ON THE INTEGRO-DIFFERENTIAL EQUATION ASSOCIATED WITH DIFFUSIVE CRACK GROWTH THEORY

by Y. A. ANTIPOV[†]

(*Department of Mathematics, Louisiana State University, Baton Rouge,* LA 70803, *USA*)

T.-J. CHUANG[‡]

(*Ceramics Division, National Institute of Standards and Technology, Gaithersburg,* MD 20899, *USA*)

and H. GAO[§]

(*Max Planck Institute for Metals Research, Seestrasse* 92, *Stuttgart* 70174, *Germany*)

[Received 4 November 2002]

Summary

At high temperatures, polycrystalline materials often suffer creep fracture under prolonged loading conditions. Microstructural examinations reveal that nucleation, propagation and linkage of interfacial cracks normal to the principal stress directions are responsible for the premature failure. To simulate service conditions, a semi-infinite crack is considered to grow, in steady state, along a grain boundary via a coupled process of surface and grain-boundary diffusion within an elastic bi-crystal subjected to a remote constant applied stress. Governing equations based on equilibrium and Hooke's law obeyed within the adjoining grains, and matter conservation and Fick's diffusion laws prevailing at both crack surfaces and the interface are employed to derive the singular integro-differential equation for the normal stress distribution along the interface ahead of the moving crack tip.

Using the Mellin transformation, the integral equation is first converted to a functional-difference equation (a Carleman boundary-value problem), and then solved analytically via an approach based on the theory of the Riemann–Hilbert problem on a curve. Asymptotic behaviours of the stress solutions at both ends (that is, near the crack tip as well as in the far field) are provided. Excellent agreement is reached when the full analytical solutions are compared with the existing numerical solutions.

The stress solutions permit the far-field loading intensity to be connected with the boundary conditions containing the parameter of crack velocity at the crack tip, thereby making it possible to predict the crack-growth rate for a given applied stress. The stress solutions in analytical form have the merit, over the numerical form, that they will facilitate the future solution scheme when the analysis is extended to tackle crack growth in the transient creep stage wherein both stresses and near-tip crack shapes are changing continuously with time.

1. Introduction

This paper aims to provide analytical solutions for a class of integro-differential equations that emerge from the diffusional crack-growth theory. Crack-like cavities at grain boundaries are often

[†] ⟨antipov@math.lsu.edu⟩
[‡] ⟨chuang@nist.gov⟩
[§] ⟨hjgao@mf.mpg.de⟩

Q. Jl Mech. Appl. Math. (2003) 56 (2), 289–310

detected in creep rupture specimens of polycrystalline materials such as metals, metal alloys and ceramics. It is believed that their growth and coalescence lead to premature failure. In particular, the growth rate of such cavities may become the dominant contributor to creep life, and development of predictive capability of the crack-growth rate as a function of applied stress, which makes life prediction possible, will be a challenging task for the design engineers and manufacturers of structural components in high temperature, load-bearing environments.

A literature review shows that the kinetics of cavity growth may involve coupled mass transport at crack surfaces and grain boundary (that is, surface and grain-boundary self-diffusion); see, for example, the books by Riedel (**1**) and Ashby and Brown (**2**). In (**3**) Chuang *et al.* reviewed the subject of diffusive cavitation along grain interfaces and laid down the conditions under which the growth of crack-like cavities prevails. In general, crack-like or slit shapes are favoured if the applied stress and grain-boundary diffusivity are larger than the sintering stress and surface diffusivity, respectively. Likewise, if the service time approaches the later stage in the growth phase, creep cavities tend to turn into crack-like shapes. Under these circumstances, the crack travels at a subcritical speed in a steady-state fashion along the grain boundary (gb). It is then appropriate to treat the crack as semi-infinite, propagating at a constant speed in an infinite elastic bicrystal under plane-strain conditions. This case has been considered by Chuang (**4**) and Chuang *et al.* (**5**) who solved the coupled problem of diffusion and elastic deformation, leading to a specific kinetic law for subcritical crack growth. The essential part of their work involved solving a singular integral equation. Unfortunately, only numerical solutions were provided in their work. It is inconvenient to expand the coverage to the non-steady-state regime based on their work in numerical form. To extend the analysis, closed form solutions are desirable. Accordingly, the main objective of the present paper is to present complete analytical solutions in closed form, thereby facilitating future work on more general time-dependent problems.

