

DIFFRACTION OF A PLANE WAVE BY A CIRCULAR CONE WITH AN IMPEDANCE BOUNDARY CONDITION*

Y. A. ANTIPOV†

Abstract. We consider the boundary-value problem for the Helmholtz equation in a circular cone with an impedance boundary condition on its face. A new approach for its solution is proposed. The scheme of solution includes applying the Kontorovich–Lebedev transform, derivation of a second-order difference equation in a strip of a complex variable, and reduction of the latter to an integral equation of the convolution type with variable coefficients. It is also shown that the equation is equivalent to a 2×2 matrix Riemann–Hilbert problem with a discontinuous coefficient. We analyze the behavior of the solution of the integral equation at the ends of the contour and construct an approximate solution using a collocation method. The diffraction coefficient is found in terms of the solution of the integral equation. Numerical results for the diffraction coefficient and comparative analysis of the results for the impedance cone with the limiting cases of the acoustically soft and hard cones are reported. A full high frequency asymptotic expansion for the scattering field in the region where no reflected waves are observed is derived for the impedance, soft, and hard cones.

Key words. scattering, impedance cone, diffraction coefficients, difference equation, integral equation, Riemann–Hilbert problem

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1. Introduction. Acoustic scattering of sound waves by obstacles is important in many different applications such as underwater acoustics, detection and recognition of targets, noise control, etc. Acoustically soft and hard obstacles such as spheres; circular, elliptic, hyperbolic, and parabolic cylinders; wires, strips, wedges, spheroids, cones, discs, and paraboloids were analyzed in [8]. A problem of the scattering of waves by an acoustically soft and hard cone with Dirichlet and Neumann boundary conditions on its face, respectively, admits an exact solution by applying either the Sommerfeld integral representation or the Kontorovich–Lebedev integral transformation [8, 14]. Both methods use the fundamental ideas of the separation of variables method. For a noncircular cone, it is possible to construct an exact solution (in terms of the Lamé functions) only in the case of an elliptic cone [19]. By the method based on Sommerfeld’s integral, potential theory, and numerical solution of boundary integral equations, a solution and diffraction coefficients for a cone of arbitrary smooth cross-section were found in [26, 2, 3]. Diffraction by an arbitrarily shaped cone was analyzed in [15] using the Kontorovich–Lebedev transform.

It is worth pointing out that acoustically soft or rigid objects are a mathematical idealization not found in nature. In general, the surface of a reflecting obstacle has a definite normal impedance $Z_s = p/v_n$ that is a complex number, and it is neither zero nor infinity. Here p is the acoustic pressure at the surface, and v_n is the component of the particle velocity normal to the surface, pointing into the obstacle characterized by the impedance Z_s . Such sound-absorbent materials as perforated plates (a honeycomb panel, for instance) or porous materials (foam, masonry with open pores) have to be treated as objects with an “impedance surface.” The limit value of the surface

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†Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK (masya@maths.bath.ac.uk).

impedance $Z_s = 0$ corresponds to a limp obstacle with a pressure-release surface. For an acoustically rigid surface the impedance is infinite. Various experimental methods of measuring the surface impedance and its link with the absorption coefficient of porous materials were discussed in [22, 25].

If the face of a cone is not completely hard but reacts to an incident pressure wave with a normal surface impedance Z_s , then the wave field satisfies a boundary condition for a linear combination of a function and its normal derivative [21, 24]. Mathematically, problems of scattering by obstacles with impedance boundary conditions on their surfaces are more complicated than those with Dirichlet and Neumann boundary conditions for the acoustically soft and hard surfaces, respectively. We mention a solution for the case of scalar plane wave scattering by an impedance wedge [20] and for a semi-infinite plane with different impedance boundary conditions on each face [4]. Scattering of an electromagnetic plane wave by an impedance wedge was analyzed in [6]. The exact solutions in the aforementioned papers are based on either the theory of a first-order difference equation in a strip of a complex variable [20, 6] or a matrix Wiener–Hopf problem [4].

Bernard [5] considered the problem of diffraction by an impedance circular cone using the Sommerfeld and Kontorovich–Lebedev integral representations and the theory of the Maliuzhinets difference equation and reduced the problem to an integral equation on the infinite imaginary axis, whose solution reveals the exponential decays at infinity. Bernard analyzed the kernel of the equation, showed that the characteristic part of the kernel is the Dixon kernel (see, e.g., [27]), and proved the convergence of an approximate solution of the integral equation to the exact solution. He also studied the existence and uniqueness of the solution of the integral equation. The leading asymptotic term for the diffraction coefficient in the case of a narrow impedance cone was found in [7].

The main objectives of the present paper are:

- (i) to propose a simple direct method of derivation of an integral equation for an impedance obstacle based on the Kontorovich–Lebedev transform,
- (ii) to study the asymptotic properties at infinity of the solution of the integral equation on the imaginary axis and to construct an effective numerical solution of the integral equation,
- (iii) to derive formulae for the diffraction coefficients efficient for numerical purposes.

The structure of the present paper is as follows. We formulate the problem, discuss the physical sense of the impedance boundary condition, and introduce a class of solutions in section 2. In section 3, we use the Kontorovich–Lebedev transform to reduce the original boundary-value problem for the Helmholtz equation in a cone to a boundary-value problem with two shifts of the theory of analytic functions. In section 4, applying the Sokhotski–Plemelj formulae for a strip enables us to derive a basic integral equation of the convolution type with variable coefficients. An exact solution of the equation is not feasible. It is shown that this equation is reducible to a singular integral equation on the interval $(0, 1)$ with a fixed singularity at the initial point:

$$(1.1) \quad \chi_0(x) + \frac{K_0(x)}{2\pi} \int_0^1 \chi_0(y) \left(\frac{1}{y+x} - \frac{y}{yx+1} \right) dy = -h_0(x), \quad 0 < x < 1.$$

The kernel possesses an immovable singularity at the point $x = y = 0$, the Hölder condition for the function $K_0(x)$ fails in a neighborhood of the point $x = 0$, and the func-

tion $h_0(x)$ has a power-logarithmic singularity at zero: $h_0(x) = O(x^\varkappa \sqrt{-\log x})$, $x \rightarrow 0$ ($-\frac{1}{2} < \varkappa < 0$). In section 5, the case of an axially symmetric incident wave is analyzed. An explicit expression for the right-hand side of the equation is determined (section 5.1). In addition, the behavior of the singular integral

$$(1.2) \quad \Sigma(x) = \int_0^1 y^\varkappa \sqrt{\log \frac{1}{y}} \frac{dy}{y+x}, \quad \Re(\varkappa) > -1, \quad \text{as } x \rightarrow +0$$

in a neighborhood of the point $x = 0$, is analyzed (section 5.2). Here and throughout we use the notation $\Re(z)$ for the real part of a complex number z . An asymptotic expansion of the singular integral (1.2) is needed to establish the convergence of the integral representation for the wave field and the diffraction coefficients. In section 6, we derive the numerical scheme (a version of the collocation method) for the integral equation, which leads to a linear algebraic system of the second order. We find an exact formula (in terms of the solution of the integral equation) and an approximate formula (in terms of the solution of the algebraic system) for the diffraction coefficient in section 7.1. The formulae act in the domain where no reflected waves are observed. In particular cases of the Neumann and Dirichlet boundary conditions, the classical formulae [8] are rederived (sections 7.2, 7.3). Numerical results and new physical effects induced by the impedance condition with those of acoustically soft and hard cones are discussed in sections 7.4 and 7.5, respectively. A full asymptotic expansion for the far acoustic field is presented in section 8. In Appendix A, we show an alternative approach to analysis of the original problem, namely, formulating the problem as a 2×2 matrix Riemann–Hilbert problem with a discontinuous coefficient. It is reducible to a problem with a continuous matrix coefficient [10] that admits formulation of an existence factorization theorem. In Appendix B, we give another simple derivation of the integral equation, based on the classical Sokhotski–Plemelj formula for a contour in the complex plane. An analytical continuation of the solution of the boundary-value problem is constructed in Appendix C. The continuation of the solution can be used for evaluating the diffraction coefficient in the singular domain where reflected waves are observed.

2. Formulation of the problem. Throughout this paper we consider an inviscid, compressible fluid undergoing irrotational motion that can be described within the framework of linear acoustics. The flow domain $\{0 < r < \infty, -\pi \leq \phi \leq \pi, 0 \leq \theta < \alpha\}$, where $\pi/2 < \alpha < \pi$, is the exterior of a circular cone \mathbf{C} . The cone is assumed to be impenetrable. This means that transmitted waves of compressional and shear type in the interior of the cone are neglected. The surface of the cone possesses the normal impedance Z_s . The system is activated by an incident wave of fixed angular frequency ω travelling in the direction $\theta = \theta_0$ with $0 \leq \theta_0 < \alpha$ (Figure 1), at the sound speed c , such that its potential is $e^{i\omega t} u^{inc}$, where

$$(2.1) \quad u^{inc} = e^{ikr(\cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos\phi)},$$

where $k = \omega/c$ is termed the wave number and we assume that $-\pi < \arg k < 0$.

Since the flow is irrotational, there exists such a complex potential U^{total} that the velocity vector \mathbf{v} and the sound pressure p can be expressed as follows:

$$(2.2) \quad \mathbf{v} = \nabla U^{total}, \quad p = -\rho \frac{\partial}{\partial t} U^{total},$$

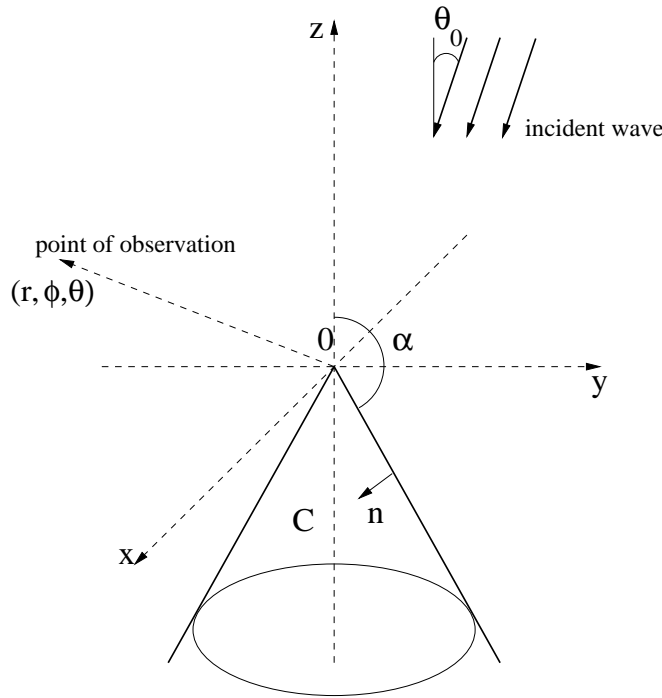


FIG. 1. *Geometry of the problem.*

where U^{total} is the total wave field, ρ is the density of the fluid, and the symbol ∇ denotes the gradient operator in spherical coordinates (r, ϕ, θ)

$$(2.3) \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right).$$

The angular frequency of the incident wave is fixed; therefore

$$(2.4) \quad U^{total} = e^{i\omega t}U, \quad \mathbf{v} = e^{i\omega t}\nabla U, \quad p = -i\omega\rho e^{i\omega t}U.$$

The wave field U satisfies the three-dimensional Helmholtz equation

$$(2.5) \quad \Delta U + k^2U = 0$$

in the exterior of the conical obstacle. Here the symbol Δ denotes the Laplacian

$$(2.6) \quad \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

Let us derive the boundary condition on the surface of the cone. Since the acoustic impedance of the surface is Z_s , the ratio of the pressure p to the normal component of the fluid velocity directed into the cone is given by (see [13, 25])

$$(2.7) \quad \left. \frac{p}{\mathbf{v} \cdot \mathbf{n}} \right|_{\partial \mathbf{C}} = Z_s,$$

where \mathbf{n} is the unit normal to the surface $\partial \mathbf{C}$ pointing into the medium in the cone \mathbf{C} . The parameter Z_s is complex, and its real part is positive in the case of absorbing

boundaries. This means that the cone responds passively to the incident wave. The surface possessing this intrinsic property is said to be locally reacting if Z_s is a function of a point of the surface [24]. If the quantity $\mathbf{v} \cdot \mathbf{n} - Z_s^{-1}p$ is not equal to zero, then the surface is called active [21], with the driving velocity v_ω defined by $\mathbf{v} \cdot \mathbf{n} - Z_s^{-1}p = v_\omega e^{i\omega t}$. We will consider the passive area with $v_\omega = 0$. By substituting (2.4) and the relation

$$(2.8) \quad \mathbf{v} \cdot \mathbf{n}|_{\partial \mathbf{C}} = \frac{e^{i\omega t}}{r} \frac{\partial}{\partial \theta} U \Big|_{\theta=\alpha}$$

into the boundary condition (2.7), we get

$$(2.9) \quad \left(\frac{1}{r} \frac{\partial U}{\partial \theta} + ik\beta U \right)_{\theta=\alpha} = 0, \quad 0 < r < \infty, \quad -\pi \leq \phi \leq \pi.$$

Equation (2.9) is called the impedance boundary condition. The parameter β is the acoustic admittance and is expressible in terms of the surface impedance Z_s : $\beta = \rho c / Z_s$. The acoustic admittance is complex, and its real part is positive for absorbing boundaries. The imaginary part can be arbitrary. When $\beta = 0$ (the Neumann boundary condition), the boundary is rigid and the cone is acoustically hard. If $\beta = \infty$, we arrive at the Dirichlet boundary condition that corresponds to an acoustically soft cone.

