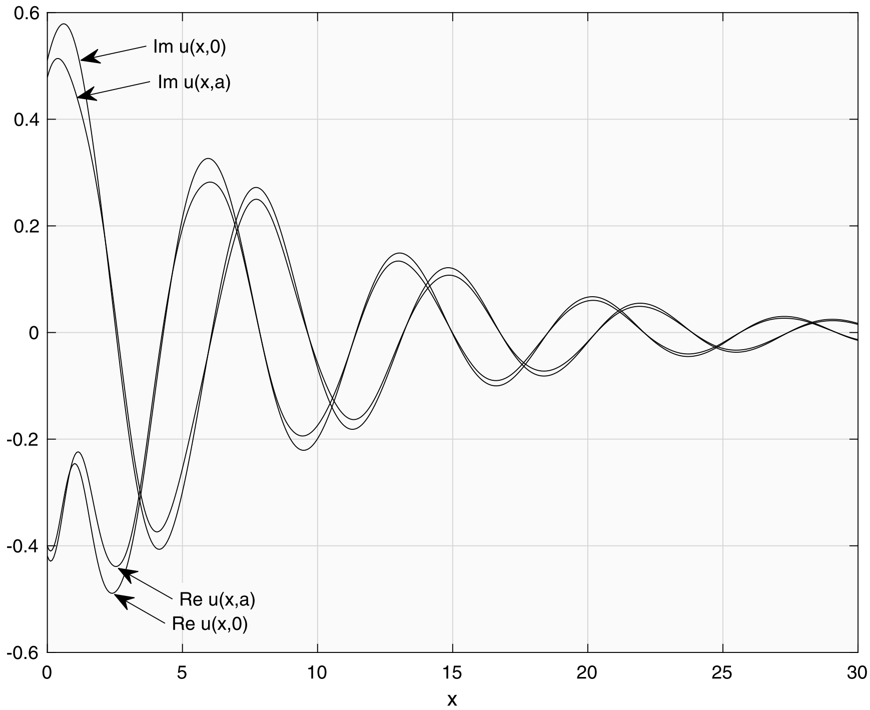

Fig. 3. The real and imaginary parts of the complex potential u in the horizontal sides in case (i) when $0 \le x \le 30a$, $a = 1$.

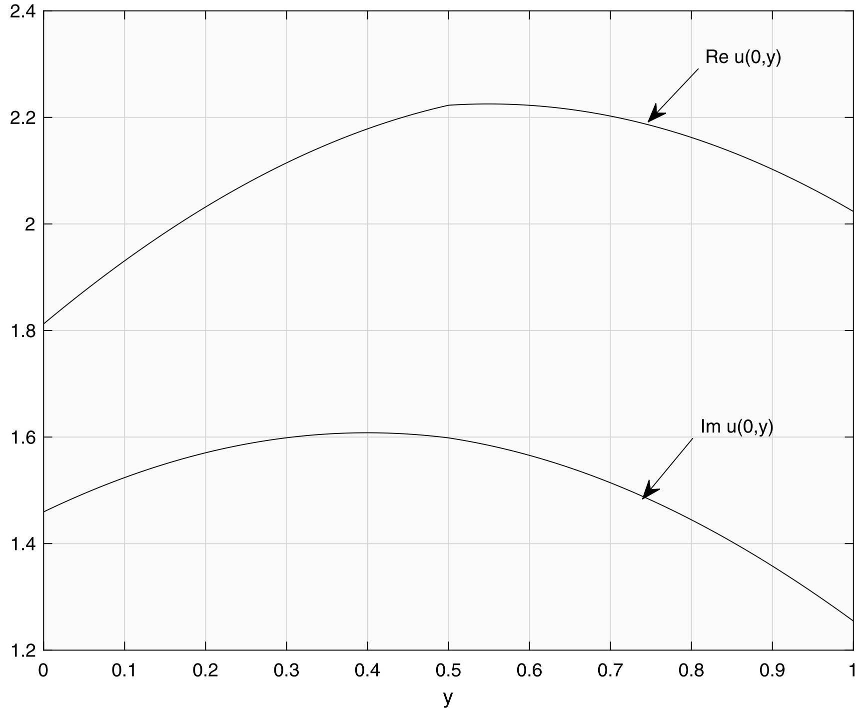

Fig. 4. The real and imaginary parts of the complex potential u in the vertical wall in case (iii) when $a = 1$, $\gamma_0 = 0.5$, $\gamma_1 = 0.05$, $\alpha_0 = 0.5\alpha_2$, $\alpha_1 = 2\alpha_2$, $\alpha_2 = \sqrt{\gamma_0}k$, $\mu_0 = 0.5\mu_2$, $\mu_1 = 2\mu_2$, and $\mu_2 = \gamma_1 k^2$.