The program of the current paper is as follows: first, we will briefly describe the diffusional crack growth theory, and then using the governing mechanics and physical laws derive the integral equation in section 2. Next, in section 3, in order to solve the integral equation, we convert it into a functional-difference equation, via the Mellin transformation technique and the Cauchy theorem. We then proceed to tackle the mathematical problem by the method based on the theory of the Riemann–Hilbert problem on an arc (see Antipov and Gao (**6**)). In section 4, we present the asymptotic solution near the crack tip. The far-field solutions are presented in section 5. A complete closed-form solution is given in section 6. Finally in section 7, numerical evaluations are made for the integral representation using the numerical quadrature. The full distribution of stress ahead of the crack tip is determined. The complete solutions are shown to agree with the asymptotic solutions both near the crack tip and at the far field. Moreover, excellent agreement is reached when our analytical solutions are compared with the numerical solutions given by Chuang (**4**), including stress solutions and stress intensity versus crack-velocity relations. Concluding remarks are given at the end in section 8. The Appendix extends the solution scheme to a more general class of integral equations which may or may not have any physical implications.

2. Formulation of the physical problem

2.1 *Description of the crack-growth model by diffusion*

Let us consider a semi-infinite crack travelling in steady state at a constant, unknown a priori, velocity V along a grain boundary between two dissimilar grains under the action of a remote applied stress σ_a (Fig. 1).

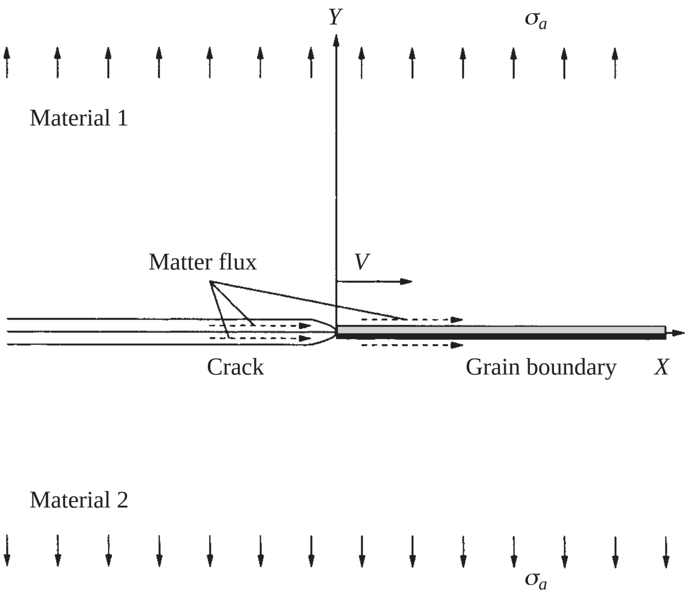

Fig. 1 A diffusive crack-growth model wherein a semi-infinite crack is considered to grow at a constant speed V driven by the applied stress σ_a. The model seeks to establish the functional relationship between σ_a and V

Here we use a rectangular Cartesian coordinate system with its origin coincident with the moving crack tip and the positive X-axis aligned with the planar grain boundary. In this way, all field variables in reference to the moving Cartesian coordinate system will become time independent at the steady state. The two grains on either side of the grain boundary are assumed to be linearly elastic with distinct mechanical and physical properties. Our goal in developing this model is to predict the crack velocity V for a given applied stress σ_a. The mechanism of crack growth is based on a coupled process of surface diffusion and grain-boundary diffusion, since at temperatures higher than one-half the melting point, stress assisted mass diffusion should be activated: under the action of the remote principal stress, atoms at crack surfaces near the tip are driven by the curvature gradient in the presence of surface free energy towards the crack tip and into the grain boundary. Once there, they are further driven away from the tip via grain-boundary diffusion due to the gradient of the normal stress. The primary contributor to the local stress along the gb is of course the applied stress. The induced elastic stress field in the presence of the crack exhibits stress concentration at the tip, and thus becomes non-uniform. Moreover, mass transport in the gb may induce residual stresses or dislocational misfit stresses which alter the elastic stress field. It turns out, as will be shown later, that solving the true stress distribution (that is, elastic stresses superimposed with residual stresses) is a necessary step to achieving the final goal of establishing the functional relationship between V and σ_a or stress-intensity factor K_I.

2.2 *Governing equations*

2.2.1 *Derivation of the singular integro-differential equation for stress.* As discussed in section 2.1, the real stress consists of two terms: the elastic stress and the dislocational misfit stress due to diffusion. To formulate the latter term, consider an edge dislocation residing at an arbitrary location $X = X_0$ in the gb ahead of the crack tip. The elastic stress field generated by this dislocation and by the applied stress has the following expression (**7**):

$$\sigma_{yy} + i\sigma_{xy} = \sigma_a - \pi i \lambda \beta (b_y + i b_x)\delta(X - X_0) - \frac{\lambda(b_y + i b_x)}{X - X_0}, \tag{2.1}$$

where $i = \sqrt{-1}$, b_x and b_y are the Burger's vectors in the X and Y directions, respectively, $\delta(X)$ is the Dirac delta function and λ and β are Dundurs's constants. To simulate the residual stress, let us consider an arbitrary distribution of dislocations along the X-axis under the action of a remote stress σ_a such that $db_y = \Delta_y' dX$, $db_x = \Delta_x' dX$. The normal stress at any location X can be integrated to yield the following form:

$$\sigma_{yy}(X) = \sigma_a - \lambda \int_{-\infty}^{\infty} \frac{\Delta_y'(X_0) dX_0}{X - X_0} + \pi \lambda \beta \Delta_x'(X). \tag{2.2}$$