Remark. It is clear that if ω is changed by $-\omega$, then β in (2.9) has to be changed by $-\beta$.

It is convenient to introduce a new scattering potential u , whereby $U = u^{inc} + u$. Since the incident potential is a solution of the Helmholtz equation, the new potential satisfies the following boundary-value problem:

$$(2.10) \quad \Delta u + k^2 u = 0, \quad 0 < r < \infty, \quad -\pi \leq \phi \leq \pi, \quad 0 \leq \theta < \alpha,$$

$$(2.11) \quad \left(u + \frac{1}{ik\beta r} \frac{\partial u}{\partial \theta} \right)_{\theta=\alpha} = g(r, \phi), \quad 0 < r < \infty, \quad -\pi \leq \phi \leq \pi,$$

where

$$(2.12) \quad g(r, \phi) = - \left(1 - \frac{1}{\beta} \sin \alpha \cos \theta_0 + \frac{1}{\beta} \cos \alpha \sin \theta_0 \cos \phi \right) e^{ikr(\cos \alpha \cos \theta_0 + \sin \alpha \sin \theta_0 \cos \phi)}.$$

In order for the functions involved in the boundary-value problem (2.10), (2.11) to be continuous, we demand $u \in C^2(\text{ext } \mathbf{C})$, where $\text{ext } \mathbf{C}$ is the exterior of the cone \mathbf{C} . The boundary condition (2.11) implies that u is a continuous function of ϕ and r and a C^1 -continuous function of θ on any finite surface $\partial \hat{\mathbf{C}} = \{0 < r_0 \leq r \leq r_1 < \infty, -\pi \leq \phi \leq \pi, \theta = \alpha\}$. In a neighborhood of the vertex of the cone, we stipulate the classical Meixner condition

$$(2.13) \quad |u| \leq \frac{B_0 r^\gamma}{|\log r|}, \quad r \rightarrow 0,$$

where B_0 is a positive constant and $\gamma > -1/2$. As $r \rightarrow \infty$ (see [7])

$$(2.14) \quad u = O(r^{ikr \cos(\varepsilon + \pi/2)}), \quad \varepsilon = \min(\pi/2, 2\alpha - (\theta + \theta_0 + \pi/2)).$$

3. Boundary-value problem with two shifts. In this section, we aim to reduce the boundary-value problem (2.10), (2.11) to a difference equation in a strip. First, it is natural to eliminate the variable ϕ from (2.10) and the boundary condition (2.11) by taking the finite Fourier transform

$$(3.1) \quad u_m(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r, \phi, \theta) e^{im\phi} d\phi, \quad u(r, \phi, \theta) = \sum_{m=-\infty}^{\infty} u_m(r, \theta) e^{-im\phi}.$$

So the original problem (2.10), (2.11) is equivalent to the two-dimensional problem for the transform $u_m(r, \theta)$,

$$(3.2) \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_m}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u_m}{\partial \theta} \right) + \left(k^2 - \frac{m^2}{r^2 \sin^2 \theta} \right) u_m = 0,$$

$$0 < r < \infty, \quad 0 \leq \theta < \alpha,$$

$$(3.3) \quad \left(u_m + \frac{1}{ik\beta r} \frac{\partial u_m}{\partial \theta} \right)_{\theta=\alpha} = g_m(r), \quad 0 < r < \infty,$$

where

$$(3.4) \quad g_m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(r, \phi) e^{im\phi} d\phi.$$

Let us introduce the Kontorovich–Lebedev transform (see [18])

$$(3.5) \quad u_{m,s}(\theta) = \int_0^{\infty} u_m(r, \theta) H_s^{(2)}(kr) \frac{dr}{\sqrt{r}}, \quad u_m(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^{i\infty} s u_{m,s}(\theta) J_s(kr) ds.$$

Here $J_s(kr)$ and $H_s^{(2)}(kr)$ are the Bessel functions. The conditions (2.13), (2.14) guarantee the convergence of the first integral in (3.5) as $r \rightarrow 0$ and $r \rightarrow \infty$. The function $u_{m,s}(\theta)$ satisfies the following conditions:

- (i) $u_{m,s}(\theta)$ is an analytic function of s in the strip $|\Re(s)| < \delta_0$ ($0 < \delta_0 < 1/2$);
- (ii)

$$(3.6) \quad u_{m,s}(\theta) = e^{i\pi s} u_{m,-s}(\theta);$$

(iii) the integral

$$(3.7) \quad \int_{-\infty}^{\infty} |(\delta + it) u_{m,\delta+it}(\theta)| e^{-t \arg k + \pi|t|/2} dt$$

is convergent uniformly with respect to $\delta \in [-\delta_0, \delta_0]$.

Condition (iii) provides the existence of the inverse transform $u_m(r, \theta)$ in (3.5). Applying the Kontorovich–Lebedev transform to (3.2), we get the ordinary differential equation

$$(3.8) \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} u_{m,s}(\theta) \right) + \left(s^2 - \frac{1}{4} - \frac{m^2}{\sin^2 \theta} \right) u_{m,s}(\theta) = 0, \quad 0 \leq \theta \leq \alpha,$$

whose general solution, bounded at $\theta = 0$, is

$$(3.9) \quad u_{m,s}(\theta) = A_m(s)P_{s-1/2}^m(\cos \theta).$$

Here $P_{s-1/2}^m(\cos \theta)$ is the associated Legendre function of the first kind, and $A_m(s)$ is an arbitrary function to be determined from the boundary condition. To apply the Kontorovich–Lebedev transform to the boundary condition (3.3), in addition to condition (i), we analyze the integral

$$(3.10) \quad \frac{d}{d\theta} \int_0^\infty u_m(r, \theta) H_s^{(2)}(kr) \frac{dr}{r^{1/2}} \Big|_{\theta=\alpha},$$

which is an analytic function of s in the strip $-\delta_0 < \Re(s) < \delta_0$. Due to the relation

$$(3.11) \quad \frac{1}{r} H_s^{(2)}(kr) = \frac{k}{2s} [H_{s-1}^{(2)}(kr) + H_{s+1}^{(2)}(kr)]$$

for the Bessel function, condition (i), and the boundary condition (3.3), the function $\frac{d}{d\theta} u_{m,s}(\alpha)$ is analytically continuable into the strip $-1 < \Re(s) < 1$. Thus, the boundary condition (3.3) in its transformed form can be written in the form

$$(3.12) \quad u_{m,s}(\alpha) + \frac{1}{2i\beta s} \frac{d}{d\theta} [u_{m,s-1}(\alpha) + u_{m,s+1}(\alpha)] = \frac{h(s)}{2i\beta k^{1/2} s},$$

where

$$(3.13) \quad h(s) = 2i\beta k^{1/2} s \int_0^\infty g_m(r) H_s^{(2)}(kr) \frac{dr}{\sqrt{r}}.$$

Now we introduce a new function

$$(3.14) \quad \Phi(s) = k^{1/2} \frac{d}{d\theta} u_{m,s+1}(\alpha).$$

Due to formula (3.9) and the analyticity of the $\frac{d}{d\theta} u_{m,s}(\alpha)$ in the strip $-1 < \Re(s) < 1$, we obtain that the function

$$(3.15) \quad \Phi(s) = k^{1/2} \frac{d}{d\theta} P_{s+1/2}^m(\cos \alpha) A_m(s+1)$$

is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$. Comparing expressions (3.9) and (3.15) gives the transform $u_{m,s}(\theta)$ in terms of the function $\Phi(s)$:

$$(3.16) \quad u_{m,s}(\theta) = \frac{P_{s-1/2}^m(\cos \theta)}{k^{1/2} \frac{d}{d\theta} P_{s-1/2}^m(\cos \alpha)} \Phi(s-1).$$

Now we substitute formulae (3.14), (3.16) into (3.12) and obtain the Carleman-type boundary-value problem of the theory of analytic functions in a strip as follows.

Find the function $\Phi(s)$ that is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$ and satisfies the boundary condition

$$(3.17) \quad \Phi(\sigma) + K(\sigma)\Phi(\sigma-1) + \Phi(\sigma-2) = h(\sigma), \quad \sigma \in \Omega = \{-i\infty, i\infty\},$$

where

$$(3.18) \quad K(s) = \frac{2i\beta s P_{s-1/2}^m(\cos \alpha)}{\frac{d}{d\theta} P_{s-1/2}^m(\cos \alpha)},$$

and the auxiliary conditions

$$(3.19) \quad \int_{-\infty}^{\infty} |\Phi(x + it)|^2 dt < \infty,$$

uniformly with respect to $x \in [-2, 0]$, and

$$(3.20) \quad \Phi(\sigma - 1) = e^{i\pi\sigma} \Phi(-\sigma - 1), \quad \sigma \in \Omega.$$

The last relation follows from condition (ii), (3.16), and the symmetry property of the Legendre function $P_{i\tau-1/2}^m(x) = P_{-i\tau-1/2}^m(x)$. The formulated problem can be considered as a difference equation of the second order in the strip Π with the two additional conditions (3.19) and (3.20).

Remark. By a more complicated method, a similar difference equation was derived in [5].

Let us show that the coefficient $K(s)$ is discontinuous at infinity. We rewrite the expression for the function $K(s)$ as

$$(3.21) \quad K(s) = 2i\beta s \left\{ -\left(s + \frac{1}{2}\right) \cot \alpha + \frac{s - m + 1/2}{\sin \alpha} \frac{P_{s+1/2}^m(\cos \alpha)}{P_{s-1/2}^m(\cos \alpha)} \right\}^{-1}$$

and use the asymptotic expansion for the Legendre function (see [12, formula 8.721(1)])

$$P_s^m(\cos \alpha) = \frac{2}{\sqrt{\pi}} \Gamma(s+m+1) \sum_{l=0}^{\infty} \frac{\Gamma(m+l+\frac{1}{2})}{\Gamma(m-l+\frac{1}{2})} \frac{\cos[(s+l+\frac{1}{2})\alpha - \frac{\pi}{4}(2l+1) + \frac{m\pi}{2}]}{l! \Gamma(s-l+\frac{3}{2}) (2 \sin \alpha)^{l+1/2}},$$

$$(3.22) \quad |s| \rightarrow \infty, \quad 0 < \delta \leq \alpha \leq \pi - \delta < \pi,$$

and for the Γ -function

$$(3.23) \quad \frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha-\beta} \left[1 + \frac{(\alpha - \beta)(\alpha + \beta - 1)}{2z} + O\left(\frac{1}{z^2}\right) \right], \quad z \rightarrow \infty.$$

Now we can obtain

$$(3.24) \quad \frac{P_{c+i\tau+1/2}^m(\cos \alpha)}{P_{c+i\tau-1/2}^m(\cos \alpha)} = e^{\mp i\alpha} \left[1 + O\left(\frac{1}{\tau}\right) \right], \quad \tau \rightarrow \pm\infty.$$

Thus, the function $K(s)$ is discontinuous on the contour Ω :

$$(3.25) \quad K(i\tau) = \mp 2\beta \left[1 + O\left(\frac{1}{\tau}\right) \right], \quad \tau \rightarrow \pm\infty.$$

4. Derivation of the integral equation. The exact solution of problem (3.17), (3.19), (3.20) can be found for particular cases of the coefficient $K(\sigma)$ only. In general, the problem can be reduced to a 2×2 matrix Riemann–Hilbert boundary value problem (see Appendix A). To find a constructive solution of the problem, we reformulate the original problem as an integral equation. First, we rewrite (3.17) as follows:

$$(4.1) \quad \Phi(\sigma) + \Phi(\sigma - 2) = h^*(\sigma), \quad \sigma \in \Omega,$$

where

$$(4.2) \quad h^*(\sigma) = h(\sigma) - K(\sigma)\Phi(\sigma - 1).$$

The function $\Phi(\sigma)$ is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$ and satisfies the boundary condition (4.1) and the auxiliary condition (3.19). Therefore, due to the Sokhotski–Plemelj formula modified for a strip, we write

$$(4.3) \quad \Phi(s) = \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin \frac{\pi}{2}(\tau - s)}, \quad s \in \Pi,$$

$$(4.4) \quad \Phi(\sigma) = \frac{1}{2}h^*(\sigma) + \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin \frac{\pi}{2}(\tau - \sigma)}, \quad \sigma \in \Omega,$$

$$(4.5) \quad \Phi(\sigma - 2) = \frac{1}{2}h^*(\sigma) - \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin \frac{\pi}{2}(\tau - \sigma)}, \quad \sigma \in \Omega.$$

The first formula (4.3) defines the function $\Phi(s)$ in the interior of the strip Π in terms of its values on the contour $\{-1 - i\infty, -1 + i\infty\}$. The last two formulae (4.4) and (4.5) relate to the boundary values of the function $\Phi(s)$ on the contour Ω and on the contour $\{-2 - i\infty, -2 + i\infty\}$. Now letting $s = \sigma - 1$ in equality (4.3) and taking into account (4.2), we get a convolution integral equation with respect to the function $\Phi(\sigma - 1)$:

$$(4.6) \quad \Phi(\sigma - 1) + \frac{1}{4i} \int_{-i\infty}^{i\infty} \frac{K(\tau)\Phi(\tau - 1) - h(\tau)}{\cos \frac{\pi}{2}(\tau - \sigma)} d\tau = 0, \quad \sigma \in \Omega.$$