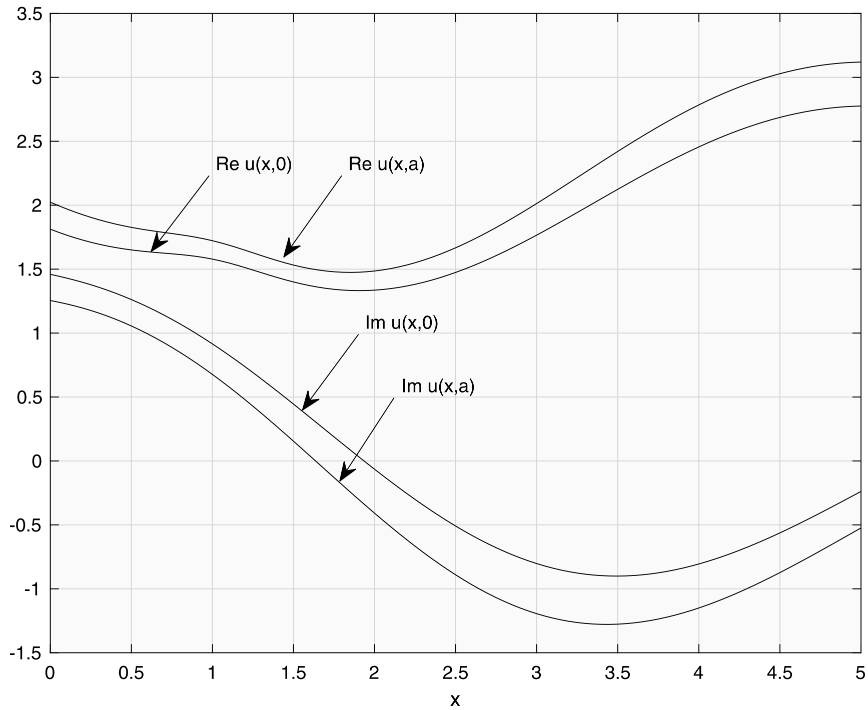

Fig. 5. The real and imaginary parts of the complex potential u in the horizontal sides $0 \le x \le 5a$, $a = 1$, in case (iii) when $\gamma_0 = 0.5$, $\gamma_1 = 0.05$, $\alpha_0 = 0.5\alpha_2$, $\alpha_1 = 2\alpha_2$, $\alpha_2 = \sqrt{\gamma_0}k$, $\mu_0 = 0.5\mu_2$, $\mu_1 = 2\mu_2$, and $\mu_2 = \gamma_1 k^2$. The function u is continuous at the corners of the semi-strip.