The shear stress becomes

$$\sigma_{xy}(X) = -\lambda \int_{-\infty}^{\infty} \frac{\Delta_x'(X_0) dX_0}{X - X_0} - \pi \lambda \beta \Delta_y'(X). \tag{2.3}$$

Assume that the flat gb cannot resist shear, that is, $\sigma_{xy} = 0$ everywhere along the whole X-axis, which is true in many realistic situations. Applying the Hilbert transformation, we have

$$\Delta_x'(X) = \frac{\beta}{\pi} \int_{-\infty}^{\infty} \frac{\Delta_y'(X_0) dX_0}{X - X_0}. \tag{2.4}$$

Substituting equation (2.4) in (2.2) and applying the Hilbert transformation again we arrive at

$$\pi^2 \lambda (1 - \beta^2) \Delta_y'(X) = \int_0^{\infty} \frac{\sigma_{yy}(X_0) dX_0}{X - X_0}. \tag{2.5}$$

Notice that the integration now starts at zero, rather than from minus infinity because of the traction-free conditions at the crack plane in the negative X-axis. An additional relation between Δ_y and σ_{yy} coming from the grain-boundary diffusion equation (Fick's law) and steady-state conditions yields

$$\Delta_y(X) = \frac{D_b \delta_b \langle \Omega \rangle}{VkT} \frac{d\sigma_{yy}}{dX}, \tag{2.6}$$

where $D_b\delta_b = (D_b\delta_b)_1 + (D_b\delta_b)_2$ is the gb diffusivity. Here a subscript j refers to material j ($j = 1, 2$), and kT has its usual meaning and $\langle \Omega \rangle$ is the effective atomic volume defined by the harmonic mean of Ω_1 and Ω_2 with a weighting factor R_1 and R_2 respectively:

$$\langle \Omega \rangle = \frac{R_1 + R_2}{R_1/\Omega_1 + R_2/\Omega_2}. \tag{2.7}$$

Here R_1 and R_2 are material properties defined as follows (**5**):

$$R_j = \frac{(D_s\delta_s\Omega)_j^{\frac{2}{3}}}{\gamma_j}, \quad j = 1, 2, \tag{2.8}$$

where $D_s\delta_s$ is surface diffusivity and γ_j is surface free energy for material j. Notice that in the special case where $\Omega_1 = \Omega_2 = \Omega$, the effective atomic volume reduces to Ω, that is, $\langle\Omega\rangle = \Omega$ even if $R_1 \neq R_2$. Finally, combination of equations (2.5) and (2.6) yields the following integro-differential equation for the unknown $\sigma(X) = \sigma_{yy}(X)$:

$$L^2\sigma''(X) = \int_0^\infty \frac{\sigma(X_0)}{X - X_0} dX_0, \tag{2.9}$$

where the integral is supposed to be performed in the sense of Cauchy principal value and a length scale L is defined by

$$L = \sqrt{\frac{\pi}{4}\left\langle\left\langle\frac{E}{1-\nu^2}\right\rangle\right\rangle \frac{D_b\delta_b\langle\Omega\rangle}{VkT}}, \tag{2.10}$$

where E, ν are Young's modulus and Poisson's ratio respectively, and $\langle\langle\cdot\rangle\rangle$ denotes the simple harmonic mean of the two materials' properties. It can be shown that when the properties of materials 1 and 2 are identical, L in relation (2.10) reduces to the expression for the single-phase case.

2.2.2 *Derivation of initial conditions at the crack-tip.* With the governing equation (2.9), it is clear that two initial conditions at $X = 0$, namely $\sigma(0)$ and the first derivative of stress with respect to X, $\sigma'(0)$, are required in order to secure unique solutions.

The first initial condition at the crack tip, $\sigma(0)$, can be derived from the continuity requirement of the chemical potential at $X = 0$. Recognizing that the crack tip is located at the juncture of the two crack surfaces and the gb, one needs to find the general expressions there. Now, the chemical potential at a free surface can be expressed as $\mu_s = \kappa\gamma_s\Omega$ if we neglect the strain-energy contribution. Here κ is the local surface curvature of the crack profile. On the other hand, the chemical potential at a gb can be expressed as $\mu_b(X) = \sigma(X)\Omega$. The surface curvature immediately adjacent to the crack tip has been obtained by Chuang and Rice (**8**) and Chuang *et al.* (**5**) from solving the constant near-tip shapes travelling at a constant V along a gb in steady state. Equating the two chemical potentials at the crack tip and substituting the curvature expressions at the tip in the equation, we arrive at the following expression for $\sigma(0)$:

$$\sigma(0) = \sqrt{\frac{2(\gamma_1 + \gamma_2 - \gamma_b)}{R_1 + R_2}}(VkT)^{\frac{1}{3}}, \tag{2.11}$$

where it is shown that the crack-tip stress is proportional to the crack velocity V to a power of one-third.

