Fundamental solution and the weight functions of the transient problem on a semi-infinite crack propagating in a half-plane

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Abstract

The two-dimensional transient problem that is studied concerns a semi-infinite crack in an isotropic solid comprising an infinite strip and a half-plane joined together and having the same elastic constants. The crack propagates along the interface at constant speed subject to time-independent loading. By means of the Laplace and Fourier transforms the problem is formulated as a vector Riemann-Hilbert problem. When the distance from the crack to the boundary grows to infinity the problem admits a closed-form solution. In the general case, a method of partial matrix factorization is proposed. In addition to factorizing some scalar functions it requires solving a certain system of integral equations whose numerical solution is computed by the collocation method. The stress intensity factors and the associated weight functions are derived. Numerical results for the weight functions are reported and the boundary effects are discussed. The weight functions are employed to describe propagation of a semi-infinite crack beneath the half-plane boundary at piecewise constant speed.

1 Introduction

The main goal of this work is to propose a new method for analyzing plane dynamic transient problems when the two modes are coupled, and the standard Wiener-Hopf method does not work. In addition to factorization of two scalar functions it employs derivation and solution of a certain system of two integral equations. The method is illustrated by the study of a crack propagating at sub-Rayleigh speed parallel to the boundary of a solid when loading is time independent. The model problem admits formulation as a vector Riemann-Hilbert problem (RHP). In the case when the crack is far away from the boundary, the problem can be modeled as propagation of a semi-infinite crack along the interface between two weakly bonded, identical and isotropic half-planes. The problem on crack growth in a plane at constant sub-Rayleigh speed due to general time-independent loading (including the case of concentrated forces applied to the crack faces) was solved exactly in [12, 13] by means of the Wiener-Hopf method. The intersonic regime including the case of concentrated forces (the fundamental solution problem) and the model problem on a suddenly stopping crack were analyzed in [15, 16]. When the crack is close to the boundary of the body, the boundary effects cannot be ignored, and the problem on a crack propagating parallel to the half-plane boundary can be considered as an adequate model. In the static case, the matrix coefficient of the RHP admits a closed-form factorization [26]. The steady-state case, when loads applied to the propagating crack move with the crack at the same constant speed, was recently analyzed in [6]. By means of the Fourier transform, the problem was mapped into a vector RHP whose matrix coefficient did not allow for an explicit factorization. The RHP was rewritten as a system of singular integral equations, and an approximate method of orthogonal polynomials for its solution was proposed. To the authors' knowledge, an analytical solution to the transient problem on a semi-infinite crack propagating along the boundary of a half-plane is not available in the literature.

In Section 2, we describe the transient model and apply the Fourier and Laplace transforms in a standard manner [13, 22, 7] in order to reduce the governing boundary-value problem to an order-2 vector RHP. Although the matrix coefficient has the same structure as in the steady-state case [6], the transient problem is much harder for the parameters α and β involved being functions of the Laplace and Fourier parameters not constants as in the steady-state problem. All our efforts availed us no results not only in factorizing the matrix coefficient of the RHP, but even in computing the partial indices of factorization [23]. The partial indices play an essential part in solvability theory of the vector RHP and in the theory of approximate Wiener-Hopf matrix factorization. According to the stability criterion for partial indices [10, 14, 23] applied to a 2×2 matrix the partial indices, integers κ_1 and κ_2, are stable if and only if $|\kappa_1 - \kappa_2| \leq 1$. If they do not satisfy this criterion, then approximate canonical Wiener-Hopf factors may not converge to the exact ones. At the same time, without knowledge of exact factorization, in general, there is no way to determine the partial indices. An example (not inspired by an applied physical problem) of unstable partial indices is given in [17]. It turns out that the partial indices associated with contact, fracture, and diffraction models available in the literature [18, 4, 5] are stable. Although this circumstance makes the determination of the partial indices of factorization not an absolute necessity but rather a desideratum, in this paper we surmount the deficiency of not knowing the partial indices by bypassing the problem of approximate matrix factorization. Instead, we propose a method of partial factorization that comprises factorization of some scalar functions and numerical solution of a certain system of integral equations.

To introduce the reader to the method presented later in the paper, in Section 3, we analyze the transient problem for a semi-infinite crack in the whole plane. In this case, the vector RHP is decoupled and solved by quadratures. We also derive exact formulas for the stress intensity factors (SIFs) and the weight functions introduced in [11] for a semi-infinite static crack in a homogeneous elastic medium. For the elastic case exact and approximate expressions for the weight functions are available in the literature for a variety of models. Exact weight functions for a static semi-infinite interfacial crack in a three-dimensional unbounded body were constructed in [2] and later on employing a different approach in [9]. The matrix factorization technique was applied in [3] for recovering exact representations of the weight functions in micropolar theory. The dynamic Mode I weight function for a semi-infinite crack in a three-dimensional homogeneous body under the conditions of steady-state normal loading by a scalar factorization method was derived in [25]. Coupling between Modes II and III in the case of shear loading was discovered and exact formulas for the steady-state Mode II and III weight functions by the matrix factorization method were derived in [19]. Mode I and II weight functions for a viscoelastic medium with different bulk and shear relaxation in the steady-state case were determined in [8]. Approximate formulas for the weight functions associated with the steady-state semi-infinite crack propagating bellow the boundary of a half-plane are also available [6]. The transient weight functions are determined for a semi-infinite crack propagating at constant speed only for an unbounded body in the elastic case for sub-Rayleigh speed [12, 13] and the transonic regime [15] and in the viscoelastic case when the constant speed of crack propagation may take any value up to the speed of dilatational waves [7]. In Section 3 of this paper, we derive the transient weight functions for a plane by a factorization method that will be later generalized for a half-plane.

In Section 4, we propose an approximate method for the vector RHP associated with the transient problem for a half-plane. First, we split the matrix coefficient into a diagonal matrix that is discontinuous at infinity and a matrix that is continuous. On factorizing the discontinuous part and recasting the vector RHP we derive a new vector RHP that is equivalent to a system of two integral equations on the interval $(-\infty, 0)$. The diagonal elements of the matrix kernel are constants, while the off-diagonal elements are continuous functions which have an order-2 zero at infinity. We show that in order to determine the Laplace transforms of the SIFs and the weight functions, it is sufficient to know the solution to the system of integral equations at the point 0 only. We describe the numerical procedure and the inversion method of the Laplace transform we applied and discuss the numerical results for the weight functions.

With the fundamental solution and weight functions being derived we proceed, in Section 5, to describe nonuniform growth of a semi-infinite crack parallel to the boundary of a half-plane. For the whole plane such an algorithm based on the fundamental solution and the solution of the model problem on a suddenly stopped crack is known [13]. We aim to generalize this procedure for a half-plane. The main feature here is to take into account the fact that the Mode I and II weight functions after the longitudinal wave reflects from the boundary and strikes the crack do not act alone anymore and the off-diagonal weight functions play a substantial part. We show that in order to determine the stresses radiated out by a suddenly stopped crack, one needs to solve a system of two Volterra convolution equations, a generalization of the single Abel equation appeared in the Freund method for the whole plane. On solving this system exactly we determine the stresses the crack needs to negate on the prospective fracture plane to proceed further. The procedure to be exposed allows for the possibility of finding the SIFs at the tip of a crack propagating at piecewise constant speed bellow the boundary and, in conjunction with the dynamic Griffith criterion, the actual nonuniform speed of crack propagation.

2 Transient problem for a half-plane as a vector RHP

2.1 Formulation

The elastic medium $\Pi = \{|x_1| < \infty,\ -\infty < x_2 < \delta\}$ through which the crack propagates comprises an infinite strip $\{|x_1| < \infty,\ 0 < x_2 < \delta\}$ and a half-plane $\{|x_1| < \infty,\ -\infty < x_2 < 0\}$ bonded together. The bonding is not perfect, and it is assumed that along the interface there is a semi-infinite crack. The faces of the crack are subjected to plane strain loading that forces the crack to propagate at constant sub-Rayleigh speed V. The presence of the weak interface encourages the crack to propagate parallel to the boundary $\{|x_1| < \infty, x_2 = \delta\}$ rather than deviate towards it (Fig. 1). The boundary of the body Π is free of traction, and the Lamé constants λ and μ and the density ρ of the strip and the half-plane are assumed to be the same. Boundary conditions of the problem are specified to the form

$$\left.\begin{array}{lll}\sigma_{j2} = -\sigma^\circ_{j2}(x_1, 0)H(t), & -\infty < x_1 < Vt, & x_2 = 0^\pm, \\ \sigma_{j2} = 0, & -\infty < x_1 < \infty, & x_2 = \delta,\end{array}\right. \quad j = 1, 2, \tag{2.1}$$

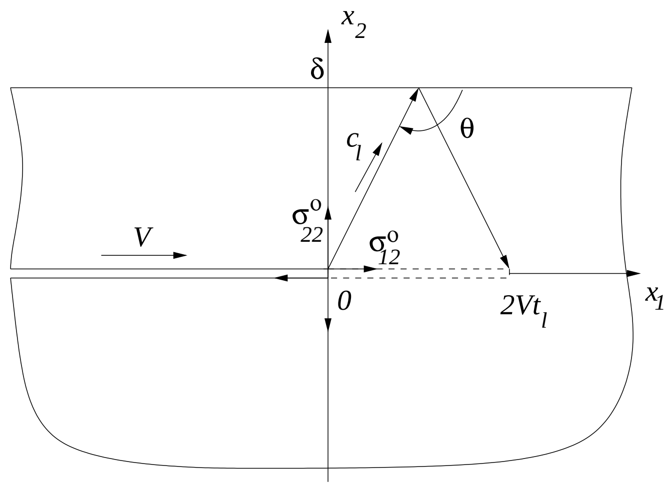

Figure 1: A semi-infinite crack propagating parallel to the boundary: the transient problem.

where σ_{12}°, σ_{22}° are prescribed functions and $H(t)$ is the unit step function.

It is advantageous to change variables from the material coordinates x_1, x_2 to the crack tip coordinates $x = x_1 - Vt$, $y = x_2$. In these coordinates, displacement potentials φ and ψ of the medium satisfy the wave equations

$$\begin{array}{l} c_l^2\hat{\alpha}^2\dfrac{\partial^2\varphi}{\partial x^2} + c_l^2\dfrac{\partial^2\varphi}{\partial y^2} + 2V\dfrac{\partial^2\varphi}{\partial x\partial t} - \dfrac{\partial^2\varphi}{\partial t^2} = 0, \\ c_s^2\hat{\beta}^2\dfrac{\partial^2\psi}{\partial x^2} + c_s^2\dfrac{\partial^2\psi}{\partial y^2} + 2V\dfrac{\partial^2\psi}{\partial x\partial t} - \dfrac{\partial^2\psi}{\partial t^2} = 0, \end{array} \quad (x,y) \in \hat{\Pi} \setminus \{-\infty < x < 0, y = 0\}, \quad t > 0, \tag{2.2}$$

where $\hat{\Pi} = \{|x| < \infty, -\infty < y < \delta\}$, subject to the zero initial conditions

$$\varphi = \psi = 0, \quad \frac{\partial\varphi}{\partial t} = \frac{\partial\psi}{\partial t} = 0, \quad (x,y) \in \hat{\Pi}, \quad t < 0. \tag{2.3}$$

Here, c_l and c_s are the longitudinal and shear wave speeds

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}}, \tag{2.4}$$

and

$$\hat{\alpha} = \sqrt{1 - v_l^2}, \quad \hat{\beta} = \sqrt{1 - v_s^2}, \quad v_l = V/c_l, \quad v_s = V/c_s. \tag{2.5}$$

In the new coordinates, the displacements u and v and the stresses σ_{xy} and σ_{yy} are expressed through the dynamic potentials φ and ψ as

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x},$$

$$\sigma_{xy} = \mu\left(\frac{2\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right),$$

$$\sigma_{yy} = \lambda\frac{\partial^2\varphi}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2\varphi}{\partial y^2} - 2\mu\frac{\partial^2\psi}{\partial x\partial y}. \tag{2.6}$$

To complete the formulation of the problem, we explicitly write the boundary conditions in the moving coordinates

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad |x| < \infty, \quad y = \delta, \quad t \geq 0,$$

$$\sigma_{xy} = -\sigma_1^\circ(x + Vt, 0) \quad \sigma_{yy} = -\sigma_2^\circ(x + Vt, 0), \quad -\infty < x < 0, \quad y = 0^\pm, \quad t \geq 0. \tag{2.7}$$

Notice that in the new coordinates the loading is time-dependent.