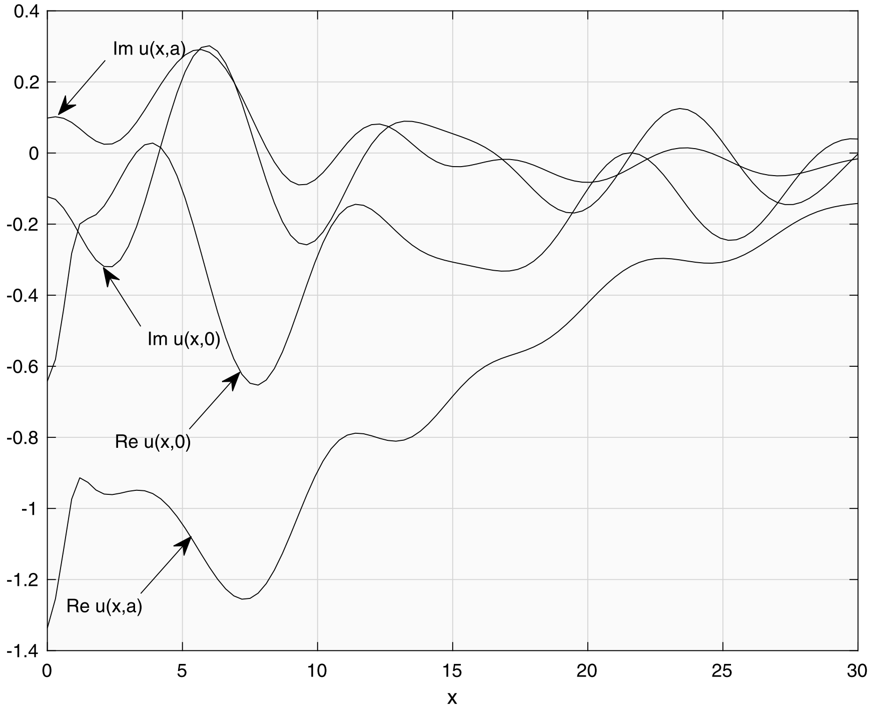

Fig. 6. The real and imaginary parts of the complex potential u in the horizontal sides when $0 \le x \le 30a$, $a = 1$ in case (iii) when $\gamma_0 = 0.5$, $\gamma_1 = 0.05$, $\alpha_0 = 0.5\alpha_2$, $\alpha_1 = 2\alpha_2$, $\alpha_2 = \sqrt{\gamma_0}k$, $\mu_0 = 5\mu_2$, $\mu_1 = 2\mu_2$, and $\mu_2 = \gamma_1 k^2$.

3. Semi-infinite waveguide: the walls are elastic plates

Our previous analysis of the Helmholtz equation (2.1) in a semi-infinite strip has been entirely limited to the case of membrane walls modeled by the third order boundary conditions (2.2). We next turn to the two-dimensional model problem of a compressible fluid bounded by elastic walls. This brings us boundary conditions with derivatives of order five. Assume that B_j and m_j are the bending stiffness and mass per unit area of the plate W_j, respectively ($j = 0, 1, 2$). As in Section 2.3, the function $g(x, y)$ is taken to be $g(x, y) = -\delta(x - x^\circ)\delta(y - y^\circ)$, (x°, y°) is an internal point of the semi-infinite strip, and the governing equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u(x, y) = -\delta(x - x^\circ)\delta(y - y^\circ), \quad 0 < x < \infty, \quad 0 < y < a, \tag{3.1}$$

For the walls modeled by thin elastic plates under flexural vibrations the boundary conditions (2.2) are replaced by (Leppington, [5])

$$\left[\left(\frac{\partial^4}{\partial x^4} - \alpha_0^4\right)\frac{\partial}{\partial y} + \mu_0\right] u = 0, \quad (x, y) \in W_0 = \{0 < x < \infty, y = 0\},$$

$$\left[-\left(\frac{\partial^4}{\partial x^4} - \alpha_1^4\right)\frac{\partial}{\partial y} + \mu_1\right] u = 0, \quad (x, y) \in W_1 = \{0 < x < \infty, y = a\},$$

$$\left[\left(\frac{\partial^4}{\partial y^4} - \alpha_2^4\right)\frac{\partial}{\partial x} + \mu_2\right] u = 0, \quad (x, y) \in W_2 = \{x = 0, 0 < y < a\}. \tag{3.2}$$

Here, $\omega = \omega_1 + i\omega_2$, $\omega_1 > 0$, $\omega_2 > 0$, $\mathrm{Re}[e^{-i\omega t}u(x, y)]$ is the fluid velocity potential introduced in Section 2.1,

$$\alpha_j^4 = \frac{m_j\omega^2}{B_j}, \quad \mu_j = \frac{\rho\omega^2}{B_j}, \quad j = 0, 1, 2. \tag{3.3}$$

We need to choose constraints at the two edges $x = 0, y = 0$ and $x = 0, y = a$. It is designated that the plates are clamped at the edges (Fig. 1), and therefore the deflections and the angles of deflection equal zero at the edges,

$$\frac{\partial u}{\partial y}(0^+, y_j) = \frac{\partial^2 u}{\partial x \partial y}(0^+, y_j) = 0, \quad j = 0, 1, \quad y_0 = 0, \quad y_1 = a,$$

Table 2
The roots z_j ($j = 0, 1, \dots, 4$) of the polynomial $Q(\eta)$ for the wave number $k = 1 + 0.1i$ and some values of the parameters γ_0 and γ_1.