To analyze the last equation, we reduce it to another integral equation on the finite segment $(0, 1)$. Because of the symmetry conditions

$$\Phi(-\sigma - 1) = e^{-i\pi\sigma}\Phi(\sigma - 1),$$

$$(4.7) \quad K(-\tau) = -K(\tau), \quad h(-\tau) = -e^{-i\pi\tau}h(\tau),$$

which follow from (3.20), (3.18), and (3.13), it is possible to write (4.6) in the form

$$(4.8) \quad \Phi(i\xi - 1) + \frac{1}{4} \int_{-\infty}^0 [K(i\eta)\Phi(i\eta - 1) - h(i\eta)] \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi)} \right) d\eta = 0,$$

$$-\infty < \xi < 0.$$

Then, introducing a new function

$$(4.9) \quad \chi(\xi) = K(i\xi)\Phi(i\xi - 1) - h(i\xi),$$

we get

$$\chi(\xi) + \frac{K(i\xi)}{4} \int_{-\infty}^0 \chi(\eta) \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi)} \right) d\eta = -h(i\xi), \quad -\infty < \xi < 0.$$

(4.10)

Finally, setting in (4.10)

$$(4.11) \quad e^{\pi\eta} = y, \quad e^{\pi\xi} = x$$

and using the relations

$$(4.12) \quad \frac{1}{\cosh \frac{\pi}{2}(\eta - \xi)} = \frac{2\sqrt{yx}}{y + x}, \quad \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi)} = \frac{2y\sqrt{yx}}{yx + 1},$$

we arrive at an integral equation that is, obviously, equivalent to the original (4.6):

$$(4.13) \quad \chi_0(x) + \frac{K_0(x)}{2\pi} \int_0^1 \chi_0(y) \left(\frac{1}{y + x} - \frac{y}{yx + 1} \right) dy = -h_0(x), \quad 0 < x < 1,$$

where

$$\chi_0(x) = \frac{1}{\sqrt{x}} \chi \left(\frac{1}{\pi} \log x \right), \quad K_0(x) = K \left(\frac{i}{\pi} \log x \right),$$

$$(4.14) \quad h_0(x) = \frac{1}{\sqrt{x}} h \left(\frac{i}{\pi} \log x \right).$$

Equation (4.13) is a singular integral equation with an immovable singularity at the point $x = 0$. The function $K_0(x)$ does not belong to the Hölder space. Indeed, in a neighborhood of the point $x = 0$, one can write (compare with (3.25))

$$(4.15) \quad K_0(x) = 2\beta + \frac{r_0}{\log x} + O \left(\frac{1}{\log^2 x} \right), \quad x \rightarrow 0, \quad r_0 = \text{const},$$

and the Hölder condition for the function $K_0(x)$ fails near the point $x = 0$. Obviously, the behavior of the function $\chi_0(x)$ at the singular point $x = 0$ is defined by the singularity of the function $h_0(x)$ at this point.

Remark. One can see that the key step in the derivation of (4.13) is formula (4.3). We note that the same equation can be obtained in another way, using the classical Sokhotski–Plemelj formula (see Appendix B).

5. Analysis of the integral equation for an axially symmetric incident wave. We shall study the axially symmetric problem of scattering of an incident plane wave

$$(5.1) \quad u^{inc} = e^{ikr \cos \theta}$$

by a circular impedance cone. In this case $\theta_0 = 0$, and the function $g(r, \phi)$ is independent of ϕ . We set $m = 0$ in (3.2), (3.3) and, obviously, $u_0 = u$. The function $g_0(r)$ is specified by

$$(5.2) \quad g(r, \phi) = g_0(r) = -\beta_0 e^{-ikr \cos \alpha_0},$$

where

$$(5.3) \quad \beta_0 = 1 - \frac{\sin \alpha}{\beta}, \quad \alpha_0 = \pi - \alpha, \quad 0 < \alpha_0 < \frac{\pi}{2}.$$

Therefore the function $h(s)$ in (3.13) is the following integral:

$$(5.4) \quad h(s) = -2i\beta k^{1/2} \beta_0 s \int_0^\infty e^{-ikr \cos \alpha_0} H_s^{(2)}(kr) \frac{dr}{\sqrt{r}}.$$

Remark. It is clear that if $\beta = \sin \alpha$, then $\beta_0 = 0$, and therefore the function $h(s) \equiv 0$. Due to the uniqueness of the solution of the integral equation (4.6) (see [5]), we arrive at the situation in which the total field coincides with the incident plane wave, and the scattering wave is identically equal to zero. If we set $\beta = \sin \zeta$, then it means that if $\zeta = \alpha$ or $\zeta = \pi - \alpha$, then the scattering field vanishes. We mention that such an angle ζ is called the Brewster glancing angle [20].

5.1. Analysis of the function $h(s)$. Next, we find an exact representation for the function $h(s)$. Using the integral representation of the Bessel function (see [12, formula 8.421(2)]),

$$(5.5) \quad H_s^{(2)}(kr) = -\frac{e^{i\pi s/2}}{\pi i} \int_{-\infty}^\infty e^{-ikr \cosh \eta - s\eta} d\eta,$$

we calculate the interior integral with respect to r , and instead of (5.4) we get

$$(5.6) \quad h(s) = \frac{2\beta\beta_0 s}{\sqrt{\pi}} e^{i\pi(s/2-1/4)} \int_{-\infty}^\infty \frac{e^{-s\eta} d\eta}{\sqrt{\cos \alpha_0 + \cosh \eta}}, \quad -\frac{1}{2} < \Re(s) < \frac{1}{2}.$$

The last integral is expressible in terms of the conical function $P_{it-1/2}(\cos \alpha_0)$ (see [12, formula 8.842(1)]). Therefore we may find an explicit expression for the function $h(s)$ through the Legendre function

$$(5.7) \quad h(s) = \frac{2i\beta\beta_0 s \sqrt{2\pi}}{\cos \pi s} e^{i\pi(s/2-3/4)} P_{s-1/2}(\cos \alpha_0).$$

Now we take into account the asymptotic expansion for large $|s|$ for the Legendre function (3.22) and obtain the asymptotics of the function $h(s)$ as $\Im(s) \rightarrow \pm\infty$ (here and elsewhere $\Im(s)$ is the imaginary part of a complex number s):

$$(5.8) \quad \begin{aligned} h(s) &\sim \frac{4\beta\beta_0 s^{1/2}}{\sqrt{\sin \alpha_0}} e^{i(3\pi/2-\alpha_0)s}, \quad s \rightarrow +i\infty, \quad \left(\arg s \rightarrow \frac{\pi}{2}\right), \\ h(s) &\sim -\frac{4i\beta\beta_0 s^{1/2}}{\sqrt{\sin \alpha_0}} e^{i(-\pi/2+\alpha_0)s}, \quad s \rightarrow -i\infty, \quad \left(\arg s \rightarrow -\frac{\pi}{2}\right), \end{aligned}$$

and therefore from (4.14) the behavior of the function $h_0(x)$ at the point $x = 0$ is defined by

$$(5.9) \quad h_0(x) = C_0 x^{-\alpha_0/\pi} \sqrt{\log \frac{1}{x}} + O\left(x^{-\alpha_0/\pi} \frac{1}{\sqrt{-\log x}}\right), \quad x \rightarrow +0,$$

where

$$(5.10) \quad C_0 = -\frac{2\sqrt{2}(i+1)\beta\beta_0}{\sqrt{\pi} \sin \alpha_0}.$$

5.2. Behavior of the solution of the integral equation at the singular point. To analyze the function $\chi(x)$ in a neighborhood of the point $x = 0$, we study the following singular integral with the density that has a power-logarithmic singularity:¹

$$(5.11) \quad \Sigma(x) = \int_0^1 y^{\varkappa} \sqrt{\log \frac{1}{y}} \frac{dy}{y+x}, \quad \Re(\varkappa) > -1, \quad \text{as } x \rightarrow +0.$$

First, we note that the function $\Sigma(x)$ is an integral of the Mellin convolution type. Indeed, if we introduce the functions

$$(5.12) \quad v_-(y) = \begin{cases} y^{\varkappa} \sqrt{\log \frac{1}{y}}, & 0 < y < 1, \\ 0, & y > 1, \end{cases} \quad w(y) = \frac{1}{1+y}$$

due to the convolution theorem, we can represent integral (5.11) as follows:

$$(5.13) \quad \Sigma(x) = \int_0^\infty v_-(y) w\left(\frac{x}{y}\right) \frac{dy}{y} = \frac{1}{2\pi i} \int_L V_-(t) W(t) x^{-t} dt,$$

where $L = \{t \in \mathbb{C} : \Re(t) = t_0 \in (\varkappa_0, 1)\}$, $\varkappa_0 = \max\{0, -\Re(\varkappa)\}$, and

$$(5.14) \quad V_-(t) = \int_0^\infty v_-(y) y^{t-1} dy = \frac{\sqrt{\pi}}{2} (\varkappa + t)^{-3/2},$$

$$(5.15) \quad W(t) = \int_0^\infty w(y) y^{t-1} dy = \frac{\pi}{\sin \pi t}.$$

Here we used formulae 4.269(3) and 3.241(2) from [12] and assumed that $\varkappa_0 < \Re(t) < 1$, $\arg(\varkappa + t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as $t \in L$. Thus, we have reduced the original integral to another one:

$$(5.16) \quad \Sigma(x) = \frac{1}{2} \pi^{3/2} \sigma\left(x, \frac{3}{2}\right),$$

¹The Mel'nik formula [11] describes the behavior of the Cauchy integral

$$\frac{1}{2\pi i} \int_l \frac{\phi(\tau) \log^\mu(\tau - a)}{(\tau - a)^\gamma (\tau - z)} d\tau$$

for positive integer values of μ and is not suitable in our case $\mu = 1/2$. Here l is an open contour with the starting point $z = a$ and an ending point $z = b$.

where

$$(5.17) \quad \sigma(x, q) = \frac{1}{2\pi i} \int_L \frac{x^{-t} dt}{(\varkappa + t)^q \sin \pi t}, \quad 0 < x < 1.$$

At first, we assume that $\Re(q) < 1$. The function $(\varkappa + t)^q$ has two branch points in the half-plane $\Re(t) < t_0$. They are $t = -\varkappa$ and the point at infinity. We cut the t -plane by a straight line from the point $t = -\varkappa$ to infinity in the domain $\Re(t) < t_0$ and fix a single-valued branch of the function $(\varkappa + t)^q$ in such a way that $-\frac{5\pi}{4} \leq \arg(\varkappa + t) \leq \frac{3\pi}{4}$. The Cauchy theorem and Jordan's lemma lead to the following relation:

$$(5.18) \quad \sigma(x, q) + I^+(x, q) + I^-(x, q) = \sigma^{(0)}(x, q),$$

where

$$I^\pm(x, q) = \mp \frac{x^\varkappa}{2\pi i} e^{-i\lambda^\pm(q-1)} \int_0^\infty \frac{x^{-r} \exp(i\lambda^\pm) dr}{r^q \sin\{\pi[r \exp(i\lambda^\pm) - \varkappa]\}},$$

$$(5.19) \quad \lambda^+ = \frac{3\pi}{4}, \quad \lambda^- = -\frac{5\pi}{4},$$

and

$$(5.20) \quad \sigma^{(0)}(x, q) = \frac{1}{\pi} \sum_{m=0}^\infty \frac{(-1)^m x^m}{(\varkappa - m)^q}, \quad -\frac{5\pi}{4} \leq \arg(\varkappa - m) \leq \frac{3\pi}{4}, \quad m = 0, 1, \dots$$

Using the asymptotic expansion

$$(5.21) \quad \frac{1}{\sin\{\pi[r \exp(i\lambda^\pm) - \varkappa]\}} = -\frac{1}{\sin \pi \varkappa} - \frac{\pi e^{i\lambda^\pm} \cot \pi \varkappa}{\sin \pi \varkappa} r + O(r^2), \quad r \rightarrow 0,$$

we obtain

$$(5.22) \quad \begin{aligned} I^\pm(x, q) = \mp \frac{x^\varkappa e^{\mp \pi i q}}{2\pi i \sin \pi \varkappa} & \left[\Gamma(1 - q) \log^{q-1} \frac{1}{x} - \pi \cot \pi \varkappa \Gamma(2 - q) \log^{q-2} \frac{1}{x} \right] \\ & + O\left(x^\varkappa \log^{q-3} \frac{1}{x}\right), \quad x \rightarrow +0. \end{aligned}$$

Then, combining (5.22) and (5.19), we have

$$(5.23) \quad \sigma(x, q) = \sigma^{(0)}(x, q) + \sigma^{(1)}(x, q),$$

where the $\sigma^{(0)}(x, q)$ is defined in (5.20) and

$$(5.24) \quad \sigma^{(1)}(x, q) = \frac{x^\varkappa \log^{q-1} \frac{1}{x}}{\Gamma(q) \sin \pi \varkappa} \left[-1 + \frac{\pi(1 - q) \cot \pi \varkappa}{\log \frac{1}{x}} + O\left(\frac{1}{\log^2 \frac{1}{x}}\right) \right], \quad x \rightarrow +0, \quad \Re(q) < 1.$$

Next, we analytically continue formula (5.23) into the domain $\Re(q) > 1, q \neq 2, 3, \dots$, and find the representation for the original function $\Sigma(x)$:

$$(5.25) \quad \Sigma(x) = \frac{1}{2} \pi^{3/2} \left[\sigma^{(0)}\left(x, \frac{3}{2}\right) + \sigma^{(1)}\left(x, \frac{3}{2}\right) \right].$$

In particular, when $-1 < \Re(\varkappa) < 0$, formula (5.25) yields

$$(5.26) \quad \Sigma(x) = -\frac{\pi x^\varkappa}{\sin \pi \varkappa} \sqrt{\log \frac{1}{x}} \left[1 + \frac{\pi \cot \pi \varkappa}{2 \log \frac{1}{x}} + O\left(\frac{1}{\log^2 \frac{1}{x}}\right) \right], \quad x \rightarrow +0.$$

Formula (5.26) enables us to define the singularity of the solution of (4.13) at the point $x = 0$. By comparing representations (5.9) and (5.26), we conclude that there is only one possibility: the solution of (4.13) behaves at the endpoint $x = 0$ as follows:

$$(5.27) \quad \chi_0(x) = A_0 x^{-\alpha_0/\pi} \sqrt{\log \frac{1}{x}} + O\left(x^{-\alpha_0/\pi} \frac{1}{\sqrt{-\log x}}\right), \quad x \rightarrow +0.$$

In any other case, the left-hand side of (4.13) will not be consistent with the right-hand side as $x \rightarrow 0$. We substitute formulae (5.9), (5.27), and (4.15) into (4.13) and get

$$(5.28) \quad A_0 \left(1 + \frac{\beta}{\sin \alpha}\right) x^{-\alpha_0/\pi} \sqrt{\log \frac{1}{x}} = -C_0 x^{-\alpha_0/\pi} \sqrt{\log \frac{1}{x}} + O\left(x^{-\alpha_0/\pi} \frac{1}{\sqrt{-\log x}}\right),$$

$$x \rightarrow +0.$$

From here we immediately find the constant A_0 ,

$$(5.29) \quad A_0 = -\left(1 + \frac{\beta}{\sin \alpha}\right)^{-1} C_0.$$

It is clear that it is possible to write down a full asymptotic expansion as $x \rightarrow 0$ for the functions $K_0(x)$, $h_0(x)$, and the integral (5.11), and therefore from (4.13) an expansion as $x \rightarrow 0$ for the unknown function $\chi_0(x)$ with the coefficients constructed in explicit form.