2.2 Vector RHP

We next aim to transform the boundary value problem to a vector RHP. On applying first the Laplace transform with respect to time

$$\begin{pmatrix} \hat{\varphi} \\ \hat{\psi} \end{pmatrix}(x, y, s) = \int_0^\infty \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(x, y, t)e^{-st}dt, \quad \operatorname{Re} s = \sigma > 0, \tag{2.8}$$

and then the Fourier transform with respect to x

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}(p, y, s) = \int_{-\infty}^\infty \begin{pmatrix} \hat{\varphi} \\ \hat{\psi} \end{pmatrix}(x, y, s)e^{ipx}dx, \quad p \in \mathbb{R}, \tag{2.9}$$

we can write the governing equations in the form

$$\frac{\partial^2 \tilde{\varphi}}{\partial y^2} - \alpha^2 \tilde{\varphi} = 0, \quad \frac{\partial^2 \tilde{\psi}}{\partial y^2} - \beta^2 \tilde{\psi} = 0, \quad y \in \{-\infty, \delta\} \setminus \{0\}, \tag{2.10}$$

where

$$\alpha^2 = \hat{\alpha}^2 p^2 + 2ipsv_l/c_l + s^2/c_l^2, \quad \beta^2 = \hat{\beta}^2 p^2 + 2ipsv_s/c_s + s^2/c_s^2. \tag{2.11}$$

To fix single branches of the two-valued functions (2.11), we cut the p-plane along lines that pass through the infinite point and join the branch points $a_\pm$ of the former function and $\beta_\pm$ of the second one,

$$a_\pm = \frac{is}{V \pm c_l} \in \mathbb{C}^\pm, \quad b_\pm = \frac{is}{V \pm c_s} \in \mathbb{C}^\pm. \tag{2.12}$$

We denote the single branches as

$$\alpha = \hat{\alpha}(p - a_-)^{1/2}(p - a_+)^{1/2}, \quad \beta = \hat{\beta}(p - b_-)^{1/2}(p - b_+)^{1/2}, \tag{2.13}$$

assuming that as $p \in L = (-\infty, +\infty)$,

$$-\pi < \arg(p - a_+) < 0, \quad 0 < \arg(p - a_-) < \pi,$$

$$-\pi < \arg(p - b_+) < 0, \quad 0 < \arg(p - b_-) < \pi. \tag{2.14}$$

In these circumstances, (2.13) implies $\operatorname{Re} \alpha > 0$ and $\operatorname{Re} \beta > 0$ as $p \in L$. Then the general solution to the differential equations (2.10), bounded as $y \to -\infty$, reads

$$\tilde{\varphi}(p, y, s) = C_0(p, s)e^{\alpha y}, \quad \tilde{\psi}(p, y, s) = D_0(p, s)e^{\beta y}, \quad -\infty < y < 0, \tag{2.15}$$

and

$$\begin{aligned} \tilde{\varphi}(p, y, s) &= C_1(p, s)\cosh(\alpha y) + C_2(p, s)\sinh(\alpha y), \\ \tilde{\psi}(p, y, s) &= D_1(p, s)\cosh(\beta y) + D_2(p, s)\sinh(\beta y), \end{aligned} \quad 0 < y < \delta. \tag{2.16}$$

It is helpful to introduce new functions representing the jumps of the tangential derivatives of the displacement components u, v on the crack faces

$$\begin{aligned} &\frac{\partial u}{\partial x}(x,0^+,t)-\frac{\partial u}{\partial x}(x,0^-,t)=\chi_1(x,t),\\ &\frac{\partial v}{\partial x}(x,0^+,t)-\frac{\partial v}{\partial x}(x,0^-,t)=\chi_2(x,t),\end{aligned}\qquad -\infty<x<0,\quad t>0. \tag{2.17}$$

Then we define the Laplace transforms with respect to time

$$\hat{\chi}_j(x,s)=\int_0^\infty \chi_j(x,t)e^{-st}dt,\quad q_j(x,s)=\int_0^\infty \sigma_j^\circ(x+Vt,0)e^{-st}dt,\quad x<0,\quad j=1,2,$$

$$\hat{\sigma}_1(x,s)=\int_0^\infty \sigma_{xy}(x,0,t)e^{-st}dt,\quad \hat{\sigma}_2(x,s)=\int_0^\infty \sigma_{yy}(x,0,t)e^{-st}dt,\quad x>0, \tag{2.18}$$

and the one-sided Fourier transforms

$$\tilde{\chi}_j^-(p,s)=\int_{-\infty}^0 \hat{\chi}_j(x,s)e^{ipx}dx,\quad \tilde{q}_j^-(p,s)=\int_{-\infty}^0 q_j(x,s)e^{ipx}dx,$$

$$\tilde{\sigma}_j^+(p,s)=\int_0^\infty \hat{\sigma}_j(x,s)e^{ipx}dx,\quad j=1,2. \tag{2.19}$$

In order to derive the governing vector RHP, we apply the Laplace and Fourier transforms to the six boundary conditions (2.7) and equations (2.17), use the notations (2.18) and (2.19), and eliminate the functions $C_j(p,s)$ and $D_j(p,s)$ $(j=0,1,2)$. The two equations left comprise the vector RHP

$$\begin{pmatrix} \tilde{\sigma}_1^+(p,s)\\ \tilde{\sigma}_2^+(p,s)\end{pmatrix}=\mu i G(p,s)\begin{pmatrix} \tilde{\chi}_1^-(p,s)\\ \tilde{\chi}_2^-(p,s)\end{pmatrix}+\begin{pmatrix} \tilde{q}_1^-(p,s)\\ \tilde{q}_2^-(p,s)\end{pmatrix},\quad p\in L. \tag{2.20}$$

The functions $\tilde{\sigma}_j^+(p,s)$ are analytic in the upper p-half-plane and vanish as $p\to\infty$, while the functions $\tilde{\chi}_j^-(p,s)$ and $\tilde{q}_j^-(p,s)$ are analytic in the lower p-half-plane and also vanish at infinity. The matrix coefficient of the problem is defined by

$$G(p,s)=\begin{pmatrix} g_{11}(p,s) & ig_{12}(p,s)\\ -ig_{12}(p,s) & g_{22}(p,s)\end{pmatrix}, \tag{2.21}$$

$$\begin{aligned} g_{11}(p,s)&=\frac{e^{-(\alpha+\beta)\delta}}{2\beta(p^2-\beta^2)p}\left[R_1\sinh\{(\alpha+\beta)\delta\}-R_2\sinh\{(\alpha-\beta)\delta\}+\frac{2\Delta}{R_1}\right],\\ g_{12}(p,s)&=\frac{4R_2(p^2+\beta^2)}{R_1(p^2-\beta^2)}e^{-(\alpha+\beta)\delta}\sinh^2\frac{(\alpha-\beta)\delta}{2},\\ g_{22}(p,s)&=\frac{e^{-(\alpha+\beta)\delta}}{2\alpha(p^2-\beta^2)p}\left[R_1\sinh\{(\alpha+\beta)\delta\}+R_2\sinh\{(\alpha-\beta)\delta\}+\frac{2\Delta}{R_1}\right],\end{aligned}$$

$$\begin{aligned} &\Delta=R_1^2\sinh^2\frac{(\alpha+\beta)\delta}{2}-R_2^2\sinh^2\frac{(\alpha-\beta)\delta}{2},\\ &R_1=(p^2+\beta^2)^2-4\alpha\beta p^2,\quad R_2=(p^2+\beta^2)^2+4\alpha\beta p^2.\end{aligned} \tag{2.22}$$

The matrix $G(p,s)$ resembles its analogue in the steady-state case [6]. However, in the steady-state problem, α and β are constants, while in the transient case, they are functions of p and s.

3 Transient problem for a plane

In this section, we develop explicit representations for the solution and the weight functions of the transient problem on propagation of a semi-infinite crack in the particular case $\delta=\infty$. Although the solution to this problem is known [13], our solution has a different form. It is used as a building block for the approximate procedure proposed in the next section for the solution of the problem on a crack in a half-plane.

3.1 Scalar RHP

On passing to the limit $\delta\to\infty$ in (2.22) we arrive at the following two separate equations:

$$\tilde\sigma_j^+(p,s)=\mu i g_j(p,s)\tilde\chi_j^-(p,s)+\tilde q_j^-(p,s),\quad p\in L,\quad j=1,2. \tag{3.1}$$

that are scalar RHPs with the coefficients

$$g_1(p,s)=\frac{R_1}{2\beta(p^2-\beta^2)p},\quad g_2(p,s)=\frac{R_1}{2\alpha(p^2-\beta^2)p}. \tag{3.2}$$

Assume first that s is real and positive and let $s=c_l s'$ and $p=p's'$. Then we observe that

$$\alpha(p,s)=s'\tilde\alpha(p'),\quad \beta(p,s)=s'\tilde\beta(p'), \tag{3.3}$$

where

$$\tilde\alpha(p')=\hat\alpha(p'-a'_-)^{1/2}(p'-a'_+)^{1/2},\quad \tilde\beta=\hat\beta(p'-b'_-)^{1/2}(p'-b'_+)^{1/2},$$
$$a'_\pm=\frac{i}{v_l\pm 1}\in\mathbb{C}^\pm,\quad b'_\pm=\frac{i}{v_l\pm c_s/c_l}\in\mathbb{C}^\pm. \tag{3.4}$$

The branches of the functions $(p'-a'_\pm)^{1/2}$ and $(p'-b'_\pm)^{1/2}$ are chosen as it is done in (2.14) for the original functions. For simplicity, rename p' as p and write the problems (3.1) as

$$\tilde\sigma_j^+(ps',c_ls')=\mu i\tilde g_j(p)\tilde\chi_j^-(ps',c_ls')+\tilde q_j^-(ps',c_ls'),\quad p\in L,\quad j=1,2, \tag{3.5}$$

with the coefficients $\tilde g_j(p)$ independent of s',

$$\tilde g_j(p)=\frac{\tilde R_1}{2\tilde\beta(p^2-\tilde\beta^2)p},\quad \tilde g_2(p)=\frac{\tilde R_1}{2\tilde\alpha(p^2-\tilde\beta^2)p},\quad \tilde R_1=(p^2+\tilde\beta^2)^2-4\tilde\alpha\tilde\beta p^2. \tag{3.6}$$

The functions $\tilde g_j(p)$ to be factorized have the following asymptotics at infinity and zero:

$$\tilde g_j(p)=\mp\gamma_j+O\left(\frac{1}{p}\right),\quad p\to\pm\infty,\quad \tilde g_j(p)\sim-\frac{\gamma_j^\circ}{p},\quad p\to 0, \tag{3.7}$$

where

$$\gamma_1=\frac{R_0}{2\hat\beta v_s^2},\quad \gamma_2=\frac{R_0}{2\hat\alpha v_s^2},\quad \gamma_1^\circ=\frac{1}{2c_s},\quad \gamma_2^\circ=\frac{c_l}{2c_s^2},\quad R_0=4\hat\alpha\hat\beta-(1+\hat\beta^2)^2. \tag{3.8}$$