z_j	$\gamma_0 = 5, \gamma_1 = 1$	$\gamma_0 = 1, \gamma_1 = 0.1$	$\gamma_0 = 1, \gamma_1 = 1$	$\gamma_0 = .1, \gamma_1 = 1$
z_0	$-1.806 - 0.04917i$	$-1.414 - 0.09353i$	$-1.441 + 0.008144i$	$-1.319 + 0.1075i$
z_1	$-0.02056 + 1.256i$	$0.08369 + 0.3690i$	$0.02245 + 0.7374i$	$0.02935 + 0.5892i$
z_2	$1.809 + 0.1846i$	$1.416 + 0.1184i$	$1.448 + 0.2135i$	$1.320 + 0.2936i$
z_3	$-0.09151 - 0.7698i$	$-0.3550 - 0.2809i$	$-0.6625 - 0.5350i$	$-0.8586 - 0.5673i$
z_4	$0.1083 - 0.6219i$	$0.2701 - 0.1131i$	$0.6330 - 0.4240i$	$0.8280 - 0.4229i$

$$\frac{\partial u}{\partial x}(0, 0^+) = \frac{\partial^2 u}{\partial x \partial y}(0, 0^+) = 0, \quad \frac{\partial u}{\partial x}(0, a^-) = \frac{\partial^2 u}{\partial x \partial y}(0, a^-) = 0. \tag{3.4}$$

On following the procedure presented in Section 2 we apply the Laplace transform (2.9) to the boundary value problem (3.1) to (3.3), integrate by parts and deduce the one-dimensional boundary value problem (2.10), where

$$f(y) = u_x(0, y) - i\eta u(0, y) - e^{i\eta x^\circ}\delta(y - y^\circ), \quad \tilde{\mu}_j(\eta) = \frac{\mu_j}{\alpha_j^4 - \eta^4},$$
$$\tilde{g}^j(\eta) = \frac{c_{j0} - i\eta c_{j1}}{\alpha_j^4 - \eta^4}, \quad j = 0, 1, \tag{3.5}$$

and

$$c_{j0} = \frac{\partial^4 u}{\partial x^3 \partial y}(0^+, y_j), \quad c_{j1} = \frac{\partial^3 u}{\partial x^2 \partial y}(0^+, y_j), \quad j = 0, 1. \tag{3.6}$$

Since the only difference between the problem (2.10) obtained in the previous section and the one derived here is the form of the functions $\tilde{\mu}_j(\eta)$, $f(y)$ and $\tilde{g}^j(\eta)$, we still have the relations (2.22) to (2.24). The next step of the procedure of Section 2 is to apply the Laplace transform to the boundary condition on the vertical wall, the third condition in (3.2). This brings us to Eq. (2.25) with the following notations adopted for the problem under consideration:

$$\hat{\mu}_2(\zeta) = \frac{\mu_2}{\alpha_2^4 - \zeta^4}, \quad \hat{g}^2(i\zeta) = \frac{c_{20} + \zeta c_{21} - (c_{30} + \zeta c_{31})e^{-a\zeta}}{\alpha_2^4 - \zeta^4}, \tag{3.7}$$

where

$$c_{20} = u_{xyyy}(0, 0^+), \quad c_{21} = u_{xyy}(0, 0^+), \quad c_{30} = u_{xyyy}(0, a^-), \quad c_{31} = u_{xyy}(0, a^-). \tag{3.8}$$

Analogously to Section 2 the functions $\Phi_1^\pm(\eta) = \tilde{u}(\pm\eta, 0)$ and $\Phi_2^\pm(\eta) = \tilde{u}(\pm\eta, a)$ solve the symmetric Riemann–Hilbert problem (2.30), (2.31) with the coefficient:

$$H(\eta) = -\frac{\eta + i\hat{\mu}_2(\zeta)}{\eta - i\hat{\mu}_2(\zeta)} = -\frac{\eta[\alpha_2^4 - (\eta^2 - k^2)^2] + i\mu_2}{\eta[\alpha_2^4 - (\eta^2 - k^2)^2] - i\mu_2}. \tag{3.9}$$

Remarkably, the coefficient $H(\eta)$ and its Wiener–Hopf factors share the main features of those derived for the membrane walls of the wave guide. Let $Q(\eta) = \eta[(\eta^2 - k^2)^2 - \alpha_2^4] + i\mu_2$, and z_j ($j = 0, 1, \dots, 4$) be the zeros of this polynomial. Then $H(\eta) = Q(\eta)/Q(-\eta)$, and $\eta = -z_j$ are the zeros of the denominators in (3.9). It turns out that for all realistic values of the problem parameters two zeros, $z_1 = \eta_1$ and $z_2 = \eta_2$, lie in the upper half-plane $\mathbb{C}^+$, and two zeros, $z_3 = -\eta_3$ and $z_4 = -\eta_4$, are located in the lower half-plane $\mathbb{C}^-$ ($\text{Im}\,\eta_j > 0$, $j = 1, 2, 3, 4$). As for the fifth zero, z_0, there are three possible cases,

(i) $z_0 = -\eta_0 \in \mathbb{C}^-$,
(ii) $z_0 = \eta_0 \in \mathbb{R}$, and
(iii) $z_0 = \eta_0 \in \mathbb{C}^+$.