Now, it is easy to define the behavior of the function $\chi(\xi)$ as $\xi \rightarrow -\infty$, and the basic function $\Phi(\sigma - 1)$ as $\sigma \rightarrow \pm i\infty$. Indeed, relations (4.14), (4.11), (4.15) and (5.9), (5.27) yield

$$(5.30) \quad \chi(\xi) = \sqrt{\pi} A_0 e^{(\pi/2 - \alpha_0)\xi} \left\{ \sqrt{-\xi} + O\left(\frac{1}{\sqrt{-\xi}}\right) \right\}, \quad \xi \rightarrow -\infty,$$

$$h(\xi) = \sqrt{\pi} C_0 e^{(\pi/2 - \alpha_0)\xi} \left\{ \sqrt{-\xi} + O\left(\frac{1}{\sqrt{-\xi}}\right) \right\}, \quad \xi \rightarrow -\infty \quad \left(0 < \alpha_0 < \frac{\pi}{2}\right),$$

and therefore the function $\Phi(\sigma - 1)$ decays exponentially as $\sigma \rightarrow \pm i\infty$:

$$(5.31) \quad \Phi(\sigma - 1) = \frac{\sqrt{\pi} C_0}{2(\sin \alpha + \beta)} e^{(\pi/2 - \alpha_0)\xi} \left\{ \sqrt{-\xi} + O\left(\frac{1}{\sqrt{-\xi}}\right) \right\}, \quad \sigma = i\xi \rightarrow -i\infty,$$

$$\Phi(\sigma - 1) = \frac{\sqrt{\pi} C_0}{2(\sin \alpha + \beta)} e^{(-3\pi/2 + \alpha_0)\xi} \left\{ \sqrt{\xi} + O\left(\frac{1}{\sqrt{\xi}}\right) \right\}, \quad \sigma = i\xi \rightarrow +i\infty,$$

with the constant C_0 defined explicitly in (5.10).

6. Numerical solution of the integral equation. Despite the singularity of the integral equation (4.10),² it can be solved numerically rather efficiently. We have established the exponential decay of the functions $\chi(\xi)$ and $h(\xi)$ as $\xi \rightarrow -\infty$, and therefore even the following simple interpolation should provide a good rate of convergence of an approximate solution to the exact one. Let us consider a graded mesh

$$(6.1) \quad x_n = -\nu n^\delta, \quad n = 0, 1, \dots, N,$$

for some grading exponent $\delta \in (0, 1)$ and a positive constant ν . (It can be more convenient to take $x_n = -\nu(n)n^{\delta(n)}$ with $\nu(n) \rightarrow \nu_0 > 0$, $\delta(n) \rightarrow \delta_0 \in (0, 1)$, as $n \rightarrow \infty$.) Then $0 = x_0 > x_1 > x_2 > \dots > x_N \rightarrow -\infty$, $N \rightarrow \infty$. We introduce the notations

$$(6.2) \quad \chi(\xi_m) = X_m \quad \text{as} \quad \xi_m = \frac{x_{m-1} + x_m}{2}, \quad m = 1, 2, \dots, N,$$

and approximate the function $\chi(\xi)$ by a piecewise constant function $\chi(\xi) = X_m$, $\xi \in (x_m, x_{m-1})$. Then we discretize (4.10) as follows:

$$X_n + \frac{K(i\xi_n)}{4} \sum_{m=1}^N X_m \int_{x_m}^{x_{m-1}} \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi_n)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi_n)} \right) d\eta = -h(i\xi_n),$$

$$(6.3) \quad n = 1, 2, \dots, N.$$

Thus, for the coefficients X_k we get the following system of algebraic equations:

$$(6.4) \quad X_n + \sum_{m=1}^N c_{nm} X_m = b_n, \quad n = 1, 2, \dots, N,$$

where

$$c_{nm} = \frac{K(i\xi_n)}{4} \int_{x_m}^{x_{m-1}} \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi_n)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi_n)} \right) d\eta,$$

$$(6.5) \quad b_n = -h(i\xi_n).$$

The integrals in the representation of the coefficients c_{nm} can be estimated explicitly:

$$(6.6) \quad c_{nm} = \frac{K(i\xi_n)}{\pi} (t_{nm}^{(1)} + t_{nm}^{(2)}),$$

where

$$(6.7) \quad t_{nm}^{(1)} = \tan^{-1} \frac{\lambda_{m-1}}{\mu_n} - \tan^{-1} \frac{\lambda_m}{\mu_n},$$

²Equation (4.10) is defined on the unbounded set $\xi \in (-\infty, 0)$, and the kernel of the corresponding integral equation (4.13) on the segment $(0, 1)$ has an immovable singularity at the point $x = y = 0$.

$$(6.8) \quad t_{nm}^{(2)} = \frac{\lambda_m - \lambda_{m-1}}{\mu_n} + \frac{1}{\mu_n^2} [\tan^{-1}(\lambda_{m-1}\mu_n) - \tan^{-1}(\lambda_m\mu_n)],$$

$$\lambda_m = e^{\pi x_m/2} \quad (m = 0, 1, \dots, N), \quad \mu_n = \sqrt{\lambda_n \lambda_{n-1}} \quad (n = 1, 2, \dots, N),$$

$$(6.9) \quad 0 < \lambda_m \leq 1, \quad 0 < \mu_n < 1.$$

We note that, for small μ_n , instead of formula (6.8) it is more convenient to use the series representation of the function $\tan^{-1} x$, namely,

$$(6.10) \quad t_{nm}^{(2)} = -\frac{1}{3}\mu_n(\lambda_{m-1}^3 - \lambda_m^3) + \frac{1}{5}\mu_n^3(\lambda_{m-1}^5 - \lambda_m^5) - \dots$$

One may conclude that $t_{nm}^{(2)} \rightarrow 0$ as $n \rightarrow \infty$ (m is fixed) and $m \rightarrow \infty$ (n is fixed). The rate of the decay is exponential. For calculation of the coefficients $t_{nm}^{(1)}$, the following two series representations are useful:

$$(6.11) \quad t_{nm}^{(1)} = \frac{\lambda_{m-1}}{\mu_n} - \frac{\lambda_m}{\mu_n} - \frac{1}{3} \left(\frac{\lambda_{m-1}}{\mu_n} \right)^3 + \frac{1}{3} \left(\frac{\lambda_m}{\mu_n} \right)^3 - \dots \quad \text{as } \frac{\lambda_m}{\mu_n} \rightarrow 0,$$

$$t_{nm}^{(1)} = -\frac{\mu_n}{\lambda_{m-1}} + \frac{\mu_n}{\lambda_m} + \frac{1}{3} \left(\frac{\mu_n}{\lambda_{m-1}} \right)^3 - \frac{1}{3} \left(\frac{\mu_n}{\lambda_m} \right)^3 + \dots \quad \text{as } \frac{\lambda_m}{\mu_n} \rightarrow \infty.$$

Thus, the coefficients c_{nm} , b_n decay exponentially as $n \rightarrow \infty$ (m is fixed):

$$c_{nm} = O\left(e^{\pi x_n/2}\right), \quad n \rightarrow \infty \quad (x_n \rightarrow -\infty), \quad m \text{ is fixed},$$

$$(6.12) \quad b_n = O\left(e^{(\pi/2 - \alpha_0)x_n} \sqrt{-x_n}\right), \quad n \rightarrow \infty.$$

If n is fixed and $m \rightarrow \infty$, then $c_{nm} = O(e^{\pi x_m/2})$. For the chosen ν and δ in (6.1), denote by

$$(6.13) \quad \mathbf{X}^* = (X_1^*, X_2^*, \dots, X_N^*, \dots)$$

the exact solution of an infinite system of algebraic equations corresponding to the limit case $N = \infty$ of system (6.4):

$$(6.14) \quad X_n^* + \sum_{m=1}^{\infty} c_{nm} X_m^* = b_n, \quad n = 1, 2, \dots$$

It is clear that for $0 < \delta < 1$ there exists such an integer $M > 0$ that

$$(6.15) \quad \sum_{m=1}^{\infty} |c_{nm}| < 1, \quad n = M, M + 1, \dots,$$

and the coefficients b_n decay exponentially at infinity. Therefore, by the theory of approximate methods [16], the existence and the uniqueness of the solution of the integral equation [5] (when the real part of the surface impedance is positive), an approximate solution $\mathbf{X}^{(N)} = (X_1, X_2, \dots, X_N, 0, 0, \dots)$ of the infinite system (6.14) converges to the exact solution $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_N^*, \dots)$ in the space m of all bounded sequences.

7. Diffraction coefficient.

7.1. An impedance cone. We now turn to the wave field u , the solution of problem (2.10), (2.11). Applying the inverse Kontorovich–Lebedev transform (3.5) for $m = 0$, we get

$$(7.1) \quad u(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^{i\infty} su_s(\theta)J_s(kr)ds.$$

Since the function $u_s(\theta)$ satisfies the equality $u_s(\theta) = e^{i\pi s}u_{-s}(\theta)$ (see (3.6)), then

$$(7.2) \quad u(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^0 su_s(\theta)[J_s(kr) - e^{-i\pi s}J_{-s}(kr)]ds,$$

where the function $u_s(\theta)$ is defined in terms of the solution of the integral equation (4.10):

$$(7.3) \quad u_s(\theta) = \frac{1}{k^{1/2}} \frac{P_{s-1/2}(\cos \theta)\Phi(s-1)}{\frac{d}{ds}P_{s-1/2}(\cos \alpha)} = \frac{P_{s-1/2}(\cos \theta)[\chi(-is) + h(s)]}{2i\beta k^{1/2}sP_{s-1/2}(\cos \alpha)}.$$

Therefore the scattering wave field is given by

$$(7.4) \quad u(r, \theta) = -\frac{1}{4i\beta\sqrt{r}k^{1/2}} \int_{-i\infty}^{i\infty} \frac{P_{s-1/2}(\cos \theta)[\chi(-is) + h(s)]}{P_{s-1/2}(\cos \alpha)} J_s(kr)ds.$$

Let us find the diffraction coefficient $D(\theta)$ (the scattering diagram), an important characteristic in the geometric theory of diffraction [17]. The coefficient $D(\theta)$ is introduced as follows,

$$(7.5) \quad u(r, \theta) = \frac{D(\theta)}{kr} e^{-ikr} + O\left(\frac{1}{(kr)^2}\right), \quad kr \rightarrow \infty,$$

and it describes the high frequency asymptotics of the solution. To find the diffraction coefficient, we use the behavior of the Bessel function at infinity,

$$(7.6) \quad J_{\pm s}(kr) = \sqrt{\frac{2}{\pi kr}} \cos\left(kr \mp \frac{\pi s}{2} - \frac{\pi}{4}\right) + O\{(kr)^{-3/2}\}, \quad kr \rightarrow \infty,$$

and from formula (7.2) we obtain

$$(7.7) \quad u(r, \theta) = \frac{(1-i)e^{-ikr}}{2\sqrt{\pi kr}} \int_{-i\infty}^0 su_s(\theta)e^{-i\pi s/2} \sin \pi s ds + O\left(\frac{1}{(kr)^2}\right), \quad kr \rightarrow \infty.$$

One needs to study the convergence of this integral. The behavior of the function $\Phi(s-1)$ as $s \rightarrow -i\infty$ was analyzed in section 5.2:

$$(7.8) \quad \Phi(s-1) = O\{e^{-(\pi/2-\alpha_0)is}(is)^{1/2}\}, \quad s \rightarrow -i\infty.$$

From formulae (3.16), (3.22), and (7.8) we easily deduce the asymptotics

$$(7.9) \quad su_s(\theta)e^{-i\pi s/2} \sin \pi s = O(s^{1/2}e^{is(\pi+\theta-2\alpha)}), \quad s \rightarrow -i\infty.$$

Therefore the integral (7.7) converges if

$$(7.10) \quad \theta < 2\alpha - \pi.$$

Now we substitute the expression (7.3) for the function $u_s(\theta)$ into the asymptotic formula (7.7) and compare the new formula with the classical representation (7.5). It yields the expression for the diffraction coefficient:

$$(7.11) \quad D(\theta) = \frac{i+1}{4\beta\sqrt{\pi}} \int_{-\infty}^0 \frac{P_{i\xi-1/2}(\cos \theta)}{P_{i\xi-1/2}(\cos \alpha)} [\chi(\xi) + h(i\xi)] e^{\pi\xi/2} \sinh \pi\xi d\xi.$$

The integral is improper, it converges uniformly in the domain $0 \leq \theta < 2\alpha - \pi$, and the rate of the convergence is exponential. The function $\chi(\xi)$ is the solution of (4.10).