We emphasize that in the sub-Rayleigh regime, $V<c_R$, c_R is the Rayleigh speed, and the parameter R_0 is positive. Employing the relations (3.7) we split the coefficients of the RHPs as

$$\tilde g_j(p)=-\gamma_j\coth(\pi p)g_j^\circ(p). \tag{3.9}$$

Because the coefficients of the Riemann-Hilbert problem, $\tilde{g}_1$ and $\tilde{g}_2$, have a first order infinity at the point $p = 0 \in L$, to bypass this point, we deform the contour L. There are two possibilities to do so. One of them is to replace L by $L_\varepsilon = L' \cup C_\varepsilon^- \cup L''$, where $L' = \{-\infty < p_1 \le -\varepsilon, p_2 = 0\}$, $L'' = \{\varepsilon \le p_1 < +\infty, p_2 = 0\}$, $C_\varepsilon^- = \{|p| = \varepsilon, p_2 < 0\}$, $p = p_1 + ip_2$. The second one is to replace L by $\tilde{L}_\varepsilon = L' \cup C_\varepsilon^+ \cup L''$, where $C_\varepsilon^+ = \{|p| = \varepsilon, p_2 > 0\}$. Consider the first case. In Appendix A, we analyze the second possibility and show that the final solution is independent of the way we deform the contour. The contour L_ε splits the p-plane into two domains $D^+ \ni 0$ and D^-. It is important to notice that the new functions $g_j^\circ(p)$ can be easily factorized $g_j^\circ(p) = \Omega_j^+(p)/\Omega_j^-(p)$, $p \in L_\varepsilon$, in terms of the Cauchy integrals

$$\Omega_j^\pm(p) = \lim_{\tilde{p}\to p \in L_\varepsilon, \tilde{p}\in D^\pm} \Omega_j(\tilde{p}), \quad \Omega_j(\tilde{p}) = \exp\left\{\frac{1}{2\pi i}\int_{L_\varepsilon} \frac{\ln g_j^\circ(\tau)}{\tau - \tilde{p}} d\tau\right\}, \quad \tilde{p} \in D^\pm, \tag{3.10}$$

since the functions $\ln g_j^\circ(\tau)$ are Hölder-continuous on the contour, $g_j^\circ(\tau) = 1 + O(1/\tau)$, $\tau \to \pm\infty$, positive at zero, and the increment of the argument of $g_j^\circ(\tau)$ equals zero as τ traverses the whole contour L_ε.

After factorizing the function $\coth(\pi p)$ in terms of the Gamma-functions

$$\coth(\pi p) = \frac{iK^+(p)}{K^-(p)}, \quad K^+(p) = \frac{\Gamma(1 - ip)}{\Gamma(1/2 - ip)}, \quad K^-(p) = \frac{\Gamma(1/2 + ip)}{\Gamma(ip)}, \tag{3.11}$$

it is possible to transform the boundary condition (3.1) of the RHP to the form

$$\frac{\tilde{\sigma}_j^+(ps', c_l s')}{K^+(p)\Omega_j^+(p)} - \Psi_j^+(p, s') = \frac{\mu\gamma_j \tilde{\chi}_j^-(ps', c_l s')}{K^-(p)\Omega_j^-(p)} - \Psi_j^-(p, s'), \quad p \in L_\varepsilon, \tag{3.12}$$

where

$$\Psi_j^\pm(p, s') = \lim_{\tilde{p}\to p \in L_\varepsilon, \tilde{p}\in D^\pm} \Psi_j(\tilde{p}, s'), \quad \Psi_j(\tilde{p}, s') = \frac{1}{2\pi i}\int_{L_\varepsilon} \frac{\tilde{q}_j^-(\tau s', c_l s')}{K^+(\tau)\Omega_1^+(\tau)} \frac{d\tau}{\tau - \tilde{p}}, \quad \tilde{p} \in D^\pm. \tag{3.13}$$

On applying the continuity principle and the Liouville theorem and employing the asymptotics

$$\begin{aligned} &K^\pm(p) \sim (\mp ip)^{1/2}, \quad \Omega_j^\pm(p) \sim 1, \quad \Psi_j^\pm(p, s') = O(p^{-1}), \\ &\tilde{\sigma}_j^+(ps', c_l s') = O(p^{-1/2}), \quad \tilde{\chi}_j^-(ps', c_l s') = O(p^{-1/2}), \quad p \to \infty, \end{aligned} \tag{3.14}$$

we find that the solution is unique and given by

$$\begin{aligned} &\tilde{\sigma}_j^+(ps', c_l s') = K^+(p)\Omega_j^+(p)\Psi_j^+(p, s'), && p \in D^+, \\ &\tilde{\chi}_j^-(ps', c_l s') = (\mu\gamma_j)^{-1} K^-(p)\Omega_j^-(p)\Psi_j^-(p, s'), && p \in D^-. \end{aligned} \tag{3.15}$$

Passage to the limit $\varepsilon \to 0^+$ shows that the functions $\tilde{\chi}_j^-(-i\varepsilon s', c_l s') \to 0$ that is consistent with the fact that the displacement jumps vanish as $x \to -\infty$.

3.2 Stress intensity factors and the weight functions

We now turn our attention to the SIFs, $K_I(t)$ and $K_{II}(t)$, determined as

$$\sigma_{xy}(x, 0, t) \sim \frac{K_{II}(t)}{\sqrt{2\pi}} x^{-1/2}, \quad \sigma_{yy}(x, 0, t) \sim \frac{K_I(t)}{\sqrt{2\pi}} x^{-1/2}, \quad x \to 0^+. \tag{3.16}$$

The same relations hold for the Laplace-transformed stresses and SIFs

$$\hat{\sigma}_{xy}(x,s) \sim \frac{\hat{K}_{II}(s)}{\sqrt{2\pi}} x^{-1/2}, \quad \hat{\sigma}_{yy}(x,s) \sim \frac{\hat{K}_{I}(s)}{\sqrt{2\pi}} x^{-1/2}, \quad x \to 0^+. \tag{3.17}$$

By the Abelian theorem [20] and in view of (2.19),

$$\tilde{\sigma}_1^+(ps', c_l s') \sim \tilde{K}_{II}(s') e^{i\pi/4} (2ps')^{-1/2}, \quad \tilde{\sigma}_2^+(ps', c_l s') \sim \tilde{K}_{I}(s') e^{i\pi/4} (2ps')^{-1/2},$$
$$p \to \infty, \ \arg p \in (0,\pi). \tag{3.18}$$

On the other hand, analysis of the solution (3.15) shows that

$$\tilde{\sigma}_j^+(ps', c_l s') \sim -e^{-i\pi/4} \Psi_j^\circ(s') p^{-1/2}, \quad p \to \infty, \ \arg p \in (0,\pi), \tag{3.19}$$

where

$$\Psi_j^\circ(s') = \frac{1}{2\pi i} \int_L \frac{\tilde{q}_j^-(\tau s', c_l s') d\tau}{K^+(\tau)\Omega_j^+(\tau)}. \tag{3.20}$$

The consistency of formulas (3.18) and (3.19) gives the desired expressions for the Laplace transforms of the SIFs

$$\hat{K}_{II}(s') = \sqrt{2} i \Psi_1^\circ(s')(s')^{1/2}, \quad \hat{K}_{I}(s') = \sqrt{2} i \Psi_2^\circ(s')(s')^{1/2}. \tag{3.21}$$

These expressions are obtained under the assumption that s is real and positive. By continuing analytically $\hat{K}_I(s)$ and $\hat{K}_{II}(s)$ from the real positive semi-axis to the domain $\operatorname{Re} s > 0$ we define them in the whole right half-plane and may apply the inverse Laplace transform to recover the SIFs

$$K_j(t) = \frac{1}{2\pi i} \int_{\mathcal{L}} \hat{K}_j\left(\frac{s}{c_l}\right) e^{st} ds, \quad j = I, II, \tag{3.22}$$

$\mathcal{L} = \{\operatorname{Re} s = \sigma > 0, |\operatorname{Im} s| < \infty\}$.

The SIFs can also be expressed through the weight functions, $W_I(\cdot,t)$ and $W_{II}(\cdot,t)$, by

$$K_I(t) = \int_{-\infty}^{Vt} \sigma_{22}^\circ(x_1,0,t) W_I(x_1,t) dx_1, \quad K_{II}(t) = \int_{-\infty}^{Vt} \sigma_{12}^\circ(x_1,0,t) W_{II}(x_1,t) dx_1. \tag{3.23}$$

The weight functions $W_I(x_0,t)$ and $W_{II}(x_0,t)$ coincide with the SIFs determined for the special case of loading, $\sigma_{j2}^\circ(x_1,0,t) = \delta(x_1 - x_0)$, $j = 1,2$, where $\delta(x)$ is the Dirac δ-function. In this case the functions $q_j(x,s)$ and $\tilde{q}_j^-(p,s)$ introduced in (2.18) and (2.19) become

$$q_j(x,s) = \frac{1}{V} e^{s(x-x_0)/V}, \quad \tilde{q}_j^-(ps', c_l s') = \frac{e^{-s' x_0/v_l}}{c_l s'(1 + ipv_l)}. \tag{3.24}$$

The integrals (3.20) can be evaluated exactly

$$\Psi_j^\circ(s') = \frac{e^{-s x_0/v_l}}{iVs' K^+(i/v_l)\Omega_j^+(i/v_l)}, \quad j = 1,2. \tag{3.25}$$

Consequently, the Laplace transforms of the weight functions have the form

$$\hat{W}_I(x_0,s') = \frac{\sqrt{2} e^{-s' x_0/v_l}}{V\sqrt{s'} K^+(i/v_l)\Omega_2^+(i/v_l)}, \quad \hat{W}_{II}(x_0,s) = \frac{\sqrt{2} e^{-s' x_0/v_l}}{V\sqrt{s'} K^+(i/v_l)\Omega_1^+(i/v_l)}, \tag{3.26}$$

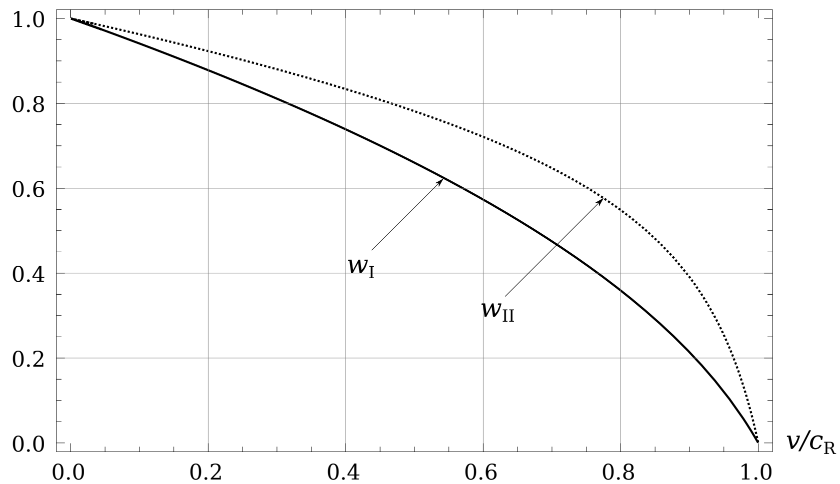

Figure 2: Graphs of the functions w_I and w_{II} versus V/c_R for $\nu = 0.3$ in the case of an unbounded plane.

Now we can evaluate the inverse Laplace transforms exactly

$$W_I(x_0,t) = \sqrt{\frac{2}{\pi(Vt-x_0)}}w_I, \quad W_{II}(x_0,t) = \sqrt{\frac{2}{\pi(Vt-x_0)}}w_{II} \tag{3.27}$$

where

$$w_j = \frac{\Gamma(1/2+1/v_l)}{\sqrt{v_l}\Gamma(1+1/v_l)\Omega^+_{3-j}(i/v_l)}, \quad j = I, II(1,2). \tag{3.28}$$

Graphs of the dimensionless functions w_I and w_{II} versus the dimensionless speed V/c_R for $\nu = 0.3$ is shown in Fig. 2. The graph of the function w_I is in good agreement with the one presented in [13], p. 349.