In Table 2, we show the roots of the polynomial $Q(\eta)$ for some values of its parameters. The following notations are adopted: $\alpha_2^4 = \gamma_0 k^2$ and $\mu_2 = \gamma_1 k^2$, where $\gamma_0 = m_2 c^2/B_2$ and $\gamma_1 = \rho c^2/B_2$.

In view of the properties of the zeros and poles of the function $H(\eta)$ we split the function $H(\eta)$ as $H(\eta) = H^+(\eta)/H^-(\eta)$, $-\infty < \eta < \infty$, where

$$H^+(\eta) = \frac{(\eta + \eta_0)(\eta + \eta_3)(\eta + \eta_4)}{(\eta + \eta_1)(\eta + \eta_2)}, \quad H^-(\eta) = -\frac{(\eta - \eta_0)(\eta - \eta_3)(\eta - \eta_4)}{(\eta - \eta_1)(\eta - \eta_2)} \tag{3.10}$$

in case (i) and

$$H^+(\eta) = \frac{(\eta + \eta_3)(\eta + \eta_4)}{(\eta + \eta_0)(\eta + \eta_1)(\eta + \eta_2)}, \quad H^-(\eta) = -\frac{(\eta - \eta_3)(\eta - \eta_4)}{(\eta - \eta_0)(\eta - \eta_1)(\eta - \eta_2)} \tag{3.11}$$

in cases (ii) and (iii).

It is seen that the asymptotics of the factors $H^+(\eta)$ and $H^-(\eta)$ at infinity is the same as for the membrane walls model and therefore the solution has the form

$$\Phi_j^\pm(\eta) = H^\pm(\eta)[b_j + \Psi_j^\pm(\eta)], \quad \eta \in C^\pm, \quad j = 0, 1. \tag{3.12}$$

where $\Psi(\eta)$ is determined by (2.45) and (2.48), $b_j = 0$ in case (i), b_j are expressed through $\Psi(-\eta_0)$ by (2.51) in case (ii), and b_j are free constants in case (iii). Notice that now formula (2.51) reads

$$b_j = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f_j(\tau)(\tau+\eta_1)(\tau+\eta_2)d\tau}{(\tau+\eta_3)(\tau+\eta_4)}, \quad j = 0, 1. \tag{3.13}$$

The functions $f_j(\eta)$ are given by (2.33), (2.24), where $\tilde{\mu}_0$, $\tilde{\mu}_1$, and $\hat{\mu}_2$ have to be replaced by their expressions in (3.5) and (3.7). The functions $f_j(\eta)$ possess eight free constants c_{j0} and c_{j1}, $j = 0, 1, 2, 3$, and for their determination we have the same number of additional conditions (3.4). Similarly to Section 2.3 these edge conditions can be rewritten as a system of eight linear algebraic equations for the eight constants c_{j0} and c_{j1}. The clamping edge conditions (3.4) guarantee that the derivatives u_y and u_{yx} vanish when $x \to 0^+$ along the horizontal walls W_0 and W_1, and the functions u_y and u_{xy} tend to zero as $y \to 0^+$ and $y \to a^-$ along the vertical wall W_2. As for the function $u(x, y)$, in general, it is discontinuous in cases (i) and (ii). In case (iii), as in Section 2.4, it is possible to achieve the continuity of the function $u(x, y)$ and therefore the continuity of the pressure distribution at the corners of the semi-strip. This can be done by fixing the remaining free constants b_1 and b_2 on satisfying the compatibility conditions (2.101).