We found that the second initial condition, the expression for the first derivative of stress with respect to X at $X = 0$, $\sigma'(0)$, can be obtained by a combination of the grain-boundary diffusion equation, conservation of mass and steady-state conditions at the crack tip $X = 0$. The result is (**5**)

$$\sigma'(0) = \sqrt{2(R_1 + R_2)(\gamma_1 + \gamma_2 - \gamma_b)}\frac{(VkT)^{\frac{2}{3}}}{D_b\delta_b\langle\Omega\rangle}, \tag{2.12}$$

where γ_j $(j = 1, 2)$ is the surface free energy for material j; γ_b is the grain-boundary free energy and $D_b\delta_b$ is the grain-boundary diffusivity. The properties $\langle\Omega\rangle$ and R_j have been defined in formulae (2.7) and (2.8). The relation (2.12) indicates that the first derivative of stress at the tip is proportional to $V^{\frac{2}{3}}$.

Having obtained the governing integral equation in (2.9), together with the initial conditions given in formulae (2.11) and (2.12), we are now in a position to solve the initial-value problem. Before we do this task, it is desirable to simplify the equations further by reformulating the parameters involved in non-dimensional quantities. So, let us define the following dimensionless parameters for the coordinate X and stress $\sigma(X)$:

$$x = \frac{X}{L}, \quad f(x) = \frac{\sigma(x)}{\sigma(0)}. \tag{2.13}$$

Then, the integral equation (2.9) becomes

$$f''(x) = \int_0^\infty \frac{f(t)dt}{x - t}, \quad 0 < x < \infty. \tag{2.14}$$

The first initial condition, by definition (2.13), is simply

$$f(0) = 1, \tag{2.15}$$

and the second initial condition (2.12) has the form

$$f'(0) = \alpha, \tag{2.16}$$

where α depends on V and is dimensionless:

$$\alpha = \frac{L\sigma'(0)}{\sigma(0)} = \sqrt{\frac{\pi\langle\langle E/(1-\nu^2)\rangle\rangle}{4D_b\delta_b\langle\Omega\rangle}}\frac{R_1 + R_2}{(VkT)^{\frac{1}{6}}}. \tag{2.17}$$

It can be seen that in addition to V, α is also a function of materials mechanical properties, E and ν, physical properties $\langle\Omega\rangle$, transport properties R_j and $D_b\delta_b$, and the absolute temperature T.

In the following sections, we will present the analytical solutions for the unknown function $f(x)$ based on the system of governing equations (2.14) to (2.16), recognizing that from the principles of linear elastic fracture mechanics, the far-field behaviour of $f(x)$ must behave like $Cx^{-\frac{1}{2}}$ (C =const) as $x \to \infty$.

3. Functional-difference equation

Consider the integro-differential equation

$$f''(x) = \frac{\lambda}{\pi}\int_0^\infty \frac{f(t)dt}{x - t}, \quad 0 < x < \infty, \tag{3.1}$$

subject to the conditions

$$f(0) = 1, \quad f'(0) = \alpha, \quad |f(x)| \leqslant Cx^{-\beta}, \quad x \to \infty, \tag{3.2}$$

where $\lambda, \alpha, C, \beta$ are constants and $C > 0, \beta > 0$. The singular integral is understood in the sense of the Cauchy principal value. The function $f(x)$ and its derivatives $f'(x)$, $f''(x)$ are assumed to satisfy the Hölder condition on every finite segment $[b, B] : f(x) \in \mathcal{H}[b, B]$ $(0 < b < B < \infty)$, and $f'(x)$ is bounded as $x \to +0$.

In this section we reduce the integro-differential equation to a functional-difference equation. A general discussion of equation (3.1) is included in the Appendix. Here we focus attention on the physically relevant range of the parameters λ, β, that is, $\lambda > 0$ and $\beta = \frac{1}{2}$. Introduce the Mellin transform of the function $f(x)$

$$F(s) = \int_0^\infty f(t)t^{s-1}dt. \tag{3.3}$$

The conditions (3.2) imply its analyticity in the strip $0 < \mathrm{Re}(s) < \beta$. Assuming that s belongs to this strip and integrating by parts yield

$$F(s) = \frac{1}{s(s+1)} \int_0^\infty f''(t)t^{s+1}dt, \quad 0 < \mathrm{Re}(s) < \beta. \tag{3.4}$$