Let us find an approximate formula for the diffraction coefficient, based on an approximate solution of (4.10) constructed in section 6. We split the interval $(-\infty, 0)$ as in section 6 and obtain from (7.11)

$$(7.12) \quad D^{(N)}(\theta) = \frac{i+1}{4\beta\sqrt{\pi}} \left[\sum_{m=1}^N X_m \int_{x_m}^{x_{m-1}} \Lambda(\xi, \theta) d\xi + \int_{-\infty}^0 h(i\xi) \Lambda(\xi, \theta) d\xi \right],$$

where $D^{(N)}(\theta)$ is an approximate value of the diffraction coefficient $D(\theta)$ and

$$(7.13) \quad \Lambda(\xi, \theta) = \frac{P_{i\xi-1/2}(\cos \theta)}{P_{i\xi-1/2}(\cos \alpha)} e^{\pi\xi/2} \sinh \pi\xi.$$

One can simplify this formula further, using an approximate formula for integrals (7.12):

$$(7.14) \quad D^{(N)}(\theta) = \frac{i+1}{4\beta\sqrt{\pi}} \sum_{m=1}^N (X_m - b_m) \Lambda(\xi_m, \theta) (x_{m-1} - x_m).$$

In general, the diffraction coefficient is complex. Let us find a real characteristic that is closely related to the far field coefficient $D(\theta)$, the bistatic radar cross section, or the scattering cross section (see [8]):

$$(7.15) \quad \sigma(\theta) = \lim_{r \rightarrow \infty} 4\pi r^2 \frac{|u|^2}{|u^{inc}|^2}.$$

Formulae (5.1) and (7.15) yield

$$(7.16) \quad \sigma(\theta) = \frac{\lambda^2}{\pi} |D(\theta)|^2,$$

where $\lambda = 2\pi k^{-1}$ is the wavelength. To compute the total scattering cross section σ_T

$$(7.17) \quad \sigma_T = \frac{1}{2} \int_0^\alpha \sigma(\theta) \sin \theta d\theta,$$

one needs to know the diffraction coefficient for all values of $\theta \in (0, \alpha)$. We note that integral (7.11) is convergent only for $0 \leq \theta < 2\alpha - \pi$; namely, in the region where no

reflected waves are observed. The direction $\theta = 2\alpha - \pi$ is singular for the diffraction coefficient: it is infinite. However, it is possible to evaluate the diffraction coefficient for the rest values of θ : $2\alpha - \pi < \theta < \alpha$ (in the region of the reflected waves). Such an approach for acoustically soft and hard cones was developed in [23] and generalized for arbitrary smooth convex soft and hard cones in [3]. The idea for the calculation of the diffraction coefficient for an impedance cone

$$(7.18) \quad D(\theta) = \frac{1-i}{2\sqrt{\pi}} \int_{-i\infty}^0 \frac{sP_{s-1/2}(\cos \theta)}{\frac{d}{d\theta}P_{s-1/2}(\cos \alpha)} e^{-\pi is/2} \sin \pi s \Phi(s-1) ds$$

is based on the analytical continuation of the function $\Phi(s)$ into the strips $\Pi_m = \{s \in \mathbb{C} : -2 + 2m < \Re(s) < 2m\}$ ($m = 0, 1, 2, \dots$) (see Appendix C) and the results of [23], [3]. Realization of this idea, however, is not trivial and will be presented elsewhere. Here we confine ourselves to evaluating the diffraction coefficient and the scattering cross section in the domain where no reflected waves are observed.

7.2. Acoustically soft cones. Let us consider the particular case of the problem (2.10), (2.11) when $\beta = \infty$, i.e., the Dirichlet problem for the Helmholtz equation in a cone (axially symmetric case): $u = -u^{inc}$ as $0 < r < \infty$, $\theta = \alpha$. This situation corresponds to an acoustically soft cone. The scattering wave field is defined by the Kontorovich–Lebedev integral

$$(7.19) \quad u(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^{i\infty} s u_s(\theta) J_s(kr) ds,$$

where

$$(7.20) \quad u_s(\theta) = A(s) P_{s-1/2}(\cos \theta),$$

with the function $A(s)$ defined from the boundary condition

$$(7.21) \quad A(s) P_{s-1/2}(\cos \alpha) = - \int_0^\infty e^{-ikr \cos \alpha_0} H_s^{(2)}(kr) \frac{dr}{\sqrt{r}}$$

by the equality

$$(7.22) \quad A(s) = \frac{\sqrt{2\pi}}{k^{1/2}} e^{i\pi s/2 - 3\pi i/4} \frac{P_{s-1/2}(\cos \alpha_0)}{\cos \pi s P_{s-1/2}(\cos \alpha)}.$$

To define the diffraction coefficient, we use the asymptotic formula (7.7), which may be transformed into the form

$$(7.23) \quad u(r, \theta) = -\frac{(1+i)e^{-ikr}}{2\sqrt{\pi kr}} \int_{-\infty}^0 \xi u_{i\xi}(\theta) e^{\pi\xi/2} \sinh \pi\xi + O\left(\frac{1}{(kr)^2}\right), \quad kr \rightarrow \infty.$$

If we compare formulae (7.5), (7.23) and take into account relations (7.20), (7.22), we finally obtain the expression for the diffraction coefficient

$$(7.24) \quad D_S(\theta) = i \int_0^\infty \frac{\xi \tanh \pi\xi P_{i\xi-1/2}(-\cos \alpha)}{P_{i\xi-1/2}(\cos \alpha)} P_{i\xi-1/2}(\cos \theta) d\xi$$

that coincides with the classical formula [8]. We note that the last integral converges in the same domain as in the case of the impedance cone, namely $0 < \theta < 2\alpha - \pi$. The scattering cross section $\sigma(\theta)$ is expressed in terms of the diffraction coefficient by formula (7.16).

7.3. Acoustically hard cones. In the case of the Neumann condition on the surface of the cone (the obstacle is hard), we get $\beta = 0$ and

$$(7.25) \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{1}{r} \frac{\partial u^{inc}}{\partial \theta} \quad \text{as } \theta = \alpha,$$

and for the function $A(s)$ in relation (7.20) instead of (7.21) we have

$$(7.26) \quad A(s) \frac{d}{d\theta} P_{s-1/2}(\cos \theta) = -\frac{d}{d\theta} \int_0^\infty e^{ikr \cos \theta} H_s^{(2)}(kr) \frac{dr}{\sqrt{r}} \quad \text{as } \theta = \alpha.$$

Therefore

$$(7.27) \quad A(s) = \frac{\sqrt{2\pi}}{k^{1/2}} e^{i\pi s/2 - 3\pi i/4} \frac{\frac{d}{d\theta} P_{s-1/2}(-\cos \theta)}{\cos \pi s \frac{d}{d\theta} P_{s-1/2}(\cos \theta)} \quad \text{as } \theta = \alpha.$$

The diffraction coefficient is given by (see also [8])

$$(7.28) \quad D_{\mathcal{H}}(\theta) = i \int_0^\infty \frac{\xi \tanh \pi \xi \frac{d}{d\theta} P_{i\xi-1/2}(-\cos \alpha)}{\frac{d}{d\theta} P_{i\xi-1/2}(\cos \alpha)} P_{i\xi-1/2}(\cos \theta) d\xi,$$

where

$$(7.29) \quad \frac{d}{d\theta} P_{s-1/2}(\pm \cos \alpha) = -\frac{s+1/2}{\sin \alpha} [\mp P_{s+1/2}(\pm \cos \alpha) + \cos \alpha P_{s-1/2}(\pm \cos \alpha)].$$

As before, the integral converges in the region where no reflected waves are observed.

7.4. Numerical results. Numerical computations are implemented for the case of the axially symmetric incident wave (5.1) creating the scattering wave (7.4). The following parameters of the mesh (6.1) are chosen: $\delta = 3$, $\nu = 1.5 \cdot 10^{-5}$, and the number of the equations in the linear system (6.4) is taken to be $N = 200$ (although even $N = 100$ gives a good approximation). It is clear that, to increase N , we need to take $\delta(n) < 1$ as $n \gg 1$. In the table below, we report the values of the diffraction coefficient for $\alpha = \frac{5\pi}{6}$ for the acoustically soft cone (the Dirichlet problem: $\beta = \infty$), the hard cone (the Neumann problem: $\beta = 0$), and the two cases of the impedance condition: $\beta = 1$ and $\beta = 1 + i$.

	$\beta = \infty$ [2]	$\beta = \infty$	$\beta = 1$	$\beta = 1 + i$		$\beta = 0$
θ	$2\pi i f$	$-iD$	$-iD$	$\Re(D)$	$\Im(D)$	$-iD$
0	0.2399	0.2399	0.03486	-0.05287	0.05152	-0.05217
$3\pi/24$	0.2576	0.2577	0.03839	-0.05776	0.05697	-0.05824
$6\pi/24$	0.3231	0.3232	0.05196	-0.07620	0.07812	-0.08117
$9\pi/24$	0.4966	0.4967	0.09110	-0.1269	0.1399	-0.1520
$12\pi/24$	1.0860	1.0862	0.2423	-0.3063	0.3843	-0.4590
$15\pi/24$	8.3073	8.2856	2.4441	-2.5373	3.9465	-6.0754

In the second column of the table, the notation $f = -(2\pi)^{-1}D(\theta)$ from [2] is used. The coefficient $\Im(f)$ was calculated in [2] for the same angle $\alpha = \frac{5\pi}{6}$ and $\beta = \infty$ (the

soft cone), and we adduce them to compare with our own results (the third column of the table). The results of calculations for the limiting case of the acoustically soft cone are in good agreement with the corresponding results of [2]. Additionally, it is seen that as θ approaches the singular line $2\alpha - \pi$ (in this case it is $2\pi/3$), the diffraction coefficient grows to infinity for three types of the boundary conditions.

In Figure 2, we present the graph of the diffraction coefficient $-iD$ versus θ for $\alpha = \frac{5\pi}{6}$ for some real values of the parameter β . The set of curves shows how the imaginary part of the diffraction coefficient ($\Re(D) = 0$ if $\Im(\beta) = 0$) grows as $\theta \rightarrow 2\alpha - \pi$: $-iD(\theta) \rightarrow -\infty$ as $\beta < \sin \alpha$, $-iD(\theta) \rightarrow +\infty$ as $\beta > \sin \alpha$. In this case, $\sin \alpha = 1/2$ and $2\alpha - \pi = 2\pi/3$. At the point $\beta = \sin \alpha$ in a neighborhood of the point $\theta = 2\alpha - \pi$ the curves $-iD(\theta)$ are unstable: for $\beta \rightarrow \sin \alpha + 0$ we have $-iD(\theta) \rightarrow +\infty$, and for $\beta \rightarrow \sin \alpha - 0$ we get $-iD(\theta) \rightarrow -\infty$ as $\theta \rightarrow 2\alpha - \pi - 0$. The point $\theta = 0$ is the global minimum for the modulus of the diffraction coefficient $D(\theta)$.

It was mentioned in section 5 that the value $\beta = \sin \alpha$ was critical for the coefficient $D(\theta)$: $D(\theta) = 0$ for all θ . If $\beta < \sin \alpha$, then $-iD(\theta) < 0$, and for $\beta > \sin \alpha$ the coefficient $-iD(\theta)$ is positive for all values of $\theta \in [0, 2\alpha - \pi)$. The situation becomes evident in Figure 3, which shows the dependence of $-iD(\theta)$ upon the acoustic admittance β ($\Im(\beta) = 0$). It is seen that all curves intersect the axis $-iD(\theta) = 0$ at the same point, namely, $\beta = 1/2 = \sin \alpha$.

The graphs in Figure 4 represent the coefficient $-iD(\theta)$ as a function of θ ($0 < \theta < 2\alpha - \pi$) for $\alpha = \frac{7\pi}{12}$, $\alpha = \frac{3\pi}{4}$, and $\alpha = \frac{11\pi}{12}$ when $\beta = 1$. The coefficient $-iD(\theta)$ for “blunt” cones is greater than that for narrow cones. For a narrow cone the diffraction coefficient vanishes. This fact is in good agreement with the approximate formula

$$(7.30) \quad -iD(\theta) = \frac{(\pi - \alpha)\beta}{2(\cos \theta + 1)^2}$$

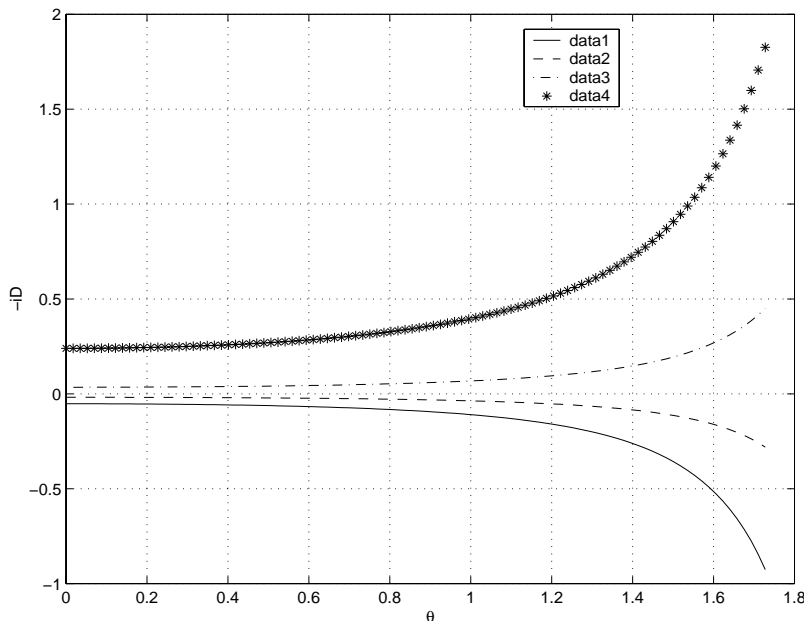


FIG. 2. Diffraction coefficient $-iD(\theta)$ versus θ with $\alpha = \frac{5\pi}{6}$ for the following values of the acoustic admittance β : 10^{-4} (data1), 0.3 (data2), 1 (data3), and 10^4 (data4).