4. Conclusion

We have developed further the method of integral transforms and made it applicable to the Helmholtz equation in a semi-infinite strip $\{0 < x < \infty, 0 < y < a\}$ with higher order impedance boundary conditions. It has been shown that if the orders of the tangential derivatives in the functionals of boundary conditions are even numbers, then the problem reduces to two symmetric scalar Riemann–Hilbert problems which share the same coefficient, $H(\eta)$, and possess different right-hand sides. The coefficient $H(\eta)$ is a rational function $q(\eta)/q(-\eta)$, where $q(\eta)$ is a degree-n polynomial, and $n-1$ is the order of the tangential derivative in the side $\{x = 0, 0 < y < a\}$ of the semi-infinite strip. In the case $n = 3$, the corresponding boundary value problem for the Helmholtz equation models acoustic wave propagation in a semi-infinite waveguide whose walls are membranes, and if $n = 5$, then the walls are elastic plates. It turns out that the right-hand sides of the Riemann–Hilbert problems associated with the membranes and elastic plates possess four and eight free constants, respectively. We have shown how these constants can be fixed by the conditions at the two edges of the structure. It has been discovered that, in addition to these expected free constants, the solution may or may not have two more free constants. This depends on the index of the Riemann–Hilbert problems; that index is determined by the location of the zeros of the polynomial $q(\eta)$ and, ultimately, by the three parameters, k, γ_0, and γ_1. In the membrane case, k is the wave number, $\gamma_0 = m_2c^2/T_2$, $\gamma_1 = \rho c^2/T_2$, c is the sound speed in the fluid, ρ is the mean fluid density, and m_2 and T_2 are the mass per unit area and the surface tension of the finite vertical wall of the semi-strip, respectively. In the plate case, the meaning of the parameters is the same except for T_2. It has to be replaced by B_2, the bending stiffness of the plate $x = 0, 0 < y < a$. For acceptable values of the parameters, the index κ of the symmetric Riemann–Hilbert problems is either -1, and the solution is unique, or 1, and then the solution to each Riemann–Hilbert problem has its own free constant. We have shown that if $\kappa = -1$, then the solution satisfies the edge conditions, but the pressure distribution $p(x, y, t)$ is discontinuous at the two corners. In the case $\kappa = 1$, the two free constants available can be fixed such that the function $p(x, y, t)$ is continuous at the vertices $x = y = 0$ and $x = 0, y = a$.

The approach we presented has been verified for the case of order-1 impedance boundary conditions by deriving the solution by this method and also by the method of finite integral transforms when the kernel of the integral transform can be constructed by solving the associated Sturm–Liouville problem (for higher order impedance boundary conditions this approach is inapplicable).

We also remark that the method works if the governing PDE is of order two, has only even order derivatives, and the tangential derivatives in the generalized impedance boundary conditions are of an even order. If the functional of the boundary conditions has tangential derivatives of an odd order, as in the Poincaré boundary value problem, then the problem is transformed into an order-2 vector Riemann–Hilbert problem whose coefficient is explicitly factorizible only in some particular cases.

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Appendix

In the appendix we wish to show that this method, when applied to the first order impedance boundary conditions on each side of the half-strip, recovers the solution that coincides with the one obtained by the classical method of integral transforms. Consider the boundary value problem for the Helmholtz equation

$$u_{xx} + u_{yy} + k^2u = g(x, y), \quad 0 < x < \infty, \quad 0 < y < a,$$

$$-u_y+\mu_0 u=0,\quad y=0,\quad u_y+\mu_1 u=0,\quad y=a,\quad 0<x<\infty,$$

$$-u_x+\mu_2 u=g_2(y),\quad x=0,\quad u\to 0,\quad x\to\infty,\quad 0<y<a. \tag{A.1}$$

By the method of Section 2 we reduce the problem to the two symmetric Riemann–Hilbert problems (2.30) to (2.33), where the functions $\hat{\mu}_j$ need to be replaced by the parameters μ_j. Since the function

$$H(\eta)=-\frac{\eta+i\mu_2}{\eta-i\mu_2} \tag{A.2}$$

has only one zero $\eta=-i\mu_2\in\mathbb{C}^-$ and one pole $\eta=i\mu_2\in\mathbb{C}^+$, in contrast to the previous applications of the method, we do not have distinct cases of the zeros and poles location. The solution to the resulting Riemann–Hilbert problems always exists and it is unique,

$$\Phi_j^\pm(\eta)=\pm(\eta\pm i\mu_2)\Psi_j^\pm(\eta),\quad \eta\in\mathbb{C}^\pm,\quad j=0,1, \tag{A.3}$$

where

$$\Psi_j^\pm(\eta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_j(\tau)}{\tau+i\mu_2}\frac{d\tau}{\tau-\eta}=\frac{1}{\pi i}\int_0^{\infty}\frac{f_j(\tau)}{\tau+i\mu_2}\frac{\tau d\tau}{\tau^2-\eta^2},\quad \eta\in\mathbb{C}^\pm. \tag{A.4}$$