Therefore, by the inverse Mellin transform, we have

$$f''(x) = \frac{1}{2\pi i} \int_\Omega s(s+1)F(s)x^{-s-2}ds, \quad \Omega = \{\mathrm{Re}(s) = c \in (0, \beta)\}. \tag{3.5}$$

On the other hand, by the Mellin convolution theorem we get

$$\frac{1}{\pi} \int_0^\infty \frac{f(t)dt}{x-t} = -\frac{1}{2\pi i} \int_\Omega \cot \pi s F(s)x^{-s}ds. \tag{3.6}$$

Next, continue analytically the function $F(s)$ outside the strip of convergence of the integral (3.3), namely for $\mathrm{Re}(s) \geqslant \beta$ and $\mathrm{Re}(s) \leqslant 0$. This continuation defines the integral (3.3) in these half-planes in the generalized sense. Now we introduce a new function

$$\Phi(s) = \lambda \cot \pi s F(s), \tag{3.7}$$

which is required to be analytic everywhere in the strip $\Pi = \{c < \mathrm{Re}(s) < c + 2\}$. In addition, we impose the following condition:

$$\int_{-\infty}^\infty |\Phi(\tau + it)|^2 dt \leqslant C \quad (C = \mathrm{const}) \tag{3.8}$$

uniformly with respect to $\tau \in [c, c+2]$. The analyticity of the function $\Phi(s)$ in the strip Π and the Cauchy theorem give

$$\int_{\Omega} \Phi(s) x^{-s} ds = \int_{\Omega} \Phi(s+2) x^{-s-2} ds. \tag{3.9}$$

Now it is possible to rewrite the original integro-differential equation as follows:

$$\int_{\Omega} \left[\Phi(s+2) + \frac{1}{\lambda} s(s+1) \tan \pi s \Phi(s) \right] x^{-s-2} ds = 0, \quad 0 < x < \infty, \tag{3.10}$$

which is equivalent to the functional-difference equation

$$\Phi(\sigma+2) + K(\sigma)\Phi(\sigma) = 0, \quad \sigma \in \Omega, \tag{3.11}$$

where $K(\sigma) = \lambda^{-1}\sigma(\sigma+1)\tan \pi\sigma$. The function $\Phi(s)$ is analytic in the strip Π and satisfies the condition (3.11).

REMARK 1. Alternatively, instead of the function $\Phi(s)$ defined by (3.7), it is also possible to introduce the function $\Phi_*(s) = s(s+1)F(s)$, analytic in the strip $\Pi_* = \{c-2 < \mathrm{Re}(s) < c\}$. Then equation (3.11) with the shift to the right becomes the following functional-difference equation with the shift to the left: $\Phi_*(\sigma) + K(\sigma)\Phi_*(\sigma-2) = 0\,(\sigma \in \Omega)$, with the same function $K(\sigma)$ as in (3.11). Both functions $\Phi(s)$ and $\Phi_*(s)$ can be used in the procedure being presented and, finally, lead to the same result.

Because the coefficient of equation (3.11), the function $K(\sigma)$, grows at infinity and is discontinuous as $\sigma \to c \pm i\infty$, it is desirable to transform this equation to a new one whose coefficient is continuous and tends to 1 as $\sigma \to c \pm i\infty$. By using the identities

$$\sigma(\sigma+1) = \frac{\Gamma(\sigma+2)}{\Gamma(\sigma)}, \quad \frac{1}{\lambda} = \frac{(\sqrt{\lambda})^{\sigma}}{(\sqrt{\lambda})^{\sigma+2}}, \quad \tan\frac{\pi\sigma}{4} = \frac{\sin\frac{1}{4}\pi\sigma}{\sin\frac{1}{4}\pi(\sigma+2)}, \tag{3.12}$$

equation (3.11) becomes

$$\Phi_0(\sigma+2) + K_0(\sigma)\Phi_0(\sigma) = 0, \quad \sigma \in \Omega, \tag{3.13}$$

where

$$\Phi_0(s) = \frac{\lambda^{s/2}}{\Gamma(s)} \sin\frac{\pi s}{4} \Phi(s), \quad K_0(s) = \cot\frac{\pi s}{4} \tan \pi s. \tag{3.14}$$

It is clear that the new coefficient is continuous and bounded at infinity:

$$K_0(\sigma) = 1 + O(e^{-\pi|t|/2}), \quad \sigma = c + it, \quad t \to \pm\infty, \tag{3.15}$$

and the increment of the argument of the function $K_0(\sigma)$ as σ traverses the contour Ω equals zero.