4 Approximate solution of the transient problem for a half-plane

Since the structure of the matrix $G(p,s)$ given by (2.21), (2.22) does not allow for its explicit factorization by the methods currently available in the literature, we propose a method of partial factorization. This technique eventually leads to a system of two integral equations convenient for the determination of the SIFs and numerical implementation.

4.1 System of integral equations

We show first that the direct use of the convolution theorem reduces the boundary condition of the vector RHP (2.20) to a system of integral equations not convenient for numerical purposes. The system has the form

$$\int_{-\infty}^{0} K(x-\xi,s)\begin{pmatrix}\hat\chi_1(\xi,s)\\ \hat\chi_2(\xi,s)\end{pmatrix}d\xi = -\frac{1}{\mu}\begin{pmatrix}q_1(x,s)\\ q_2(x,s)\end{pmatrix}, \quad -\infty < x < 0, \tag{4.1}$$

where

$$K(\xi,s)=\frac{i}{2\pi}\int_{-\infty}^{\infty}G(p,s)e^{-ip\xi}dp. \tag{4.2}$$

On analyzing the asymptotic behavior of the entries of $G(p,s)$ as $p\to 0$ and $p\to\infty$ we discover that

$$g_{jj}(p,s)\sim -\gamma_j\operatorname{sgn}p\left[1+\frac{r_j}{p}+O\left(\frac{1}{p^2}\right)\right],\quad j=1,2,$$
$$g_{12}(p,s)\sim r_0e^{-2\hat{\beta}\delta|p|},\quad p\to\pm\infty, \tag{4.3}$$

where r_j ($j=0,1,2$) are nonzero constants, and γ_j are the positive constants given by (3.8). As $p\to 0$, $g_{jj}(p,s)\sim -\tilde{\gamma}_jp^{-1}$, $g_{12}(p,s)\sim -\tilde{\gamma}_0$, ($\tilde{\gamma}_j$ are positive constants). To clarify the structure of the diagonal kernels, we represent the functions $g_{jj}(p,s)$ as

$$g_{jj}(p,s)=-\gamma_j[\coth(\pi p)+g^\circ_{jj}(p,s)], \tag{4.4}$$

where

$$g^\circ_{jj}(p,s)=\frac{r_j}{|p|}+O\left(\frac{1}{p^2}\right),\quad p\to\pm\infty,\quad g^\circ_{jj}(p,s)\sim\frac{\hat{\gamma}_j}{p},\quad p\to 0, \tag{4.5}$$

$\hat{\gamma}_j$ are constants. Because of the integral

$$\int_{-\infty}^{\infty}\coth(\pi p)e^{-ip\zeta}dp=-i\coth\frac{\zeta}{2}, \tag{4.6}$$

this ultimately brings us to the system of singular integral equations

$$\int_{-\infty}^{0}\left[\coth\frac{\xi-x}{2}+k_{jj}(x-\xi,s)\right]\hat{\chi}_j(\xi,s)d\xi$$
$$+\int_{-\infty}^{0}k_{j\,3-j}(x-\xi,s)\hat{\chi}_{3-j}(\xi,s)d\xi=-\frac{2\pi}{\mu\gamma_j}q_j(x,s),\quad -\infty<x<0,\quad j=1,2. \tag{4.7}$$

The functions $k_{jj}(\zeta,s)$ have a logarithmic singularity, and the functions $k_{j\,3-j}(\zeta,x)$ are bounded at $\zeta=0$. As $\zeta\to\infty$, all the kernels decay, $k_{ij}=O(\zeta^{-1})$. Difficulties will arise, however, if we try to use this system for computations. This is for the slow convergence of the integrals in (4.7) due to the presence of the function $\coth\zeta/2$ bounded as $\zeta\to\infty$.

To avoid dealing with such kernels, we propose another approach. First, we recast the system (4.1) into a different form. As in section 3.1, we split the diagonal entries of the matrix G

$$g_{jj}(p,s)=-\gamma_j\coth(\pi p)\breve{g}_{jj}(p,s),\quad j=1,2, \tag{4.8}$$

factorize the function $\coth(\pi p)$ as in (3.11) and the functions $\breve{g}_{jj}(p,s)$ as follows:

$$\breve{g}_{jj}(p,s)=\frac{\Omega^+_{jj}(p,s)}{\Omega^-_{jj}(p,s)},\quad p\in L_\varepsilon,$$
$$\Omega_{jj}(p,s)=\exp\left\{\frac{1}{2\pi i}\int_{L_\varepsilon}\frac{\ln\breve{g}_{jj}(\tau,s)d\tau}{\tau-p}\right\},\quad p\in D^\pm. \tag{4.9}$$

Introduce new functions

$$\breve{\sigma}^+_j(p,s)=\frac{\tilde{\sigma}^+_j(p,s)}{K^+(p)\Omega^+_{jj}(p,s)},\qquad \breve{\chi}^-_j(p,s)=\frac{\mu\tilde{\chi}^-_j(p,s)}{K^-(p)\Omega^-_{jj}(p,s)},$$
$$\breve{q}_j(p,s)=\frac{\tilde{q}^-_j(p,s)}{K^+(p)\Omega^+_{jj}(p,s)},\qquad j=1,2. \tag{4.10}$$

After the partial factorization has been implemented, in the new notations, the original vector RHP (2.20) reads

$$\begin{pmatrix} \check{\sigma}_1^+(p,s) \\ \check{\sigma}_2^+(p,s) \end{pmatrix} = \begin{pmatrix} \gamma_1 & \check{g}_1(p,s) \\ \check{g}_2(p,s) & \gamma_2 \end{pmatrix} \begin{pmatrix} \check{\chi}_1^-(p,s) \\ \check{\chi}_2^-(p,s) \end{pmatrix} + \begin{pmatrix} \check{q}_1(p,s) \\ \check{q}_2(p,s) \end{pmatrix}, \quad p \in L_\varepsilon, \tag{4.11}$$

where

$$\check{g}_1(p,s) = -\frac{ig_{12}(p,s)\Omega_{22}^-(p,s)}{\coth(\pi p)\Omega_{11}^+(p,s)}, \quad \check{g}_2(p,s) = \frac{ig_{12}(p,s)\Omega_{11}^-(p,s)}{\coth(\pi p)\Omega_{22}^+(p,s)}, \tag{4.12}$$

and the functions $\Omega_{jj}^\pm(p,s)$ are defined by the Sokhotski-Plemelj formulas

$$\Omega_{jj}^\pm(p,s) = \exp\left\{\pm\frac{1}{2}\ln \check{g}_{jj}(p,s) + P.V.\Omega_{jj}(p,s)\right\}, \quad p \in L_\varepsilon. \tag{4.13}$$

Assume first that x is negative. On applying the convolution theorem to (74) we conclude that the vector RHP (4.11) yields

$$\gamma_1\chi_1^*(x,s) + \int_{-\infty}^0 k_1^*(x-\xi,s)\chi_2^*(\xi,s)d\xi = -q_1^*(x,s),$$

$$\gamma_2\chi_2^*(x,s) + \int_{-\infty}^0 k_2^*(x-\xi,s)\chi_1^*(\xi,s)d\xi = -q_2^*(x,s), \quad -\infty < x < 0. \tag{4.14}$$

Here,

$$\chi_j^*(x,s) = \frac{1}{2\pi}\int_{L_\varepsilon} \check{\chi}_j^-(p,s)e^{-ipx}dp, \quad k_j^*(x,s) = \frac{1}{2\pi}\int_{L_\varepsilon} \check{g}_j(p,s)e^{-ipx}dp,$$

$$q_j^*(x,s) = \frac{1}{2\pi}\int_{L_\varepsilon} \check{q}_j(p,s)e^{-ipx}dp, \quad j=1,2. \tag{4.15}$$

Because of the asymptotics (4.3) of the function $g_{12}(p,s)$, the functions $\check{g}_j(p,s)$ decay exponentially as $p \to \pm\infty$. It also follows from (4.12) that the functions $\check{g}_j(p,s)$ are continuously differentiable on the whole real axis and therefore $|k_j^*(x,s)| \le cx^{-2}$ when $x \to -\infty$ (c is a function of s and independent of x). By the Riemann-Lebesgue lemma, the functions $\chi_j^*(x,s) \to 0$ as $x \to -\infty$.

If $x > 0$, then the convolution theorem applied to (4.11) gives

$$\int_{-\infty}^0 k_1^*(x-\xi,s)\chi_2^*(\xi,s)d\xi = \sigma_1^*(x,s) - q_1^*(x,s),$$

$$\int_{-\infty}^0 k_2^*(x-\xi,s)\chi_1^*(\xi,s)d\xi = \sigma_2^*(x,s) - q_2^*(x,s), \quad 0 < x < \infty, \tag{4.16}$$

where

$$\sigma_j^*(x,s) = \frac{1}{2\pi}\int_{L_\varepsilon} \check{\sigma}_j^+(p,s)e^{-ipx}dp, \quad j=1,2. \tag{4.17}$$

On employing the continuity of the convolutions $k_1 * \chi_2^*$ and $k_2 * \chi_1^*$ and concatenating equations (4.14) and (4.16) at $x=0$ we establish the important relations

$$\gamma_j\chi_j^*(0^-,s) = -\sigma_j^*(0^+,s) + q_j^*(0^+,s) - q_j^*(0^-,s), \quad j=1,2, \tag{4.18}$$

to be used in the next section for computing the weight functions.

4.2 Weight functions

Because the mode-I and mode-II are coupled we have four weight functions, $W_{I,I}$, $W_{I,II}$, $W_{II,I}$, and $W_{II,II}$. Through them, the SIFs are found to be

$$K_I(t) = \int_{-\infty}^{Vt} W_{I,I}(x_1,t)\sigma_{22}^\circ(x_1,0)dx_1 + \int_{-\infty}^{Vt} W_{I,II}(x_1,t)\sigma_{12}^\circ(x_1,0)dx_1,$$

$$K_{II}(t) = \int_{-\infty}^{Vt} W_{II,I}(x_1,t)\sigma_{22}^\circ(x_1,0)dx_1 + \int_{-\infty}^{Vt} W_{II,II}(x_1,t)\sigma_{12}^\circ(x_1,0)dx_1. \tag{4.19}$$

The values of the weight functions $W_{I,I}$ and $W_{II,I}$ at a point x_0 coincide with the SIFs K_I and K_{II}, respectively if $\sigma_{22}^\circ(x_1,0) = \delta(x_1 - x_0)$ and $\sigma_{12}^\circ(x_1,0) = 0$. Similarly, if $\sigma_{22}^\circ(x_1,0) = 0$ and $\sigma_{12}^\circ(x_1,0) = \delta(x_1 - x_0)$, then the SIFs K_I, K_{II} are equal to the other two weight functions $W_{I,II}(x_0,t)$ and $W_{II,II}(x_0,t)$, respectively. As in section 3.2, the transforms of the traction components, $\tilde{\sigma}_1^+(p,s)$ and $\tilde{\sigma}_2^+(p,s)$, have the asymptotics

$$\tilde{\sigma}_1^+(p,s) \sim \tilde{K}_{II}(s)e^{i\pi/4}(2p)^{-1/2}, \quad \tilde{\sigma}_2^+(p,s) \sim \tilde{K}_I(s)e^{i\pi/4}(2p)^{-1/2},$$
$$p \to \infty,\ \arg p \in (0,\pi). \tag{4.20}$$

To express the asymptotics of these functions through the solution to the system of integral equations (4.14), we note that due to the relation (4.18), the continuity of $q_j^*(x,s)$ at $x=0$ and formulas (4.17), (4.15) and (4.12)

$$\check{\sigma}_j^+(p,s) \sim \frac{\gamma_j \chi_j^*(0^-,s)}{ip}, \quad p\to\infty, \quad p \in D^+. \tag{4.21}$$

Now we employ formulas (4.10) to discover

$$\tilde{\sigma}_j^+(p,s) \sim -e^{i\pi/4}\gamma_j\chi_j^*(0^-,s)p^{-1/2}, \quad p\to\infty, \quad p\in D^+. \tag{4.22}$$

In view of (4.20) this momentarily gives the SIFs

$$\hat{K}_I(s) = -\sqrt{2}\gamma_2\chi_2^*(0^-,s), \quad \hat{K}_{II}(s) = -\sqrt{2}\gamma_1\chi_1^*(0^-,s). \tag{4.23}$$

Therefore, to determine the Laplace transforms of the SIFs, we have to know the solution of the system (4.14) at the point $x=0$ only. This system needs to be solved for the special right-hand side,

$$q_1^*(x,s) = 0, \quad q_2^*(x,s) = \frac{e^{s(x-x_0)/V}}{VK^+(is/V)\Omega_{22}^+(is/V,s)}, \tag{4.24}$$

in the case of the weight functions $W_{I,I}$ and $W_{II,I}$ and for

$$q_1^*(x,s) = \frac{e^{s(x-x_0)/V}}{VK^+(is/V)\Omega_{11}^+(is/V,s)}, \quad q_2^*(x,s) = 0, \tag{4.25}$$

in the case of the weight functions $W_{I,II}$ and $W_{II,II}$.