Here we used the identity

$$\frac{f_j(\tau)}{\tau+i\mu_2}=-\frac{f_j(-\tau)}{-\tau+i\mu_2}. \tag{A.5}$$

We next want to specify the formulas in the following particular case:

$$g(x,y)=0,\quad 0<x<\infty,\quad 0<y<a,\quad g_2(y)=1,\quad 0<y<a. \tag{A.6}$$

Then

$$h_j(\eta)=0,\quad \hat{g}_2(i\zeta)=\frac{1-e^{-a\zeta}}{\zeta},\quad f_j(\eta)=\frac{2\eta(\zeta\sinh a\zeta+2\mu_{1-j}\sinh^2\frac{1}{2}a\zeta)}{(\eta-i\mu_2)\zeta\tilde{\Delta}(\zeta)}, \tag{A.7}$$

where

$$\tilde{\Delta}(\zeta)=(\mu_0+\mu_1)\zeta\cosh a\zeta+(\mu_0\mu_1+\zeta^2)\sinh a\zeta, \tag{A.8}$$

and therefore $f_j(\eta)$ are meromorphic functions of η.

Our interest is in having expressions for the complex potential $u(x,y)$ in the horizontal sides of the half-strip

$$u(x,y_j)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\eta x}\Phi_j^+(\eta)d\eta,\quad j=0,1. \tag{A.9}$$

On continuing analytically by means of the boundary conditions of the Riemann–Hilbert problems the functions $\Phi_j^+(\eta)$ into the lower half-plane we have

$$\Phi_j^+(\eta)=-\frac{\eta+i\mu_2}{\eta-i\mu_2}\Phi_j^-(\eta)+f_j(\eta),\quad \eta\in\mathbb{C}^-. \tag{A.10}$$

Since the functions $-(\eta+i\mu_2)(\eta-i\mu_2)^{-1}\Phi_j^-(\eta)$ are holomorphic in the lower half-plane, we derive

$$u(x,y_j)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f_j(\eta)e^{-i\eta x}d\eta,\quad 0<x<\infty,\quad j=0,1. \tag{A.11}$$

This integrals are evaluated by the theory of residues. Make the substitution $\zeta=-i\lambda$ and denote by λ_n $(n=0,1,\dots)$ the roots of the transcendental equation

$$\tan a\lambda=\frac{(\mu_0+\mu_1)\lambda}{\lambda^2-\mu_0\mu_1}. \tag{A.12}$$

These roots can be evaluated by quadratures by solving the associated Riemann–Hilbert problems [20] or by an iterative procedure. Denote by $\zeta_n=\sqrt{\lambda_n^2-k^2}$ the values which meet the condition $\operatorname{Re}\zeta_n>0$. Then the complex potential u in the sides $y=y_0=0$ and $y=y_1=a$ admits the series representations

$$u(x,y_j)=2\sum_{n=0}^{\infty}\frac{\Delta_{jn}}{\zeta_n+\mu_2}e^{-\zeta_n x},\quad j=0,1, \tag{A.13}$$

where

$$\Delta_{jn}=-\frac{\lambda_n\sin a\lambda_n+2\mu_{1-j}\sin^2\frac{1}{2}a\lambda_n}{(\mu_0+\mu_1+\mu_0\mu_1 a-a\lambda_n^2)\cos a\lambda_n-(\mu_0 a+\mu_1 a+2)\lambda_n\sin a\lambda_n}. \tag{A.14}$$

We wish next to apply an alternative method and show that the solution coincides with (A.13). Apply the integral transformation

$$u^*(x;\lambda)=\int_0^a K(y;\lambda)u(x,y)dy \tag{A.15}$$

to the boundary value problem (A.1) in order to reduce it to a one-dimensional boundary value problem. Here, λ is an eigenvalue, and the kernel $K(y;\lambda)$ is the associated eigenfunction of the Sturm–Liouville problem

$$\begin{gathered}\frac{d^2K}{dy^2}=-\lambda^2K,\quad 0<y<a,\\ \mu_0K=\frac{dK}{dy},\quad y=0,\quad -\mu_1K=\frac{dK}{dy},\quad y=a.\end{gathered} \tag{A.16}$$