At the next stage, we reduce the functional-difference equation (3.13) to a Riemann–Hilbert problem (**6**). To do this we introduce a new function $\varphi(w)$ such that

$$\varphi(w) = \frac{1}{1+w} \left(i\frac{1-w}{1+w} \right)^{-\frac{1}{2}} \Phi_0(s), \tag{3.16}$$

where

$$w = i \tan\left\{\pi\left(\frac{1}{4} + \frac{s-c}{2}\right)\right\}, \quad s = c + \frac{i}{\pi}\log\left(i\frac{1-w}{1+w}\right). \tag{3.17}$$

The conformal mapping $w(s)$ transforms the strip $\Pi \ni s$ into the complex plane $\mathbb{C} \ni w$ with the cut $\gamma = \{|w| = 1, \quad \mathrm{Im}(w) \geqslant 0\}$. The contour Ω of the s-plane is mapped onto the contour $\gamma^- = \{|w| = 1+0, \quad \mathrm{Im}(w) \geqslant 0\}$ and the contour $\Omega_1 = \{\mathrm{Re}(s) = c+2\}$ is mapped onto $\gamma^+ = \{|w| = 1-0, \quad \mathrm{Im}(w) \geqslant 0\}$. On the contour γ, the function $\log\zeta$ is real and the function $\zeta^{-\frac{1}{2}}$ is positive. Here $\zeta = i(1-w)(1+w)^{-1}$. It becomes evident that

$$(1+\eta)\varphi^+(\eta) = -e^{-\frac{1}{2}\pi i(c-\sigma)}\Phi_0(\sigma+2),$$

$$(1+\eta)\varphi^-(\eta) = e^{-\frac{1}{2}\pi i(c-\sigma)}\Phi_0(\sigma), \quad \eta \in \gamma, \quad \sigma \in \Omega. \tag{3.18}$$

The limiting values $\varphi^\pm(\eta)$ satisfy the following boundary condition of the Riemann–Hilbert problem:

$$\varphi^+(\eta) = G(\eta)\varphi^-(\eta), \quad \eta \in \gamma, \tag{3.19}$$

where

$$G(\eta) = K_0\left(c + \frac{i}{\pi}\log i\frac{1-\eta}{1+\eta}\right). \tag{3.20}$$

The function $G(\eta)$ enjoys the following properties:

$$[\arg G(\eta)]_\gamma = 0, \quad G(\eta) = 1 + O(\{1 \mp \eta\}^{\frac{1}{2}}), \quad \eta \to \pm 1, \ \eta \in \gamma, \tag{3.21}$$

where $\eta = 1$ is the starting point of the contour γ (corresponding to $\sigma = c - i\infty$) and $\eta = -1$ is the end point (the image of the point $\sigma = c + i\infty$). Therefore, the function $G(\eta)$ admits a factorization

$$G(\eta) = \frac{X^+(\eta)}{X^-(\eta)}, \quad \eta \in \gamma, \tag{3.22}$$

where $X^\pm(\eta)$ are the limiting values of the function $X(w)$ as $w \to \eta \in \gamma^\pm$. The function $X(w)$ is defined by

$$X(w) = (w-1)^p(w+1)^q e^{Y(w)}, \quad Y(w) = \frac{1}{2\pi i}\int_\gamma \frac{\log G(\eta)}{\eta - w}d\eta \tag{3.23}$$

with the integers p, q to be determined and $|\arg G(\eta)| < \pi$. The general solution of the Riemann–Hilbert problem (3.19) is given by (**9**)

$$\varphi(w) = (w-1)^p(w+1)^q e^{Y(w)} P_\kappa(w), \tag{3.24}$$

where $P_\kappa(w)$ is an arbitrary polynomial of degree κ. By the definition (3.16), $\varphi(w) = O(w^{-1})$ as $w \to \infty$. Therefore, $p + q + \kappa + 1 = 0$. Using the formula

$$|\Gamma(c+it)| \sim e^{-\pi|t|/2}|t|^{c-\frac{1}{2}}\sqrt{2\pi}, \quad |t| \to \infty, \tag{3.25}$$

and also the relations (3.14), (3.16) and (3.24) we get the asymptotic of the function $\Phi(\sigma)$ for $\sigma = c + it$ and $t \to \pm\infty$

$$\Phi(\sigma) = |t|^{c-\frac{1}{2}} e^{-3\pi|t|/4+\pi t/2} \frac{\Sigma(t)}{(i + e^{\pi t})^{q+1}(ie^{-\pi t} + 1)^p}, \tag{3.26}$$

where $\Sigma(t)$ is bounded as $t \to \pm\infty$. Analysing $\Phi(c + it)$ as $t \to +\infty$ and $t \to -\infty$ separately, we find that

$$\Phi(\sigma) = O(t^{c-\frac{1}{2}} e^{-(q+\frac{5}{4})\pi t}), \quad t \to +\infty,$$

$$\Phi(\sigma) = O((-t)^{c-\frac{1}{2}} e^{(p+\frac{5}{4})\pi t}), \quad t \to -\infty. \tag{3.27}$$