The SIFs are recovered from their Laplace transform by the inversion formula (3.22). The inversion can be implemented by computing one of the real integrals

$$K_j(t) = \frac{2e^{\sigma t}}{\pi}\int_0^\infty \mathrm{Re}\{\hat{K}_j(\sigma+i\tau)\}\cos\tau t\,d\tau,$$

$$K_j(t) = -\frac{2e^{\sigma t}}{\pi}\int_0^\infty \mathrm{Im}\{\hat{K}_j(\sigma+i\tau)\}\sin\tau t\,d\tau, \quad j=I,II, \tag{4.26}$$

and the preference should be made to the one with the better rate of convergence.

4.3 Numerical results

Here we describe the numerical procedure for evaluation of the weight functions. We recall that the weight functions coincide with the SIFs provided loading is chosen as it was described in section 4.2. Due to formulas (4.23) the Laplace transforms of the SIFs require the knowledge of the solution of the system of integral equations (4.14) at the point $x = 0$, that is $\chi_j^*(0^-, s)$, $j = 1, 2$. We find it convenient to map the system (4.14) on the semi-infinite interval into another one on the interval (-1,1). This is achieved by introducing the variables

$$\xi = \frac{\xi' - 1}{\xi' + 1}, \quad -1 < \xi' < 1, \quad x = \frac{x' - 1}{x' + 1}, \quad -1 < x' < 1. \tag{4.27}$$

The new system is easily seen to be

$$\gamma_j \mathcal{X}_j(x', s) + \int_{-1}^{1} \mathcal{K}_{3-j}(x', \xi', s)\mathcal{X}_j(\xi', s) d\xi' = -\mathcal{Q}_j(x', s), \quad -1 < x' < 1, \quad j = 1, 2, \tag{4.28}$$

where

$$\mathcal{X}_j(x', s) = \chi_j^*(x, s), \quad \mathcal{K}_j(x', \xi', s) = \frac{2k_j^*(x - \xi, s)}{(\xi' + 1)^2}, \quad \mathcal{Q}_j(x', s) = q_j^*(x, s). \tag{4.29}$$

We note that due to the asymptotics of the original kernels $k_j^*(x, s) = O(x^{-2})$ as $x \to -\infty$, the new kernels $\mathcal{K}_j(x', \xi', s)$ are bounded as $\xi' \to -1$. This circumstance implies that the system (4.28) can be approximately solved by using the collocation method with the collocation points ξ_k $(k = 1, 2, \ldots, N)$ chosen to be the zeros of the degree-N Legendre polynomial $P_N(x)$. The system of $2N$ linear algebraic equations associated with the system (4.28) has the form

$$\gamma_j \mathcal{X}_j(x_n, s) + \sum_{k=1}^{N} v_k \mathcal{K}_{3-j}(x_n, x_k, s)\mathcal{X}_{3-j}(x_k, s) = -\mathcal{Q}_j(x_n, s),$$

$$n = 1, 2, \ldots, N, \quad j = 1, 2, \tag{4.30}$$

where v_k are the Gauss-Legendre weights given by $v_k = 2(1 - x_k^2)^{-1}[P_N'(x_k)]^{-2}$.

The chief difficulty in the implementation of this procedure is the evaluation of the principal value of the integrals in (4.13), $P.V.\Omega_{jj}(p, s)$. It is helpful to recast them as integrals over the arc $l = \{|p'| = 1, \arg p' \in (-\pi/2.\pi/2)\}$

$$\Omega_{jj}(p, s) = \exp\left\{\frac{1 + p'}{2\pi i} P.V. \int_l \frac{\Gamma_j(\tau', s) d\tau'}{\tau' - p'}\right\}, \tag{4.31}$$

where

$$\Gamma_j(\tau', s) = \frac{\ln \check{g}_{jj}(\tau, s)}{1 + \tau'}, \quad \tau' = \frac{1 + i\tau}{1 - i\tau}, \quad p' = \frac{1 + ip}{1 - ip}. \tag{4.32}$$

Among numerous approximate formulas for the principal value of the Cauchy integral over a circle we choose the following one [21], p.116:

$$\Omega_{jj}(p, s) = \exp\left\{\frac{1 + p'}{2M + 1} \sum_{j=-M}^{M} \Gamma_j(e^{i\theta_j}, s) \left[\frac{1}{2} + \frac{i \sin\frac{M}{2}(\theta - \theta_j) \sin\frac{M+1}{2}(\theta - \theta_j)}{\sin\frac{1}{2}(\theta - \theta_j)}\right]\right\}, \tag{4.33}$$

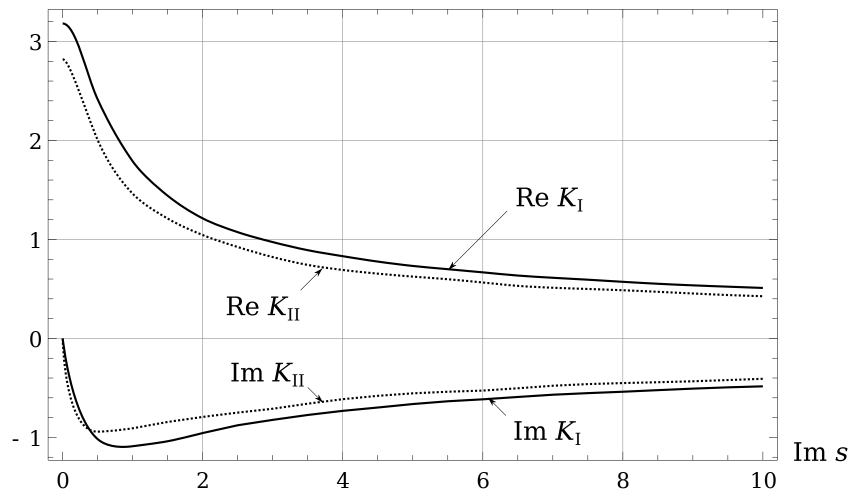

Figure 3: Graph of the functions $\operatorname{Re} K_I(s)$, $\operatorname{Re} K_{II}(s)$, $\operatorname{Im} K_I(s)$, and $\operatorname{Im} K_{II}(s)$ for $\operatorname{Re} s = 0.5$.

for being simple and proving a good accuracy. Here, $\theta = -i \ln p'$, $\theta_j = 2\pi j/(2M+1)$.

The final step in the evaluation of the weight functions or, equivalently, the SIFs K_I and K_{II} with the special loads applied, is the inversion of the Laplace transform. This can be done by applying one of the formulas in (4.26). For computations, we employ the uniform grid trapezoidal rule with $m+1$ grid points

$$K_j(t) \approx \frac{he^{s_0 t}}{\pi}\left[\operatorname{Re}\hat{K}_j(\sigma) + \operatorname{Re}\hat{K}_j(\sigma + iT)\cos Tt + 2\sum_{n=1}^{m-1}\operatorname{Re}\hat{K}_j(\sigma + inh)\cos nht\right], \quad (4.34)$$

where h is the grid spacing. Our numerical results show (Fig. 3) that the rate of convergence is slow for both, the real and imaginary parts. To accelerate the convergence, we apply the Euler summation method for alternating series. To transform (4.34) into an alternating sum, we put $h = \pi/(2t)$, $\sigma = A/(2t)$ and $T = \pi m/(2t)$, where A is a fixed real positive constant. Then [1]

$$K_j(t) \approx \frac{e^{A/2}}{2t}\left[\operatorname{Re}\hat{K}_j\left(\frac{A}{2t}\right) + \operatorname{Re}\hat{K}_j\left(\frac{A + i\pi m}{2t}\right)\cos\frac{\pi m}{2} + 2\sum_{n=1}^{m-1}(-1)^n\frac{\Delta_n}{2^{n+1}}\right], \quad (4.35)$$

where

$$\Delta_n = \sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) \operatorname{Re}\left\{\hat{K}_j\left(\frac{A + 2(n-k)\pi i}{2t}\right)\right\}. \quad (4.36)$$

In our computations, following [1], we take $A = 8\ln 10$.

Fig. 4 and 5 show how the functions $w_{i,j}(x_0, t) = \sqrt{\frac{1}{2}\pi(Vt - x_0)}W_{i,j}(x_0, t)$ and the weight functions $W_{i,j}(x_0, t)$ evolve with time. For computations, we assume x_0 to be zero, that is the time independent concentrated loads are applied at the point $x_1 = x_2 = 0$, the tip of the crack at time $t = 0$. Since the material is stress-free for $t < 0$, it is expected that, when the crack starts propagating at constant speed V, the elastic medium remains stress-free outside the disc of radius $c_l t$ centered at the point $x_1 = x_2 = 0$. At time $t_l' = \delta/c_l$, the first longitudinal wave strikes the boundary of the half-plane at the right angle, and

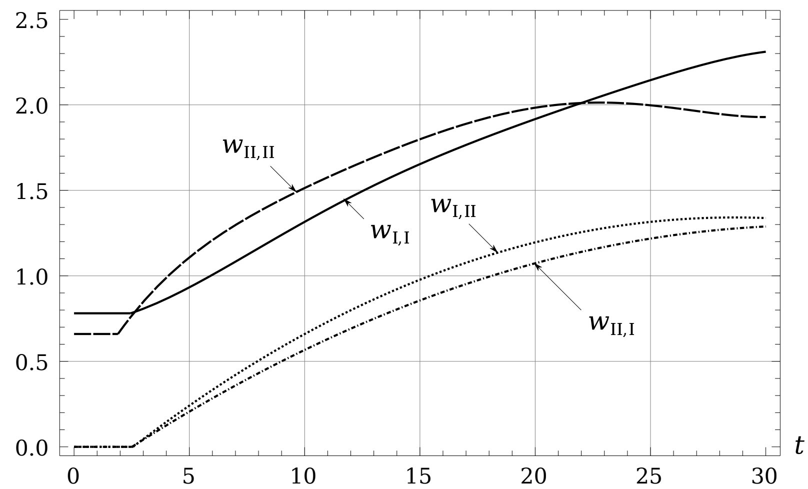

Figure 4: The functions $w_{i,j}(0,t) = \sqrt{\frac{1}{2}\pi V t} W_{i,j}(0,t)$ $(i,j = I, II)$ versus time t when $\nu = 0.3$, $\delta = 1\,\mathrm{m}$, $V = 0.5c_R\,\mathrm{m/s}$, $c_l = 1\,\mathrm{m/s}$ ($c_s \approx 0.5345\,\mathrm{m/s}$, $c_R \approx 0.4957\,\mathrm{m/s}$).