It turns out that the eigenvalues λ_n $(n=0,1,\dots)$ of the problem coincide with the roots of the transcendental equation (A.12), while the corresponding eigenfunctions which meet the condition

$$\int_0^a K(y;\lambda_n)K(y;\lambda_m)dy=\delta_{nm},\quad n,m=0,1,\dots, \tag{A.17}$$

have the form

$$K(y;\lambda_n)=\frac{\lambda_n\cos\lambda_n y+\mu_0\sin\lambda_n y}{\mu_0\sigma_n}, \tag{A.18}$$

where

$$\sigma_n=\frac{1}{2\mu_0\sqrt{\lambda_n}}[2\lambda_n(2\mu_0\sin^2 a\lambda_n+a\lambda_n^2+a\mu_0^2)+(\lambda_n^2-\mu_0^2)\sin 2a\lambda_n]^{1/2}. \tag{A.19}$$

The inverse integral transform is the series expansion in terms of the eigenfunctions

$$u(x,y)=\sum_{n=0}^{\infty}K(y;\lambda_n)u^*(x;\lambda_n). \tag{A.20}$$

On applying the integral transform (A.15) with $\lambda=\lambda_n$ to the problem (A.1) we deduce

$$\begin{gathered}\left[\frac{d^2}{dx^2}-(\lambda_n^2-k^2)\right]u^*(x;\lambda_n)=g^*(x;\lambda_n),\quad 0<x<\infty,\\ -\frac{d}{dx}u^*(0;\lambda_n)+\mu_2u^*(0;\lambda_n)=g_2^*(\lambda_n),\quad u^*(x,\lambda_n)\to 0,\quad x\to\infty,\end{gathered} \tag{A.21}$$

where $g_2^*(\lambda_n)$ is the integral transform with the kernel $K(y;\lambda_n)$ of the function $g_2(y)$. The solution to the problems (A.21) has the form

$$u^*(x;\lambda_n)=\int_0^\infty G_n(x,\xi)g^*(\xi;\lambda_n)d\xi+\frac{g_2^*(\lambda_n)}{\zeta_n+\mu_2}e^{-\zeta_n x}, \tag{A.22}$$

where $\zeta_n=\sqrt{\lambda_n^2-k^2}$, $\operatorname{Re}\zeta_n>0$, $n=0,1,\dots$, and $G_n(x,\xi)$ is the Green function

$$G_n(x,\xi)=-\frac{1}{2\zeta_n}\left(e^{-\zeta_n|x-\xi|}+\frac{\zeta_n-\mu_2}{\zeta_n+\mu_2}e^{-\zeta_n(x+\xi)}\right). \tag{A.23}$$

Now, as in the application of the first method, we take $g(x,y)\equiv 0$ and $g_2(y)\equiv 1$. Then

$$g_2^*(\lambda_n)=\frac{1}{\sigma_n}\left(\frac{1-\cos a\lambda_n}{\lambda_n}+\frac{\sin a\lambda_n}{\mu_0}\right). \tag{A.24}$$

To compare the solution derived with the one obtained by the first method, we substitute (A.22) into (A.20) and then put $y=0$ and $y=a$ to obtain

$$\begin{gathered}u(x,0)=\frac{1}{\mu_0}\sum_{n=0}^{\infty}\frac{\lambda_n e^{-\zeta_n x}}{\sigma_n^2(\zeta_n+\mu_2)}\left(\frac{1-\cos a\lambda_n}{\lambda_n}+\frac{\sin a\lambda_n}{\mu_0}\right),\\ u(x,a)=\frac{1}{\mu_0}\sum_{n=0}^{\infty}\frac{(\lambda_n\cos\lambda_n a+\mu_0\sin\lambda_n a)e^{-\zeta_n x}}{\sigma_n^2(\zeta_n+\mu_2)}\left(\frac{1-\cos a\lambda_n}{\lambda_n}+\frac{\sin a\lambda_n}{\mu_0}\right).\end{gathered} \tag{A.25}$$

By taking into account that the numbers λ_n solve Eq. (A.12) and using simple algebraic manipulations we show that these two representations are identical to formulas (A.13), (A.14).

Finally, we emphasize that the alternative method of the integral transform (A.15) works for the order-1 impedance boundary conditions (A.1). In the case of higher order boundary conditions (2.2) and (3.2) it is not seen how to construct an analogue of the kernel $K(y;\lambda)$.

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