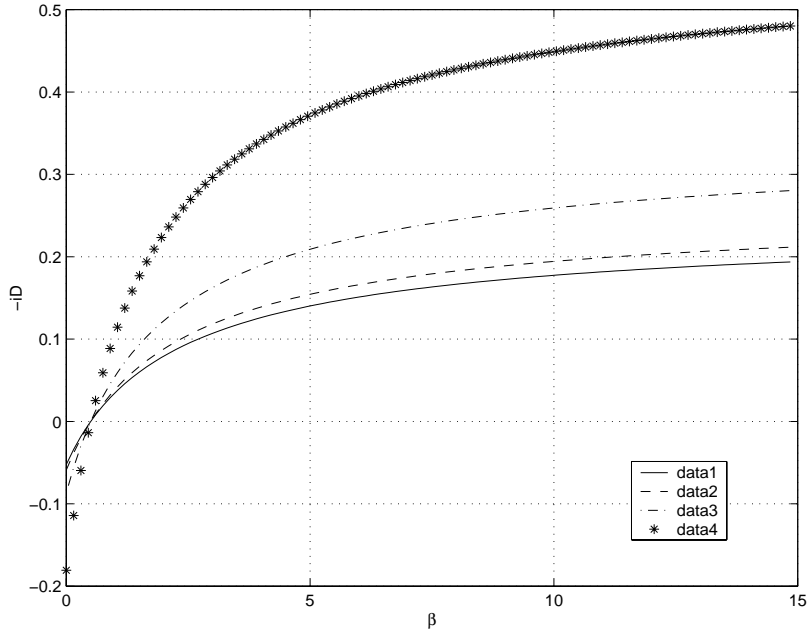


FIG. 3. Diffraction coefficient $-iD(\theta)$ versus β with $\alpha = \frac{5\pi}{6}$ for the following values of the angle θ : 0 (data1), $\varphi/5$ (data2), $2\varphi/5$ (data3), and $3\varphi/5$ (data4), where $\varphi = 2\alpha - \pi$.

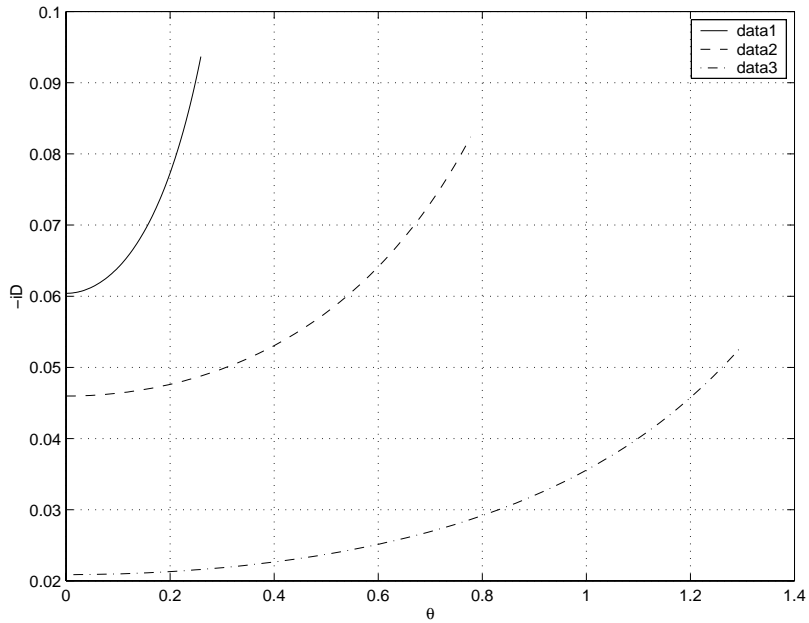


FIG. 4. Diffraction coefficient $-iD(\theta)$ versus θ with $\beta = 1$ for the following values of the angle α : $\frac{7\pi}{12}$ (data1), $\frac{3\pi}{4}$ (data2), and $\frac{11\pi}{12}$ (data3).

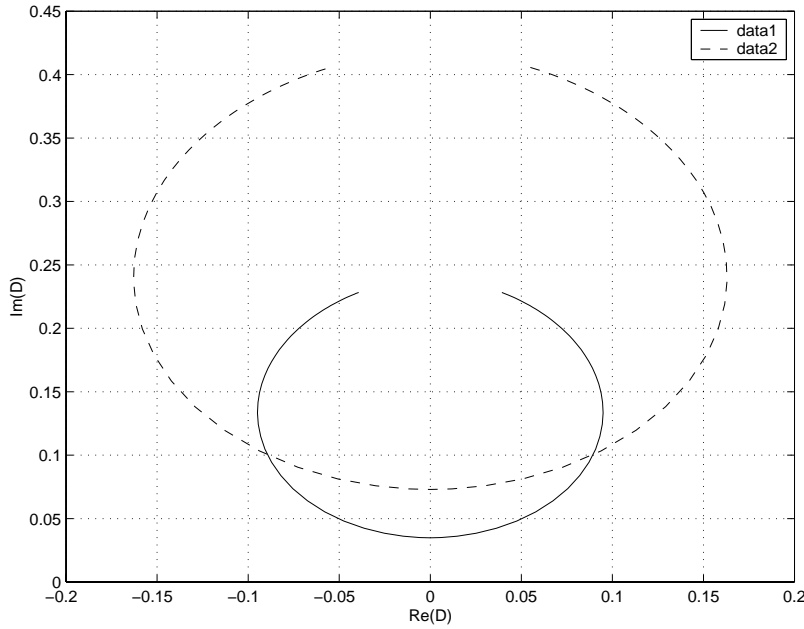


FIG. 5. Parametrically defined complex-valued diffraction coefficient $\mathfrak{J}(D)$ versus $\Re(D)$, with $\beta = 1 + iB$, $-20 < B < 20$, $\alpha = \frac{5\pi}{6}$ for $\theta = 0$ (data1), and $\theta = \frac{2\pi}{3}$ (data2).

found in [7]. The situations $\alpha = \pi$ and $\alpha = \pi/2$ are both singular: if $\alpha \rightarrow \pi$, then the cone disappears, and if $\alpha \rightarrow \pi/2$, then the problem turns into the problem of diffraction of a plane incident wave by a half-space. It is known that the leading term of the far scattering field for the two-dimensional problem differs from (7.5).

The case of complex β is illustrated by the graph in Figure 5, where $\Re(D)$ and $\mathfrak{J}(D)$ are parametrically defined as functions of β for $\theta = 0$ and $\theta = \frac{2\pi}{3}$ ($\Re(\beta) = 1$, $\alpha = \frac{5\pi}{6}$). The point traverses along the orbit from the initial point $\beta = 1 - i\infty$ to the terminal point $\beta = 1 + i\infty$ (these two point are not shown in the graph), and the direction of the motion is clockwise. The points $\beta = 1 - i\infty$ and $\beta = 1 + i\infty$ provide the same values of the diffraction coefficient corresponding to that for the Dirichlet problem.

Figure 6 shows the scattering cross section $\lambda^{-2}\sigma(\theta)$ as a dimensionless function of $\Re(\beta)$ for some fixed values of $\mathfrak{J}(\beta)$ and for $\theta = 0$. The σ has a minimum for each value of $\mathfrak{J}(\beta)$. If $\mathfrak{J}(\beta) = 0$, then this minimum is equal to zero at the point $\Re(\beta) = \sin \alpha$ ($\sin \alpha = 1/2$). As $|\mathfrak{J}(\beta)|$ increases from 0 to $+\infty$, then the point of minimum moves to the right to the point $\Re(\beta) = +\infty$.

7.5. Comparative analysis of the soft, hard, and impedance cones. The scattering field in all three cases is given by formulae (7.19), (7.20), where function $A(s)$ for the soft and hard cones is determined by (7.22) and (7.27). For the impedance cone we get

$$(7.31) \quad A_{\mathcal{I}}(s) = \left(1 - \frac{\sin \alpha}{\beta}\right) A_{\mathcal{S}}(s) + \frac{\chi(-is)}{2i\beta k^{1/2} s P_{s-1/2}(\cos \alpha)},$$

where $A_{\mathcal{I}}(s)$ and $A_{\mathcal{S}}(s)$ are the notations for the function $A(s)$ for the impedance and soft cones, respectively. Therefore the scattering field $u_{\mathcal{I}}(r, \theta)$ for the impedance cone

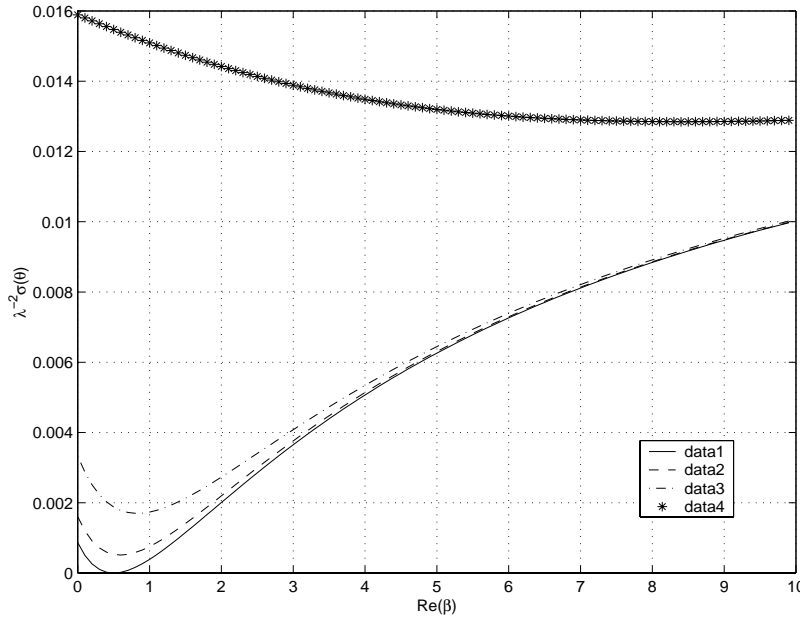


FIG. 6. Scattering cross section $\lambda^{-2}\sigma(\theta)$ versus $\Re(\beta)$, with $\theta = 0$, $\alpha = \frac{5\pi}{6}$ for the following values of $\Im(\beta)$: 0.001 (data1), 0.5 (data2), 1 (data3), and 10 (data4).

is expressible in terms of $u_S(r, \theta)$ for the acoustically soft cone, and the solution of the integral equation (4.10) is as follows:

$$(7.32) \quad u_I(r, \theta) = \left(1 - \frac{\sin \alpha}{\beta}\right) u_S(r, \theta) + u_*(r, \theta),$$

where

$$(7.33) \quad u_*(r, \theta) = \frac{i}{4\beta k^{1/2} \sqrt{r}} \int_{-\infty}^0 \frac{\chi(-is) P_{s-1/2}(\cos \theta)}{P_{s-1/2}(\cos \alpha)} [J_s(kr) - e^{-i\pi s} J_{-s}(kr)] ds.$$

Obviously, it is possible to derive a similar relation for the diffraction coefficient

$$(7.34) \quad D_I(\theta) = \left(1 - \frac{\sin \alpha}{\beta}\right) D_S(\theta) + D_*(\theta),$$

where $D_S(\theta)$ is the diffraction coefficient for the acoustically soft cone determined in (7.24), and $D_*(\theta)$ is an additional term

$$(7.35) \quad D_*(\theta) = -\frac{1+i}{4\beta\sqrt{\pi}} \int_0^\infty \frac{P_{i\xi-1/2}(\cos \theta)}{P_{i\xi-1/2}(\cos \alpha)} \chi(-\xi) e^{-\pi\xi/2} \sinh \pi\xi d\xi.$$

Here χ is the solution of the integral equation (4.10).