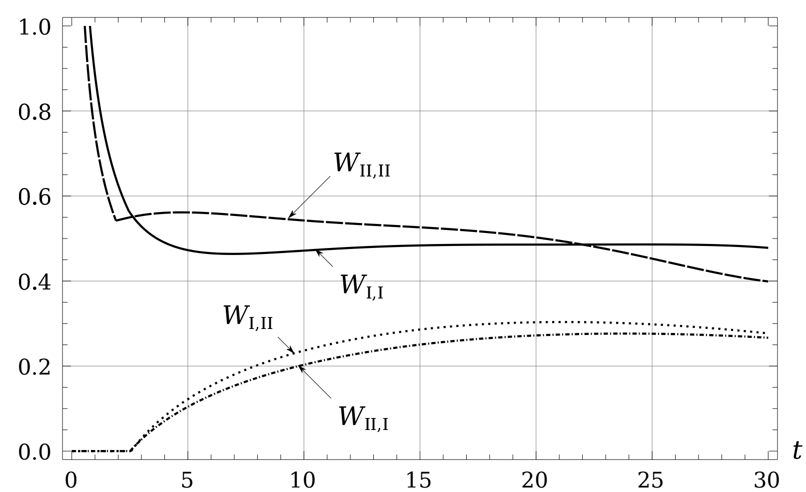

Figure 5: The weight functions $W_{i,j}(0,t)$, $(i,j = I, II)$ versus time t when $\nu = 0.3$, $\delta = 1\,\mathrm{m}$, $V = 0.5c_R\,\mathrm{m/s}$, $c_l = 1\,\mathrm{m/s}$ ($c_s \approx 0.5345\,\mathrm{m/s}$, $c_R \approx 0.4957\,\mathrm{m/s}$).

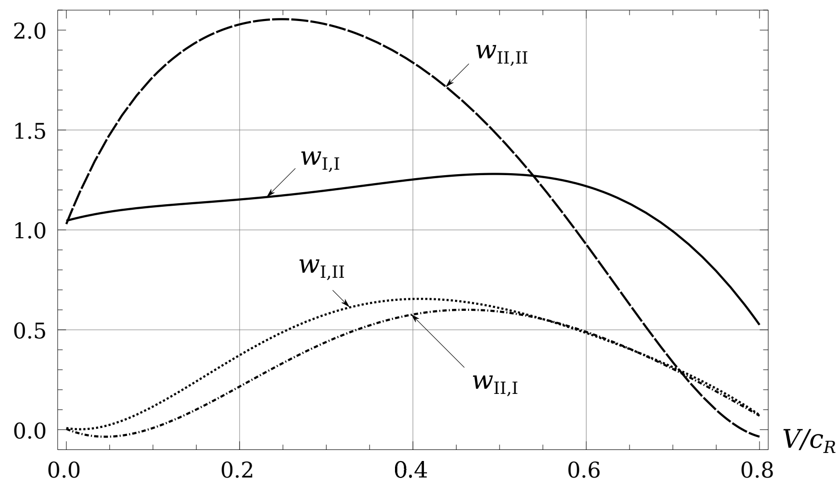

Figure 6: The functions $w_{i,j}(0,t)$ $(i,j = I, II)$ versus V/c_R when $\nu = 0.3$, $\delta = 1\,\mathrm{m}$, $t = 10s$, $c_l = 1\,\mathrm{m/s}$ ($c_s \approx 0.5345\,\mathrm{m/s}$, $c_R \approx 0.4957\,\mathrm{m/s}$).

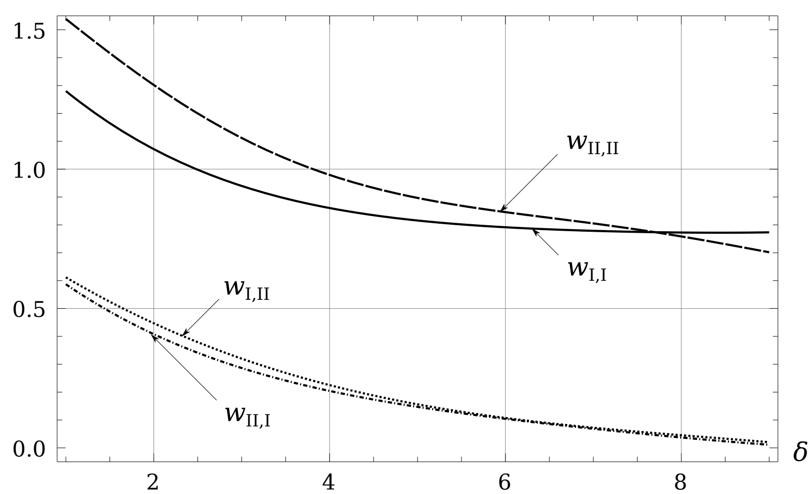

Figure 7: The functions $w_{i,j}(0,t)$ $(i,j = I, II)$ versus the distance δ from the crack to the half-plane boundary when $\nu = 0.3$, $V = 0.5c_R\,\mathrm{m/s}$, $c_l = 1\,\mathrm{m/s}$, $t = 10\,\mathrm{s}$ ($c_s \approx 0.5345\,\mathrm{m/s}$, $c_R \approx 0.4957\,\mathrm{m/s}$).

at time $t = 2t'_0$, it returns to the origin $x_1 = y_1 = 0$. By that time, the crack tip has run the distance $2Vt'_l$, and the distortion caused by the reflected wave reaches the crack tip at time $t^*_l > 2t'_l$ (for $\delta >> 1$, $t^*_l \sim 4t'_1$). The shear waves propagate slower, and the corresponding time, when the shear wave incident normally alters the SIFs, is greater than $2\delta/c_s > t^*_l$. Due to other longitudinal waves reflected from the boundary at acute angles, the actual time when the boundary affects the SIFs may be smaller than t^*_l. The time when the reflected longitudinal wave strikes the crack at its tip can be quickly evaluated. Let this wave hit the boundary of the half-plane at time $t = t_l$ at angle θ ($\theta \in (\pi/2, \pi)$ is measured from the incident wave direction to the boundary of the half-plane) (Fig. 1). Then the reflected wave strikes the crack tip at time $t = 2t_l$. By that time, the crack has run the distance $2Vt_l$, and therefore, $\sqrt{c_l^2 t_l^2 - \delta^2} = Vt_l$. This implies

$$t_l = \frac{\delta}{\sqrt{c_l^2 - V^2}}, \quad \theta = \frac{\pi}{2} + \tan^{-1} \frac{1}{\sqrt{1/v_l^2 - 1}}. \tag{4.37}$$

For the example used for drawing Fig. 4 and 5, $\delta = 1\,\text{m}$, $V = 0.5c_R$, and $c_R \approx 0.4957\,\text{m/s}$. Simple calculations show that $2t_l \approx 2.0644\,\text{s}$ and $\theta = 1.8213$. This time is consistent with the time $2t_l \approx 2\,\text{s}$ discovered from the approximate solution. Our calculations (Fig. 4 and 5) show that for time $0 < t < 2t_l$, the functions $w_{ii}(0, t)$ ($i = I, II$) are constant and practically coincide with the parameters w_i associated with the mode-I and II weight functions for the whole plane and given by (3.28). The mixed mode functions $w_{I,II}(0, t)$ and $w_{II,I}(0, t)$ are very close to zero when $0 < t < 2t_l$. The weight functions $W_{ij}(t, 0)$ approximately equal the corresponding weight functions of the problem on the whole plane for $0 < t < 2t_l$. At time $t = 2t_l$, the graphs of the weight functions associated with the half-plane and the plane start to diverge.

The functions $w_{i,j}(0, t)$ versus the dimensionless speed V/c_R are plotted in Fig. 6. As in the case of the whole plane, the functions $w_{I,I}$ and $w_{II,II}$ tend to 1 and 0 when $V/c_R \to 0$ and $V/c_R \to 1$, respectively, while the off-diagonal functions, $w_{I,II}$ and $w_{II,I}$ tend to zero not only when $V/c_R \to 1$, but also when $V/c_R \to 0$. In the case of the whole plane, the functions w_I and w_{II} are monotonic, while in the case of the half-plane, they are not.

When the distance δ from the crack to the boundary of the half-plane decreases, all the four functions $w_{i,j}(0, t)$ grow (see Fig. 7). As it is expected, when $\delta \to \infty$, the functions $w_{i,j}$ approach their limits, the corresponding functions for the whole plane, $w_{I,II} \to 0$, $w_{II,I} \to 0$, and when $\nu = 0.3$, $w_{I,I} \to w_I = 0.781473$, $w_{II,II} \to w_{II} = 0.659882$.

5 Crack growth at nonuniform speed beneath the boundary

With the fundamental solution and weight functions at hand derived and computed in the previous sections, we come now to the problem on nonuniform motion of a semi-infinite crack parallel to the boundary of a half-plane. In order to do this, first we want to describe the motion of the crack when speed, $V(t)$, is a prescribed smooth function for $t > 0$. For this we adjust the approximate method proposed in [13] for a semi-infinite crack moving at variable speed in an unbounded body.

5.1 Piece-wise constant prescribed speed

Suppose at time $t = 0$ the crack starts moving, and its position at time t is described by $l(t)$, a continuously differentiable, nondecreasing function such that $V(t) = l'(t) < c_R$. We approximate the curve $l(t)$ by a polygonal line with the vertices (t_k, l_k), $l_k = l(t_k)$, $t_0 = 0$, $l_0 = 0$. Denote $V_k = (l_{k+1} - l_k)/(t_{k+1} - t_k)$ the corresponding constant speed during the time $t_k < t < t_{k+1}$.

Initially, as $0 < t < t_1$, the crack extends at speed $V_0 = \text{const}$ by negating the stresses $\sigma^0_{12}(x_1, 0)$ and $\sigma^0_{22}(x_1, 0)$ for $x_1 > 0$. They are determined from the solution of the static problem, P_{-1}, on a semi-infinite crack parallel to the boundary of a half-plane. This problem provides the starting point for a complete description of the nonuniform motion of the crack. An exact method of matrix Wiener-Hopf factorization for this problem was presented in [26] for the case when the forces were applied to the strip at infinity, and the boundary was free of traction. These authors reduced the problem to a homogeneous order-2 vector RHP, solved it exactly and found the SIFs. On employing their method it is possible to derive the exact solution of the inhomogeneous RHP for general loading and determine the stresses everywhere in the body including the line $x_2 = 0$ ahead of the crack. We accept that the solution to Problem P_{-1} is already available.