This means that for the conditions (3.8) to be satisfied, we need to demand $p > -5/4, q > -5/4$. Since $q = -1 - p - \kappa$, we obtain $-5/4 < p < 1/4 - \kappa$. The largest class of solution is achievable when $\kappa = 1$. Therefore $p = q = -1$. Finally, the solution of equation (3.11) is

$$\Phi(s) = -\frac{(1+i)\Gamma(s)\cos\frac{1}{2}\pi(\frac{1}{2} + s - c)Q(s)}{\sqrt{2}\lambda^{s/2}\sin\frac{1}{4}\pi s}(C_0 + C_1 w), \quad Q(s) = e^{Y(w)}, \tag{3.28}$$

where w, $Y(w)$ were defined in (3.17) and (3.23) respectively, and the coefficients C_0, C_1 are to be determined from the conditions (3.2).

4. Asymptotic solution for small x

The function $\Phi(s)$ is constructed in closed form, and its representation possesses two arbitrary constants. By (3.7) the Mellin transform of the function $f(x)$ is known as well. Therefore, the inverse Mellin transform yields the integral representation of the function $f(x)$:

$$f(x) = \frac{1}{2\pi i\lambda} \int_\Omega \Phi(s) \tan\pi s x^{-s} ds. \tag{4.1}$$

To satisfy the initial conditions (3.2) and to find an asymptotic expansion for small x, we apply the technique presented by Antipov and Gao (**6**). First, construct the analytical continuation of the function $\Phi(s)$ from the strip Π into the left half-plane $\mathrm{Re}(s) < c$. Because of the periodicity of the solution

$$\Phi(s) = (-1)^m \Phi(s + 2m), \quad s \in \Pi, \quad m = 0, \pm1, \pm2, \dots, \tag{4.2}$$

the limiting values of the solution

$$\Phi_+(s_m) = \lim_{\delta\to+0} \Phi(s + 2m + \delta), \quad \Phi_-(s_m) = \lim_{\delta\to+0} \Phi(s + 2m - \delta),$$

$$s \in \Omega_0, \quad s_m \in \Omega_m = \{s : \mathrm{Re}(s) = c + 2m\}, \quad m = 0, \pm1, \pm2, \dots, \tag{4.3}$$

are discontinuous through the contours Ω_m:

$$\Phi_-(s) = K(s - 2m)\Phi_+(s), \quad s \in \Omega_m. \tag{4.4}$$

Using this boundary condition we construct the analytical continuation of the function $\Phi(s)$ to the left:

$$\mathfrak{F}_{-1}(s) = \frac{\Phi(s)}{K(s)}, \quad s \in \Pi_{-1}, \quad \mathfrak{F}_{-2}(s) = \frac{\Phi(s)}{K(s)K(s+2)}, \quad s \in \Pi_{-2}, \quad \ldots . \tag{4.5}$$

For the nth strip in the left half-plane we obtain

$$\mathfrak{F}_{-n}(s) = \frac{\Phi(s)}{\prod\limits_{m=0}^{n-1} K(s+2m)}, \quad s \in \Pi_{-n}, \quad n = 1, 2, \ldots , \tag{4.6}$$

where $\Pi_{-n} = \{c - 2n < \mathrm{Re}(s) < c - 2n + 2\}$. By continuing the function $\Phi(s)$ to the strip Π_{-1} and using the Cauchy theorem, we get

$$f(x) = \frac{1}{2\pi i} \int\limits_{\Omega} \frac{\Phi(s) x^{-s} ds}{s(s+1)} = -\Phi(2) + x\Phi(1) + \frac{1}{2\pi i} \int\limits_{\Omega_{-1}} \frac{\Phi_{+}(s) x^{-s} ds}{s(s+1)}. \tag{4.7}$$

Here we used the relations $\Phi(0) = -\Phi(2)$ and $\Phi(-1) = -\Phi(1)$. This representation enables us to formulate the initial conditions (3.2) in terms of the function $\Phi(s)$:

$$\Phi(2) = -1, \quad \Phi(1) = \alpha. \tag{4.8}$$

The above conditions fix the constants C_0 and C_1. Finally,

$$C_0 = \frac{\sqrt{\lambda}(1-i)}{2\Delta} \left(\frac{\alpha \tan \frac{1}{2}\pi(\frac{1}{2} - c)}{Q(1) \cos \frac{1}{2}\pi(\frac{1}{2} + c)} - \frac{\sqrt{2\lambda} \tan \frac{1}{2}\pi(\frac{1}{2} + c)}{Q(2) \cos \frac{1}{2}\pi(\frac{1}{2} - c)} \right),$$

$$C_1 = \frac{\sqrt{\lambda}(1+i)}{2\Delta} \left(\frac{\alpha}{Q(1) \cos \frac{1}{2}\pi(\frac{1}{2} + c)} + \frac{\sqrt{2\lambda}}{Q(2) \cos \frac{1}{2}\pi(\frac{1}{2} - c)} \right), \tag{4.9}$$

where

$$\Delta = \tan \tfrac{1}{2}\pi(\tfrac{1}{2} - c) + \tan \tfrac{1}{2}\pi(\tfrac{1}{2} + c). \tag{4.10}$$