The integral representations of the diffraction coefficient for all three boundary conditions are convergent if $\theta < \pi - 2\alpha$, and $D(\theta) \rightarrow \infty$ if $\theta \rightarrow \pi - 2\alpha$, i.e., if θ approaches the singular line. For all narrow cones the diffraction coefficient vanishes,

and for the blunt cones ($\alpha \rightarrow \pi/2$) the diffraction coefficient grows to infinity. If the acoustic admittance is real, then the diffraction coefficient is imaginary,

$$-iD_{\mathcal{H}}(\theta) < -iD_{\mathcal{I}}(\theta) < -iD_{\mathcal{S}}(\theta) \quad \text{for all } \theta \in [0, \pi - 2\alpha),$$

and $-iD_{\mathcal{I}}(\theta)$ is a monotonic function of β . Here $D_{\mathcal{H}}$ is the diffraction coefficient for the hard cone. The coefficient $D_{\mathcal{I}}(\theta)$ vanishes if the angle of the cone coincides with the Brewster angle, i.e., if $\beta = \sin \alpha$. If β is complex, $\beta = \beta_1 + i\beta_2$, the real part of the acoustic admittance is fixed, and the imaginary part changes from $-i\infty$ to $i\infty$, then the diffraction coefficient $D_{\mathcal{I}}(\theta)$ describes a closed orbit with the south pole at the point $\beta_2 = 0$ and the north pole at the point where $\beta_2 = \pm\infty$ and $D_{\mathcal{I}}(\theta) = D_{\mathcal{S}}(\theta)$ for all θ .

There always exists a bounded domain in the β -complex plane for which the scattering cross-section $\sigma(\theta)$ is less than that for the hard cone. We note that the scattering cross-section $\sigma(\theta)$ for the impedance cone is always less than $\sigma(\theta)$ for the acoustically soft cone.

8. Asymptotic expansion for the far field. If $kr \gg 1$ and $\theta < 2\alpha - \pi$, then the scattering field admits the following representation:

$$(8.1) \quad u(r, \theta) = \frac{D(\theta)}{kr} e^{-ikr} + O\left(\frac{1}{(kr)^2}\right), \quad kr \rightarrow \infty,$$

where $D(\theta)$ is the diffraction coefficient found in the previous section. Let us construct a full asymptotic expansion for the far scattering field for $\theta_0 = 0$ and $0 < \theta < 2\alpha - \pi$. To do this, we use the integral representation for the scattering field (7.2), valid for all kr , and the asymptotic expansion for the Bessel function $J_{\pm s}(kr)$ as $kr \rightarrow \infty$,

$$(8.2) \quad \begin{aligned} J_{\pm s}(kr) = & \sqrt{\frac{2}{\pi kr}} \left\{ \cos\left(kr \mp \frac{\pi}{2}s - \frac{\pi}{4}\right) \sum_{m=0}^{n-1} \frac{(-1)^m (\pm s - 2m + 1/2)_{4m}}{(2kr)^{2m} (2m)!} \right. \\ & \left. - \sin\left(kr \mp \frac{\pi}{2}s - \frac{\pi}{4}\right) \sum_{m=0}^{n-1} \frac{(-1)^m (\pm s - 2m - 1/2)_{4m+2}}{(2kr)^{2m+1} (2m+1)!} \right\} + O\left(\frac{1}{(kr)^{2n}}\right), \quad kr \rightarrow \infty, \end{aligned}$$

that can be deduced from formula 8.451(1) of [12]. Next, we take into account that

$$(8.3) \quad \left(s - 2m \pm \frac{1}{2}\right)_{4m+1 \mp 1} = \left(-s - 2m \pm \frac{1}{2}\right)_{4m+1 \mp 1} = \prod_{l=1}^{2m+1/2 \mp 1/2} \left[s^2 - \left(l - \frac{1}{2}\right)^2 \right],$$

where $(a)_m = a(a+1)\cdots(a+m-1)$, and derive the asymptotic expansion for the linear combination $J_s(kr) - e^{-i\pi s} J_{-s}(kr)$ that is involved in the integral representation (7.2):

$$(8.4) \quad \begin{aligned} J_s(kr) - e^{-i\pi s} J_{-s}(kr) = & \frac{e^{-ikr} (i-1) \sin \pi s}{\sqrt{\pi kr} e^{i\pi s/2}} \\ & \times \sum_{m=0}^{n-1} \frac{(-1)^m}{(2m)! (2kr)^{2m}} \left\{ \left(s - 2m + \frac{1}{2}\right)_{4m} - \frac{i(s - 2m - 1/2)_{4m+2}}{2kr(2m+1)} \right\} + O\left(\frac{1}{(kr)^{2n}}\right), \end{aligned}$$

$kr \rightarrow \infty.$

By substituting this expansion into formula (7.2) for the scattering field, we obtain

$$(8.5) \quad u(r, \theta) = \sum_{j=0}^{n-1} \frac{D_j(\theta)}{(kr)^{j+1}} e^{-ikr} + O\left(\frac{1}{(kr)^{n+1}}\right), \quad kr \rightarrow \infty.$$

This is a very interesting formula, for it shows that the coordinates r and θ are separated in the asymptotic expansion and there is the common factor e^{-ikr} for all terms. The coefficients $D_j(\theta)$, which we call the generalized diffraction coefficients, are given by

$$(8.6) \quad D_{2m}(\theta) = \frac{(1-i)k^{1/2}(-1)^m}{2^{2m+1}\sqrt{\pi}(2m)!} \int_{-i\infty}^0 su_s(\theta) \sin \pi s e^{-i\pi s/2} \left(s - 2m + \frac{1}{2}\right)_{4m} ds,$$

$$D_{2m+1}(\theta) = -\frac{(1+i)k^{1/2}(-1)^m}{2^{2m+2}\sqrt{\pi}(2m+1)!} \int_{-i\infty}^0 su_s(\theta) \sin \pi s e^{-i\pi s/2} \left(s - 2m - \frac{1}{2}\right)_{4m+2} ds.$$

In effect, the generalized coefficients form a vector

$$\mathbf{D}(\theta) = (D_0(\theta), D_1(\theta), \dots, D_{n-1}(\theta)).$$

Expansion (8.5) is a Poincaré asymptotic series, and therefore the vector $\mathbf{D}(\theta)$ is always finite-dimensional. The generalized coefficients can be simplified. Setting $s = -i\xi$ and using the relations

$$(8.7) \quad \left(-i\xi - 2m \pm \frac{1}{2}\right)_{4m+1 \mp 1} = \pm \rho_{2m+1/2 \mp 1/2}(\xi),$$

where

$$(8.8) \quad \rho_j(\xi) = \prod_{l=1}^j \left[\xi^2 + \left(l - \frac{1}{2}\right)^2 \right], \quad \rho_0(\xi) = 1,$$

we get the following formula valid for odd and even indices:

$$(8.9) \quad D_j(\theta) = -\frac{i^j(i+1)k^{1/2}}{2^{j+1}j!\sqrt{\pi}} \int_0^\infty \xi u_{-i\xi}(\theta) \sinh \pi \xi e^{-\pi \xi/2} \rho_j(\xi) d\xi.$$

Substituting the relations for the function $u_s(\theta)$ for the cases $\beta = \infty$ and $\beta = 0$ gives rise to explicit expressions for the coefficients $D_j(\theta)$. If the cone is acoustically soft ($\beta = \infty$), then from (7.20) and (7.22) we obtain

$$(8.10) \quad D_j(\theta) = \frac{i^{j+1}}{2^j j!} \int_0^\infty \frac{P_{i\xi-1/2}(-\cos \alpha) P_{i\xi-1/2}(\cos \theta)}{P_{i\xi-1/2}(\cos \alpha)} \xi \tanh \pi \xi \rho_j(\xi) d\xi.$$

Next, for the acoustically hard cone ($\beta = 0$) we get

$$(8.11) \quad D_j(\theta) = \frac{i^{j+1}}{2^j j!} \int_0^\infty \frac{\frac{d}{d\theta} P_{i\xi-1/2}(-\cos \alpha) P_{i\xi-1/2}(\cos \theta)}{\frac{d}{d\theta} P_{i\xi-1/2}(\cos \alpha)} \xi \tanh \pi \xi \rho_j(\xi) d\xi.$$

If the acoustic admittance β is finite and nonzero, then the generalized diffraction coefficients obtained from (7.3) are

(8.12)

$$D_j(\theta) = -\frac{i^j(i+1)}{\beta\sqrt{\pi}2^{j+2}j!} \int_0^\infty \frac{P_{i\xi-1/2}(\cos\theta)}{P_{i\xi-1/2}(\cos\alpha)} \sinh \pi\xi e^{-\pi\xi/2} [\chi(-\xi) + h(-i\xi)] \rho_j(\xi) d\xi.$$

Formulae (8.5) and (8.10)–(8.12) are useful in a number of ways. First, the far scattering field is representable in the form of a linear combination of elementary waves:

$$(8.13) \quad u \sim D_0(\theta) \frac{e^{-ikr}}{kr} + D_1(\theta) \frac{e^{-ikr}}{(kr)^2} + \cdots + D_{n-1}(\theta) \frac{e^{-ikr}}{(kr)^n} + \cdots, \quad kr \rightarrow \infty.$$

Second, the coefficients of the expansion, the generalized diffraction coefficients, converge for $0 \leq \theta < 2\alpha - \pi$ and $D_j(\theta) \rightarrow \infty$ as $\theta \rightarrow 2\alpha - \pi$. However, as in the case $j = 0$ (see [8]), they may be calculated in the region $2\alpha - \pi < \theta < \alpha$, using the principle of analytical continuation. The technique derived in the previous sections is applicable for the generalized diffraction coefficients in the region $\theta < 2\alpha - \pi$. Indeed, the integrands in (8.10)–(8.12) differ from those in (7.24), (7.28), and (7.11) by the factor $i^j(2^j j!)^{-1} \rho_j(\xi)$ only. Finally, from the asymptotic expansion (8.5) we can go on to find the scattering field not only for $kr \rightarrow \infty$ but for reasonable finite values of kr . Obviously, if $kr \rightarrow 0$, then expansion (8.5) is divergent.

Conclusion. The boundary-value problem for the Helmholtz equation in a circular cone with an impedance boundary condition on its face has been examined using the Kontorovich–Lebedev integral transform. The problem has been reduced to a difference equation of the second order in a strip of a complex variable, and then to a convolution-type integral equation with variable coefficients on the imaginary axis subject to an additional condition of symmetry. It has been shown that the equation is equivalent to a 2×2 matrix Riemann–Hilbert problem with a discontinuous coefficient. To analyze the singularities of the solution at infinity, the original equation has been reduced to a complete singular integral equation of the second kind with the Dixon kernel. Its coefficients are variable, and the right-hand side has a power-logarithmic singularity. The behavior of the Dixon’s integral at the ends in the case of a power-logarithmic singularity in the density has been described. A numerical solution of the integral equation has been constructed.

For the diffraction coefficient an exact formula (in terms of the solution of the integral equation) and an approximate expression convenient for numerical computations have been derived. The limit cases of the Dirichlet and Neumann boundary conditions (the acoustically soft and hard cones, respectively) have been considered. It has been shown that the scattering cross-section σ as a function of the acoustic admittance β has a global minimum. The quantity σ for the impedance cone is always less than that for the acoustically soft cone, and there exists such a parameter β_* that if $|\beta| \in (0, |\beta_*|)$, then the scattering cross-section for the impedance cone is less than that for the acoustically hard cone. In addition, a full asymptotic expansion as $kr \rightarrow \infty$ of the diffraction field for the impedance, acoustically soft, and hard cones has been found.

Appendix A. A 2×2 matrix Riemann–Hilbert problem with a discontinuous coefficient. To reduce the second-order difference equation (3.17) in the strip Π to a Riemann–Hilbert problem, we develop Čerskii’s idea ([9]; see also [1]) proposed for Carleman’s problem with a shift. Let us map the strip $\Pi = \{-2 < \Re(s) < 0\}$

on a complex plane with the cut along the positive semiaxis

$$\Gamma = \{\zeta \in \mathbb{C} : \zeta = \xi, \quad \arg \zeta = 0, \quad 0 < \xi < \infty\},$$

using the “gluing” function $\zeta = e^{-\pi is}$. The upper side ($\zeta = \xi + i0, \arg \zeta = 0$) of the positive semiaxis of the ζ -plane corresponds to the contour Ω of the s -plane. The contour $\{s \in \mathbb{C} : \Re(s) = -1\}$ is mapped onto the negative semiaxis ($-\infty < \xi < 0$) of the ζ -plane, and the contour $\{\Re(s) = -2\}$ of the s -plane is mapped onto the lower side ($\zeta = \xi - i0, \arg \zeta = 2\pi$) of the contour Γ . The points $\sigma = -i\infty, \sigma = +i\infty$ are mapped onto the points $\xi = 0, \xi = \infty$, respectively. Therefore, when the point s moves along Ω from the point $-i\infty$ to $+i\infty$, the corresponding point ζ moves along the contour Γ in the positive direction from 0 to ∞ . We introduce the function

$$(A.1) \quad \omega_1(\zeta) = \frac{1}{\zeta^{1/2}} \Phi\left(\frac{i}{\pi} \log \zeta\right),$$

where $s = \frac{i}{\pi} \log \zeta$. The functions $\log \zeta$ and $\zeta^{1/2}$ are defined and analytic in the ζ -plane with the cut Γ . Moreover, on the upper side of the contour Γ , the logarithmic function is real and the $\zeta^{1/2}$ is positive. Thus, the limit values of the function $\omega_1(\zeta)$ are

$$(A.2) \quad \begin{aligned} \omega_1^+(\xi) &= \omega_1(\xi + i0) = \frac{1}{\sqrt{\xi}} \Phi\left(\frac{i}{\pi} \log \xi\right) = e^{\pi i \sigma / 2} \Phi(\sigma), \\ \omega_1^-(\xi) &= \omega_1(\xi - i0) = -\frac{1}{\sqrt{\xi}} \Phi\left(-2 + \frac{i}{\pi} \log \xi\right) = -e^{\pi i \sigma / 2} \Phi(\sigma - 2), \\ \omega_1(-\xi) &= \frac{1}{i\sqrt{\xi}} \Phi\left(-1 + \frac{i}{\pi} \log \xi\right) = -ie^{\pi i \sigma / 2} \Phi(\sigma - 1), \quad 0 < \xi < \infty. \end{aligned}$$