Coming back now to the problem on a moving crack assume that the crack suddenly stops at time $t = t_1$ at the point $x_1 = l_1, x_2 = 0$. Obviously, some stresses, $\tilde\sigma^1_{12}(x_1, 0)$ and $\tilde\sigma^1_{22}(x_1, 0)$, are radiated out along the line $x_2 = 0$, $x_1 > l_1$. These stresses are unknown *a priori* and must be determined. To continue its motion, the crack negates these unknown stresses. This results in vanishing the SIFs when $x_1 = V_0 t > l_1$,

$$K_I(t; V_0) = 0, \quad K_{II}(t; V_0) = 0, \quad V_0 t > l_1, \tag{5.1}$$

and a necessity of solving a transient problem, P_0, arises. The statement of Problem P_0 coincides with that given in Section 2 with the exception that $V = V_0$ and the boundary conditions (2.1) on the faces of the crack read

$$\sigma_{j2} = -\sigma^0_{j2}(x_1, 0)\chi_{(0,l_1)}(x_1) + \tilde\sigma^1_{j2}(x_1, 0)\chi_{(l_1,V_0 t)}(x_1), \quad -\infty < x_1 < V_0 t, \quad x_2 = 0^{\pm}, \tag{5.2}$$

with $\tilde\sigma^1_{j2}(x_1, 0)$ to be recovered from equations (5.1). Here, $\chi_{(a_1,a_2)}(x_1) = 1$ if $x_1 \in (a_1, a_2)$ and $\chi_{(a_1,a_2)}(x_1) = 0$ otherwise. To solve equations (5.1), we note the following remarkable property of the weight functions

$$W_{i,j}(x_0, t; V) = W_{i,j}(0, t - x_0/V; V), \quad i, j = I, II. \tag{5.3}$$

To show this, we recall that due to (3.24) the Laplace transforms of the loads for the weight functions are given by

$$q_j(x, s) = e^{-sx_0/V} \frac{e^{sx/V}}{V}, \quad j = 1, 2, \tag{5.4}$$

and consequently from (2.19), (4.10) and (4.15) we derive the relations

$$\begin{aligned}
&\tilde q^-_j(p, s; x_0) = e^{-sx_0/V}\tilde q^-_j(p, s; 0), \quad \check q_j(p, s; x_0) = e^{-sx_0/V}\check q^-_j(p, s; 0),\\
&\check\chi^-_j(p, s; x_0) = e^{-sx_0/V}\check\chi^-_j(p, s; 0), \quad \chi^*_j(x, s; x_0) = e^{-sx_0/V}\chi^*_j(x, s; 0),
\end{aligned} \tag{5.5}$$

The latter formula and also (4.23) imply that the Laplace transforms of the weight functions satisfy the equation

$$\hat W_{i,j}(x_0, s; V) = e^{-sx_0/V}\hat W_{i,j}(0, s; V), \quad i, j = I, II. \tag{5.6}$$

and the relation (5.3) holds. Combining these results we can write down formulas (4.19) for the SIFs in the form

$$K_I(t; V_0) = -K'(t; V_0) + \int_{l_1}^{V_0 t} [W_{I,I}(0, t - x_1/V_0; V_0)\tilde\sigma^1_{22}(x_1, 0)$$

$$+W_{I,II}(0, t - x_1/V_0; V_0)\tilde{\sigma}^1_{12}(x_1, 0)]dx_1,$$

$$K_{II}(t; V_0) = -K''(t; V_0) + \int_{l_1}^{V_0 t} [W_{II,I}(0, t - x_1/V_0; V_0)\tilde{\sigma}^1_{22}(x_1, 0)$$

$$+W_{II,II}(0, t - x_1/V_0; V_0)\tilde{\sigma}^1_{12}(x_1, 0)]dx_1, \tag{5.7}$$

where $K'(t; V_0)$ and $K''(t; V_0)$ are known functions

$$K'(t; V_0) = \int_0^{l_1} [W_{I,I}(0, t - x_1/V_0; V_0)\sigma^0_{22}(x_1, 0) + W_{I,II}(0, t - x_1/V_0; V_0)\sigma^0_{12}(x_1, 0)]dx_1,$$

$$K''(t; V_0) = \int_0^{l_1} [W_{II,I}(0, t - x_1/V_0; V_0)\sigma^0_{22}(x_1, 0) + W_{II,II}(0, t - x_1/V_0; V_0)\sigma^0_{12}(x_1, 0)]dx_1. \tag{5.8}$$

We see now that the property (5.3) allows for an exact solution of the system (5.1) by transforming it into a system of two Volterra convolution equations and applying the Laplace transform. Indeed, with a change of the variables

$$x_1 = V_0\tau' + l_1, \quad t = \tau + l_1/V_0, \tag{5.9}$$

and the functions to be found

$$\pi_I(\tau') = \tilde{\sigma}^1_{22}(V_0\tau' + l_1, 0), \quad \pi_{II}(\tau') = \tilde{\sigma}^1_{12}(V_0\tau' + l_1, 0), \tag{5.10}$$

the system (5.1) reads

$$\sum_{j=I}^{II} \int_0^{\tau} W_{i,j}(0, \tau - \tau'; V_0)\pi_i(\tau')d\tau' = \omega_i(\tau), \quad \tau > 0, \quad i = I, II, \tag{5.11}$$

where

$$\omega_I(\tau) = V_0^{-1}K'(\tau + l_1/V_0; V_0), \quad \omega_{II}(\tau) = V_0^{-1}K''(\tau + l_1/V_0; V_0). \tag{5.12}$$

The Laplace images $\hat{\pi}_i(s)$ of the unknown functions $\pi_i(\tau')$ can be easily recovered from the system of linear algebraic equations

$$\sum_{j=I}^{II} \hat{W}_{i,j}(0, s; V_0)\hat{\pi}_i(s) = \hat{\omega}_i(s), \quad i = I, II. \tag{5.13}$$

On performing the Laplace inversion we obtain

$$\pi_I(\tau') = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\hat{W}_{II,II}(0, s; V_0)\hat{\omega}_I(s) - \hat{W}_{I.II}(0, s; V_0)\hat{\omega}_{II}(s)}{\hat{W}(s; V_0)} e^{s\tau'} ds,$$

$$\pi_{II}(\tau') = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\hat{W}_{I,I}(0, s; V_0)\hat{\omega}_{II}(s) - \hat{W}_{II.I}(0, s; V_0)\hat{\omega}_I(s)}{\hat{W}(s; V_0)} e^{s\tau'} ds, \tag{5.14}$$

where $\sigma > 0$ and

$$\hat{W}(s; V_0) = \hat{W}_{I,I}(0, s; V_0)\hat{W}_{II,II}(0, s; V_0) - \hat{W}_{I,II}(0, s; V_0)\hat{W}_{II,I}(0, s; V_0). \tag{5.15}$$

Thus the stresses to be nullified for $x_1 > l_1$ have the form

$$\tilde{\sigma}^1_{22}(x_1, 0) = \pi_I\left(\frac{x_1 - l_1}{V_0}\right), \quad \tilde{\sigma}^1_{12}(x_1, 0) = \pi_{II}\left(\frac{x_1 - l_1}{V_0}\right), \quad x_1 > l_1. \tag{5.16}$$

We note that the Laplace transforms $\hat{W}_{i,j}(0,s;V_0)$ have already been determined. They are expressed through the solution at the point 0 of the system of integral equations (4.16) by (4.23) as $V = V_0$ with the loading $\sigma^\circ_{22}(x_1,0) = \delta(x_1)$ and $\sigma^\circ_{12}(x_1,0) = 0$ for the weight functions $W_{I,I}(0,t;V_0)$ and $W_{II,I}(0,t;V_0)$ and with $\sigma^\circ_{22}(x_1,0) = 0$ and $\sigma^\circ_{12}(x_1,0) = \delta(x_1)$ for the functions $W_{I,II}(0,t;V_0)$ and $W_{II,II}(0,t;V_0)$.

In addition to nullifying the stresses $\tilde{\sigma}^1_{j2}(x_1,0)$, $j = 1,2$, the solution for a suddenly stopped crack has to generate zero displacement jumps through the line $x_2 = 0$ on the segment $l_1 < x_1 < V_0 t$. In contrast to the whole plane problem, when this is possible to verify analytically [13] for the sub-Rayleigh speeds and [16] for the transonic regime), it is not visible how it can be done without deploying computer based computations. That is why this condition needs to be tested numerically when the algorithm is applied.

By employing this procedure for the next period of time, $t_1 < t < t_2$, and determining the weight functions associated with speed $V = V_1$ we can find the loads $\tilde{\sigma}^2_{i2}(x_1,0)$ $(i = 1,2)$ needed to negate the stresses generated by the crack when it suddenly stops at the point $x_1 = l_2$. The boundary conditions (5.2) for the corresponding problem P_1 read

$$\sigma_{j2} = -\sigma^1_{j2}(x_1,0)\chi_{(l_1,l_2)}(x_1) + \tilde{\sigma}^2_{j2}(x_1,0)\chi_{(l_2,V_1t)}(x_1), \quad -\infty < x_1 < V_1 t, \quad x_2 = 0^\pm, \tag{5.17}$$

where the traction components $\sigma^1_{j2}(x_1,0)$ are known

$$\sigma^1_{j2}(x_1,0) = \sigma^0_{j2}(x_1,0) + \tilde{\sigma}^1_{j2}(x_1,0), \tag{5.18}$$

while the components $\tilde{\sigma}^2_{j2}(x_1,0)$ have to be recovered from the system of two equations $K_I(t;V_1) = 0$, $K_{II}(t;V_1) = 0$, $V_1 t > l_2$, that is equivalent to the corresponding system of two Volterra equations solvable by the Laplace transform as in the previous step.

Following the pattern established above, this procedure can be continued further up to any period of time (t_k, t_{k+1}). It gives an approximate solution of the problem on motion of a semi-infinite crack beneath the boundary at piecewise constant speed $V = V_i$, $t \in (t_i, t_i + 1)$, $i = 0,1,\dots,k$, that approximates the original smooth function $V(t)$. The solution of this model problem, P, is obtained by summing up the solutions of all Problems P_i $(i = -2,-1,0,1,\dots,k)$, where P_{-2} is the elementary problem on a half-plane without a crack with somehow prescribed traction on the boundary and internally loaded; its exact solution is available in the literature, P_{-1} is the static problem for the semi-infinite crack with the traction components on the crack faces being prescribed such that they negate the corresponding stresses coming from Problem P_{-2}. Problems P_i $(i = 0,1,\dots,k-1)$ are the transient problems with the boundary conditions chosen accordingly. The last problem P_k is also a transient problem whose boundary conditions pattern is different from P_i $(i = 0,1,\dots,k-1)$ since they do not have stresses to be determined from the solution on a suddenly stopped crack. The boundary conditions have the form

$$\sigma_{j2} = -\left(\sigma^0_{j2}(x_1,0) + \sum_{i=1}^{k} \tilde{\sigma}^i_{j2}(x_1,0)\right)\chi_{(l_k,V_kt)}(x_1), \quad -\infty < x_1 < V_k t, \quad x_2 = 0^\pm, \tag{5.19}$$

Clearly, for the total problem P, the homogeneous boundary conditions on the crack faces $\{0 < x_1 < l_k, x_2 = 0^\pm\}$ are satisfied. As for the SIFs at the tip of the crack at time $t \in (t_k, t_{k+1})$, when the crack moves at speed V_k, in general, they do not vanish, depend on time and are defined by the SIFs generated by Problem P_k.

A feature of Problem P is in the presence of the boundary. As it was pointed out in the previous section, initially, when $t < 2t_l$ (t_l is given by (4.37)), and when the longitudinal wave reflected from the half-plane boundary has not reached the crack, the off-diagonal weight functions $W_{I,II}$ and $W_{II,I}$ vanish, and the diagonal functions $W_{I,I}$ and $W_{II,II}$ coincide with those associated with the problem on the whole plane with a crack. Therefore, for this short period of time, the algorithm proposed in [13] can be repeated without any changes. However, this does not mean that the actual motion of the crack in a half-plane will be the same as in the whole plane even for time $t < 2t_l$. To make this conclusion, we need to recall that the boundary conditions of Problem P_0 depend on the stresses $\sigma^0_{i2}(x_1, 0)$ $(i = 1, 2)$ generated by the static crack in the half-plane which are apparently not the same as the ones associated with the whole plane. When time exceeds $2t_l$, then, in general, all the weight functions $W_{i,j}$ are nonzero and different from those associated with the whole plane. In this case to describe the nonuniform crack motion, the algorithm we exposed needs to be applied.

5.2 Determination of propagation speed

In the previous section, the pice-wise constant speed $V(t)$ was supposed to be prescribed. Assume now that it is unknown and determine it by employing the Griffits dynamic criterion [24, 6]. Let $\delta U(t)$ be the potential energy released when the crack $S_0(t) = \{-\infty < x_1 < V_0 t,\ x_2 = 0\}$ extends to $S_0(t) + \delta S_0(t) = \{\mathbf{x} : -\infty < x_1 < V_0 t + r,\ x_2 = 0\}$, where r is small. The energy $\delta U(t)$ may be expressed as

$$\delta U(t) = \frac{1}{2}\int_0^r \{\sigma_{xy}(x, 0, t)\delta[u](x, t) + \sigma_{yy}(x, 0, t)\delta[v](x, t)\}dx. \tag{5.20}$$

Here, $[u] + \delta[u]$, $[v] + \delta[v]$ are the displacement jumps related to the extended crack, and

$$\sigma_{xy} \sim \frac{K_{II}(t)}{\sqrt{2\pi x}}, \quad \sigma_{yy} \sim \frac{K_I(t)}{\sqrt{2\pi x}}, \quad x \in (0, r), \quad r \to 0^+. \tag{5.21}$$

To find asymptotic expansions for $\delta[u]$, $\delta[v]$, we employ the relations (4.10), take into account that

$$\check{\chi}_j^-(p, s) \sim \frac{1}{ip}\chi_j^*(0^-, s), \quad p \to \infty, \quad \arg p \in (-\pi, 0), \tag{5.22}$$

and also formulas (4.23). This and the Tauberian theorem eventually bring us to

$$\hat{\chi}_1(x, s) \sim -\frac{\hat{K}_{II}(s)}{\mu\gamma_1\sqrt{-2\pi x}}, \quad \hat{\chi}_2(x, s) \sim -\frac{\hat{K}_I(s)}{\mu\gamma_2\sqrt{-2\pi x}}, \quad x \to 0^-. \tag{5.23}$$

On integrating these relations with respect to x and fixing the constant of integration by assuring that the displacement jumps vanish at the crack tip we obtain the displacement jumps $[u]$ and $[v]$ for small negative x. When the crack extends to $x = r$, these formulas give

$$[v](x, t) \sim \sqrt{\frac{2(r-x)}{\pi}}\frac{K_I(t)}{\mu\gamma_2}, \quad [u](x, t) \sim \sqrt{\frac{2(r-x)}{\pi}}\frac{K_{II}(t)}{\mu\gamma_1}, \quad x \to r^-. \tag{5.24}$$

Finally, by substituting the asymptotic relations (5.21) and (5.24) into (5.20) we find the potential energy increment when the crack extends to $S_0(t) + \delta S_0(t)$

$$\delta U(t) \sim \frac{r}{4\mu}\left(\frac{K_I^2(t)}{\gamma_2} + \frac{K_{II}^2(t)}{\gamma_1}\right), \quad r \to 0^+. \tag{5.25}$$

According to the Griffith criterion, the crack starts propagating if the energy $\delta U(t)$ equals or greater than the increase in the surface energy $2Tr$, $\delta U \geq 2Tr$, where T is the Griffith material constant. This criterion may be represented in terms of the SIFs in the form

$$\frac{K_I^2(t)}{\gamma_2} + \frac{K_{II}^2(t)}{\gamma_1} \geq 8\mu T. \tag{5.26}$$

On applying this criterion, that is on solving the transcendental equation

$$\gamma_1 K_I^2(t) + \gamma_2 K_{II}^2(t) = 8\mu\gamma_1\gamma_2 T, \tag{5.27}$$

one may predict V_0, the initial speed of crack propagation. Following the successive algorithm described in Section 5.1 and solving the associated equation (5.27) it is possible to determine all the speeds V_j $j = 1, 2, \ldots$.

6 Conclusion

We have derived the fundamental solution and the weight functions of the transient two-dimensional problem on a semi-infinite crack propagating at constant speed parallel to the boundary of a half-plane. The boundary of the half-plane is free of traction, while the crack faces are subjected to general time-independent loading. We have reduced the boundary-value problem to a vector RHP on the real axis. In the particular case, when the crack is far away from the boundary of the half-plane, the vector RHP is decoupled and solved by quadratures. In the general case, we have split the matrix coefficient into a discontinuous diagonal matrix and a continuous matrix, factorized the discontinuous part and rewritten the vector RHP as a system of two convolution equations on the segment $-\infty < x < 0$. For numerical purposes, it was recast as a system of two Fredholm integral equations on the segment $(-1, 1)$. We have derived the Laplace transforms of the SIFs and the weight functions in terms of the solution of the convolution equations at the point $x = 0$. The Laplace transform has been inverted numerically. To improve the convergence, we have applied the Euler summation method for alternating series. We have obtained numerical results for the SIFs when concentrated loads are applied to the crack faces (at time $t = 0$ at the crack tip). This model problem generates four weight functions $W_{i,j}$, $i, j = I, II$. It has been discovered that during a certain initial period of time, $0 < t < 2t_l$, the off-diagonal weight functions $W_{i,j}$, $i \neq j$, approximately equal zero, and the diagonal functions almost coincide with the ones for the case of the whole plane. For time $t > 2t_l$, the boundary effects play a significant role, and, in general, all the four weight functions do not vanish and are different from the corresponding functions associated with the whole plane. It has also been found that the dimensionless functions $w_{i,i}(0,t) = \sqrt{\frac{1}{2}\pi V t} W_{i,i}(0,t)$ $(i = I, II)$ tend to 1 and 0 as V/c_R tends to 0 and 1, respectively (V is the crack speed and c_R is the Rayleigh speed), while $w_{i,j}$ $(i \neq j)$ vanish when V/c_R approach both points, 0 and 1. We have found that w_{ij} are not monotonic functions of V/c_R and attain their local maximum in the interval $(0, V/c_R)$. As the distance δ from the crack to the boundary decreases, all the functions w_{ij} grow. We emphasize that apart from small δ our numerical method is stable for all parameters δ.

Based on the Freund approximate algorithm [13] for the problem on a semi-infinite crack propagated at a nonuniform rate in the whole plane we have developed a procedure for the case when the crack propagates also at prescribed variable sub-Rayleigh speed in a half-plane parallel to the boundary and when the boundary effects are significant. The implementation of the method requires solving a system of Volterra convolution equations whose kernels are the associated weight functions, not a single Abel integral equation as in the whole plane case. The system of Volterra equations also admits a closed-form solution. However, in the case of a half-plane, there is no analog of the remarkable formula for the Mode I SIF $K_I(l(t), V_k) = k(V_k)K_I(l(t), 0)$ in any interval $t_k < t < t_{k+1}$ derived for the whole plane [13]. There is another difference between the whole plane and half-plane solutions. The displacement jumps though the crack line $x_2 = 0$ have to vanish on the segments $l_i < x_1 < V_{i-1}t$, $i = 1, \dots, k$. This property was analytically proved in [13] for the sub-Rayleigh regime and in [16] in the transonic regime. For the half-plane problem, this condition needs to be verified numerically for each Problem P_i $(i = 0, 1, \dots, k-1)$ during the implementation of the procedure.

To compute the SIFs at time t, $2t_l < t_k < t < t_{k+1}$, for the crack in a half-plane, one needs to derive the weight functions for all intermediate speeds V_i. We have shown that initially, before the longitudinal wave reflected from the boundary strikes the crack and when the weight functions coincide with those for the whole plane, the relatively simple Freund's algorithm works. At the same time, the solution is still different since it relies on the static solution on a cracked half-plane, not the whole plane with the crack. When the first longitudinal wave reflected from the half-plane boundary reaches the crack surface moving at speed $V(t) < c_R$, the boundary substantially affects the weight functions. To determine the SIFs at the crack tip at some time $t \in (t_k, t_{k+1})$, consequently, one may employ the procedure presented that requires solving the same transient problem for different constant speeds V_i $(i = 0, 1, \dots, k)$ and a system of Volterra equations to determine at each step the loads need to be negated to make possible for the crack to advance.

As for the speeds V_j $(j = 0, 1, \dots,)$ themselves, they have been determined by applying the dynamic Griffith criterion and solving a certain transcendental equation associated with each step of the algorithm.

A Solution for a plane in the case of the contour $\tilde{L}_\varepsilon$

It is important to show that as $\varepsilon \to 0^+$ the weight functions are invariant of the way the original contour L is deformed. The contour $\tilde{L}_\varepsilon$ splits the p-plane into the domains $\tilde{D}^- \ni 0$ and $\tilde{D}^+$. In this case, $\coth(\pi p)$ in the representation (3.9) needs to be factorized as follows:

$$\coth(\pi p) = \frac{i\tilde{K}^+(p)}{\tilde{K}^-(p)}, \quad \tilde{K}^+(p) = -\frac{\Gamma(-ip)}{\Gamma(1/2 - ip)}, \quad \tilde{K}^-(p) = \frac{\Gamma(1/2 + ip)}{\Gamma(1 + ip)}. \tag{A.1}$$

Due to the fact that the asymptotics of the factors $\tilde{K}^\pm(p)$ at infinity is different from that of $K^\pm(p)$, $\tilde{K}^\pm(p) \sim \mp(\mp p)^{-1/2}$, $p \to \infty$, $p \in \tilde{D}^\pm$, the solution to the RHPs has arbitrary constants C_j,

$$\begin{aligned}
\tilde{\sigma}_j^+(ps', c_l s') &= \tilde{K}^+(p)\Omega_j^+(p)\left[C_j + \tilde{\Psi}_j^+(p, s')\right], \quad p \in \tilde{D}^+, \\
\tilde{\chi}_j(ps', c_l s') &= (\mu\gamma_j)^{-1}\tilde{K}^-(p)\Omega_j^-(p)\left[C_j + \tilde{\Psi}_j^-(p, s')\right], \quad p \in \tilde{D}^-, \\
\tilde{\Psi}_j^\pm(p, s') &= \frac{1}{2\pi i}\int_{\tilde{L}_\varepsilon} \frac{\tilde{q}_j^-(\tau s', c_l s')}{\tilde{K}^+(\tau)\Omega_j^+(\tau)} \frac{d\tau}{\tau - p}, \quad p \in \tilde{D}^\pm, \quad j = 1, 2.
\end{aligned} \tag{A.2}$$

Now, let $p = i\varepsilon \in \tilde{L}_\varepsilon$ ($\varepsilon > 0$). According to the Sokhotski-Plemelj formulas

$$\tilde{\Psi}_j^+(i\varepsilon, s') - \tilde{\Psi}_j^-(i\varepsilon, s') = \frac{\tilde{q}_j^-(i\varepsilon s', c_l s')}{\tilde{K}^+(i\varepsilon)\Omega_j^+(i\varepsilon)}. \tag{A.3}$$

As $\varepsilon \to 0^+$, $[\tilde{K}^+(i\varepsilon)]^{-1} \to 0$, while the other two functions in the right-hand side of (A.3) are bounded and nonzero. Therefore, $\tilde{\Psi}_j^\pm(i\varepsilon, s') \sim \tilde{\Psi}_j(0, s')$, $\varepsilon \to 0^+$, where $\tilde{\Psi}_j(0, s')$ is the principal value of the Cauchy integral

$$\tilde{\Psi}_j(p, s') = \frac{1}{2\pi i}\int_L \frac{\tilde{q}_j^-(\tau s', c_l s')}{\tilde{K}^+(\tau)\Omega_j^+(\tau)}\frac{d\tau}{\tau - p}, \quad p \in L, \quad j = 1, 2. \tag{A.4}$$

It is evident that $\tilde{\sigma}_j^+(ps', c_l s') \to \infty$ as $p = i\varepsilon \to 0$ unless $C_j = -\tilde{\Psi}_j(0, c_l s')$. On the other hand, this choice of the constants C_j guarantees that $\tilde{\chi}_j^-(0, c_l s') = 0$, and both displacement jumps vanish at $x = -\infty$. Simple calculations show that $\tilde{\Psi}_j(0, s') = i\tilde{\Psi}_j^\circ(s')$, where $\Psi_j^\circ(s')$ is given by (3.20). Analysis of $\tilde{\sigma}_j^+(ps', c_l s')$ as $p \to \infty$, $p \in \tilde{D}^+$, results in the asymptotics

$$\tilde{\sigma}_j^+(ps', c_l s') \sim (-ip)^{-1/2} C_j = -e^{-i\pi/4}\Psi_j^\circ(s')p^{-1/2}, \quad p \to \infty, \quad \arg p \in (0, \pi), \tag{A.5}$$

that coincides with formula (3.19), and brings us to the expressions for the SIFs and the weight functions derived in section 3.2 in the case of the contour L_ε.

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