Due to (3.17), the new function $\omega_1(\xi)$ satisfies the following boundary condition:

$$(A.3) \quad \omega_1^+(\xi) + iK_0(\xi)\omega_1(-\xi) - \omega_1^-(\xi) = h_0(\xi), \quad 0 < \xi < \infty,$$

where

$$(A.4) \quad K_0(\xi) = K\left(\frac{i}{\pi} \log \xi\right), \quad h_0(\xi) = \frac{1}{\sqrt{\xi}} h\left(\frac{i}{\pi} \log \xi\right).$$

Now setting $\omega_2(\zeta) = \omega_1(-\zeta)$, we get the function $\omega_2(\zeta)$ that is analytic in the ζ -plane with the cut along the negative semiaxis ($\xi < 0$). Then for negative ξ we have the following relations:

$$(A.5) \quad \begin{aligned} \omega_2^+(\xi) &= \omega_2(\xi + i0) = \omega_1(-\xi - i0) = \omega_1^-(-\xi), \\ \omega_2^-(\xi) &= \omega_2(\xi - i0) = \omega_1(-\xi + i0) = \omega_1^+(-\xi). \end{aligned}$$

In view of (A.3) and (A.5), the boundary condition for the function $\omega_2(\zeta)$ is

$$(A.6) \quad \omega_2^-(\xi) + iK_0(-\xi)\omega_1(\xi) - \omega_2^+(\xi) = h_0(-\xi), \quad -\infty < \xi < 0.$$

We note that the functions $\omega_1(\zeta), \omega_2(\zeta)$ satisfy the two additional “natural” boundary conditions

$$(A.7) \quad \begin{aligned} \omega_1^+(\xi) &= \omega_1^-(\xi), \quad -\infty < \xi < 0, \\ \omega_2^+(\xi) &= \omega_2^-(\xi), \quad 0 < \xi < \infty. \end{aligned}$$

Inequality (3.19), formulated for the limit values of the functions $\omega_1(\zeta), \omega_2(\zeta)$, yields

$$(A.8) \quad \int_{-\infty}^{\infty} |\omega_j^\pm(\xi)|^2 d\xi < \infty, \quad j = 1, 2,$$

i.e., $\omega_j^\pm(\xi) \in L_2(-\infty, \infty)$. Due to the Čerskiĭ theorem [9], the function $\Phi(s)$ is analytically continuable into the strip Π and satisfies condition (3.19) if and only if the functions $\omega_j(\zeta)$ can be represented as Cauchy integrals

$$(A.9) \quad \omega_j(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Sigma_j(\xi)}{\xi - \zeta} d\xi,$$

with $\Sigma_j(\xi) \in L_2(-\infty, \infty)$.

Now let us combine the boundary conditions (A.3), (A.6), and (A.7) and introduce the vector

$$(A.10) \quad \omega(\zeta) = \begin{pmatrix} \omega_1(\zeta) \\ \omega_2(\zeta) \end{pmatrix}.$$

We have reduced the difference equation of the second order in the strip Π (3.17) to the 2×2 matrix Riemann–Hilbert problem that follows.

Determine the vector-function $\omega(\zeta)$, analytic in the entire ζ -plane except for the real axis, representable by a Cauchy integral with the $L_2(\mathbb{R}_1)$ -density; its boundary values $\omega^+(\xi), \omega^-(\xi)$ are L_2 -integrable on the real axis \mathbb{R}_1 and satisfy the boundary condition

$$(A.11) \quad \omega^+(\xi) = \mathbf{G}(\xi)\omega^-(\xi) + \mathbf{f}(\xi), \quad \xi \in \mathbb{R}_1 = \{-\infty, \infty\},$$

where

$$(A.12) \quad \mathbf{G}(\xi) = \begin{pmatrix} 1 & (\operatorname{sgn} \xi + 1)a(\xi) \\ (\operatorname{sgn} \xi - 1)a(\xi) & 1 \end{pmatrix}, \quad a(\xi) = -\frac{i}{2}K_0(|\xi|),$$

$$(A.13) \quad \mathbf{f}(\xi) = \frac{1}{2}h_0(|\xi|) \begin{pmatrix} \operatorname{sgn} \xi + 1 \\ \operatorname{sgn} \xi - 1 \end{pmatrix}.$$

We note that the matrix $\mathbf{G}(\xi)$ is discontinuous at the point $\xi = 0$:

$$(A.14) \quad \mathbf{G}(+0) = \begin{pmatrix} 1 & -2i\beta \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G}(-0) = \begin{pmatrix} 1 & 0 \\ 2i\beta & 1 \end{pmatrix}.$$

Using Gakhov’s algorithm [10], one can remove the discontinuity at the point $\xi = 0$ and reduce the matrix problem (A.11) to a 2×2 matrix Riemann–Hilbert problem with a continuous coefficient.

Appendix B. An alternative derivation of the integral equation. We note that the functional equation (A.3) with condition (A.8) and therefore the matrix problem (A.11) are equivalent to an integral equation. Indeed, from (A.3) by the Sokhotski–Plemelj formulae we may write

$$(B.1) \quad \omega_1(\zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{h_0(y) - iK_0(y)\omega_1(-y)}{y - \zeta} dy, \quad \zeta \in \mathbb{C} \setminus [0, \infty).$$

Letting $\zeta = -x$ ($0 < x < \infty$), we get the integral equation with respect to the function $\omega_1(-x)$,

$$(B.2) \quad \omega_1(-x) + \frac{1}{2\pi} \int_0^\infty \frac{K_0(y)\omega_1(-y) + ih_0(y)}{y+x} dy = 0, \quad 0 < x < \infty,$$

which is reducible to both (4.6) and (4.13). To show that, we set $y = e^{-i\pi\eta}$, $x = e^{-i\pi\xi}$ in (B.2); take into account the following relations (see (A.2), (A.4), and (A.5)),

$$\omega_1(-e^{-i\pi\eta}) = -ie^{i\pi\eta/2}\Phi(\eta - 1), \quad K_0(e^{-i\pi\eta}) = K(\eta),$$

$$h_0(e^{-i\pi\eta}) = e^{i\pi\eta/2}h(\eta), \quad \frac{1}{y+x} = \frac{e^{i\pi(\eta+\xi)/2}}{2 \cos \frac{\pi}{2}(\eta - \xi)};$$

and obtain the integral equation (4.6) that is equivalent to (4.13).

Appendix C. Analytical continuation of the function $\Phi(s)$. We aim to construct the analytical continuation of the function $\Phi(s)$ from the strip Π into the whole complex plane (see also [1]). Analysis of integral (4.3) shows that the solution of Carleman’s boundary-value problem (3.17), function $\Phi(s)$, is analytic in each strip

$$(C.1) \quad \Pi_m = \{s \in \mathbb{C} : 2m - 2 < \Re(s) < 2m\}, \quad (m = 0, \pm 1, \pm 2, \dots)$$

and is discontinuous when s crosses the contours $\Omega_m = \{s \in \mathbb{C} : \Re(s) = 2m\}$. Indeed, let

$$(C.2) \quad \begin{aligned} \Phi_-(s_m) &= \lim_{\varepsilon \rightarrow +0} \Phi(s_m - \varepsilon), \\ \Phi_+(s_m) &= \lim_{\varepsilon \rightarrow +0} \Phi(s_m + \varepsilon), \quad s_m = 2m + i\eta, \quad -\infty < \eta < \infty, \end{aligned}$$

be the left and right limit values of the function $\Phi(s)$ on each contour Ω_m . From (4.3) we obtain

$$(C.3) \quad \Phi_-(s_m) = (-1)^m \Phi(\sigma), \quad \Phi_+(s_m) = (-1)^{m+1} \Phi(\sigma - 2), \quad \sigma \in \Omega, \quad s_m \in \Omega_m.$$

By substituting the last two formulae into the boundary condition (3.17), we get the relation for the limit values of the function $\Phi(s)$ on the contour Ω_m :

$$(C.4) \quad \Phi_+(s_m) = \Phi_-(s_m) - (-1)^m h^*(s_m - 2m), \quad s_m \in \Omega_m.$$

This formula indicates the discontinuity of the function $\Phi(s)$ on each contour Ω_m and gives rise to the analytical continuation of the function $\Phi(s)$ from the strip Π to the strips Π_1, Π_2, \dots (to the right),

$$(C.5) \quad \mathfrak{F}_n(s) = \Phi(s) + \sum_{j=0}^{n-1} (-1)^j h^*(s - 2j), \quad s \in \Pi_n, \quad n = 1, 2, \dots,$$

and to the left,

$$(C.6) \quad \mathfrak{F}_{-n}(s) = \Phi(s) - \sum_{j=1}^n (-1)^j h^*(s + 2j), \quad s \in \Pi_{-n}, \quad n = 1, 2, \dots,$$

where $\Phi(s)$ is the integral (4.3).

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REFERENCES

- [1] Y. A. ANTIPOV AND H. GAO, *Exact solution of integro-differential equations of diffusion along a grain boundary*, Quart. J. Mech. Appl. Math., 53 (2000), pp. 645–674.
- [2] V. BABICH, D. DEMENT'EV, AND B. SAMOKISH, *On the diffraction of high-frequency waves by a cone of arbitrary shape*, Wave Motion, 21 (1995), pp. 203–207.
- [3] V. M. BABICH, D. B. DEMENT'EV, B. A. SAMOKISH, AND V. P. SMYSHLYAEV, *On evaluation of the diffraction coefficients for arbitrary “nonsingular” directions of a smooth convex cone*, SIAM J. Appl. Math., 60 (2000), pp. 536–573.
- [4] P. G. BARTON AND A. D. RAWLINS, *Acoustic diffraction by a semi-infinite plane with different face impedances*, Quart. J. Mech. Appl. Math., 52 (1999), pp. 469–487.
- [5] J. M. L. BERNARD, *Méthode Analytique et Transformées Fonctionnelles pour la Diffraction d'ondes Par une Singularité Conique*, Rapport CEA-R-5764 Editions Dist/Saclay, CEA/Saclay, Gif-sur-Yvette, France, 1997.
- [6] J. M. L. BERNARD, *Diffraction at skew incidence by an anisotropic impedance wedge in electromagnetism theory: A new class of canonical cases*, J. Phys. A, 31 (1998), pp. 595–613.
- [7] J. M. L. BERNARD AND M. A. LYALINOV, *The leading asymptotic term for the scattering diagram in the problem of diffraction by a narrow circular impedance cone*, J. Phys. A, 32 (1999), L43–L48.
- [8] J. J. BOWMAN, T. B. A. SENIOR, AND P. L. E. USLENGHI, *Electromagnetic and Acoustic Scattering by Simple Shapes*, Hemisphere Publishing Corporation, New York, 1987.
- [9] JU. I. ČERSKIĬ, *A normally solvable equation of smooth transition*, Soviet Math. Dokl., 11 (1970), pp. 55–59.
- [10] F. D. GAKHOV, *Riemann boundary value problem for a system of n pairs of functions*, Russian Math. Surveys, 7 (1952), pp. 3–54.
- [11] F. D. GAKHOV, *Boundary Value Problems*, Pergamon Press, Oxford, 1963.
- [12] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Academic Press, Orlando, FL, 1980.
- [13] P. FILIPPI, D. HABAUT, J.-P. LEFEBVRE, AND A. BERGASSOLI, *Acoustics: Basic Physics, Theory and Methods*, Academic Press, San Diego, 1999.
- [14] D. S. JONES, *The Theory of Electromagnetism*, Pergamon Press, Oxford, 1964.
- [15] D. S. JONES, *Scattering by a cone*, Quart. J. Mech. Appl. Math., 50 (1997), pp. 499–523.
- [16] L. V. KANTOROVICH AND G. P. AKILOV, *Functional Analysis*, Pergamon Press, Oxford, 1982.
- [17] J. B. KELLER, *The geometrical theory of diffraction*, J. Opt. Soc. Amer., 52 (1962), pp. 116–130.
- [18] M. I. KONTOROVICH AND N. N. LEBEDEV, *On a method of solution of some problems of diffraction theory and relative problems*, J. Experiment. Theoret. Phys., 8 (1938), pp. 1192–1206.
- [19] L. KRAUS AND L. M. LEVINE, *Diffraction by an elliptic cone*, Comm. Pure Appl. Math., 14 (1961), pp. 49–68.
- [20] G. D. MALIUZHINETS, *Excitation, reflection and emission of surface waves from a wedge with given face impedances*, Sov. Phys. Dokl., 3 (1958), pp. 752–755.
- [21] P. M. MORSE AND K. U. INGARD, *Theoretical Acoustics*, McGraw–Hill, New York, 1968.
- [22] P. M. MORSE AND R. H. BOLT, *Sound waves in rooms*, Rev. Modern Phys., 16 (1944), pp. 69–150.
- [23] B. G. NIKOLAYEV, *Wave processes in diffraction by a perfectly reflected cone in the axisymmetric case*, J. Sov. Math., 3 (1975), pp. 125–141.
- [24] A. D. PIERCE AND R. N. THURSTON, *Underwater Scattering and Radiation*, Physical Acoustics 22, Academic Press, Boston, 1993.
- [25] E. SKUDRZYK, *The Foundations of Acoustics*, Springer-Verlag, Vienna, 1971.
- [26] V. P. SMYSHLYAEV, *The high-frequency diffraction of electromagnetic waves by cones of arbitrary cross sections*, SIAM J. Appl. Math., 53 (1993), pp. 670–688.
- [27] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1948.

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