Continuation of the function $\Phi_{+}(s)$ $(s \in \Omega_{-1})$ into the next strip Π_{-2} and use of the Cauchy theorem give

$$f(x) = 1 + \alpha x + R_{-2} + R_{-3} + \frac{\lambda}{2\pi i} \int\limits_{\Omega_{-2}} \frac{\Phi_{+}(s) x^{-s} ds}{s(s+1)(s+2)(s+3) \tan \pi s}, \tag{4.11}$$

where

$$R_{-2} = \operatorname*{res}_{s=-2} \frac{\lambda \Phi(s) x^{-s}}{s(s+1)(s+2)(s+3) \tan \pi s} = \frac{\lambda x^2}{2\pi} \left[\Phi'(-2) + \Phi(-2) \left(\frac{1}{2} - \log x \right) \right],$$

$$R_{-3} = \operatorname*{res}_{s=-3} \frac{\lambda\Phi(s)x^{-s}}{s(s+1)(s+2)(s+3)\tan\pi s} = -\frac{\lambda x^3}{6\pi}\left[\Phi'(-3) + \Phi(-3)\left(\frac{11}{6} - \log x\right)\right]. \quad (4.12)$$

In the next strip Π_{-3}, the analytical continuation $\mathfrak{F}_{-3}(s)$ has two poles ($s = -4, s = -5$) of the third order. Computing the residues and applying the relations (4.2) yield the following representation for the solution:

$$\begin{aligned} f(x) = {} & 1 + \alpha x + \frac{\lambda x^2}{2\pi}\left[\Phi'(2) + \log x - \frac{1}{2}\right] - \frac{\lambda x^3}{6\pi}\left[\Phi'(1) + \alpha\left(\frac{11}{6} - \log x\right)\right] \\ & - \frac{\lambda^2 x^4}{24\pi^2}\left[\frac{1}{2}\Phi''(2) + \Phi'(2)\left(\frac{13}{12} - \log x\right) - \frac{259}{144} - \frac{\log^2 x}{2} + \frac{13}{12}\log x\right] \\ & + \frac{\lambda^2 x^5}{120\pi^2}\left[\frac{1}{2}\Phi''(1) + \Phi'(1)\left(\frac{137}{60} - \log x\right) + \alpha\left(\frac{12019}{3600} + \frac{\log^2 x}{2} - \frac{137}{60}\log x\right)\right] + S(x), \end{aligned} \quad (4.13)$$

where

$$S(x) = \frac{\lambda^3}{2\pi i}\int_{\Omega_{-3}} \frac{\Phi_-(s)x^{-s}ds}{\tan^3 \pi s \prod_{m=0}^{7}(s+m)} = O(x^6 \log^3 x), \quad x \to 0. \quad (4.14)$$

It is clear that in each strip Π_{-m} ($m = 1, 2, \ldots$) there are two poles $s = -2m+2$ and $s = -2m+1$ of the mth order. To evaluate the residues at these points, we need to compute the derivatives $\Phi^{(k-1)}(1)$ and $\Phi^{(k-1)}(2)$ ($k = 1, 2, \ldots, m$).

5. Asymptotic solution for large x

The analytical continuation of the solution, the function $\mathfrak{F}_n(s)$, in the strips $\Pi_n = \{c + 2n < \operatorname{Re}(s) < c + 2n + 2\}$ is

$$\mathfrak{F}_n(s) = \Phi(s)\prod_{m=1}^{n} K(s - 2m), \quad s \in \Pi_n, \quad n = 1, 2, \ldots. \quad (5.1)$$

It enables us to find an asymptotic expansion for $x \to \infty$. Using the Cauchy theorem transforms (4.1) into

$$f(x) = \frac{x^{-\frac{1}{2}}}{\pi\lambda}\Phi\left(\frac{1}{2}\right) + \frac{x^{-3/2}}{\pi\lambda}\Phi\left(\frac{3}{2}\right) + \frac{1}{2\pi i\lambda^2}\int_{\Omega_1}(s-1)(s-2)\tan^2\pi s\,\Phi_+(s)x^{-s}ds. \quad (5.2)$$

The points $s = 5/2$ and $s = 7/2$ are poles of the second order in the strip Π_1. Evaluating the corresponding residues gives

$$\begin{aligned} f(x) = & \frac{x^{-\frac{1}{2}}}{\pi\lambda}\left[\Phi\left(\frac{1}{2}\right)+\frac{1}{x}\Phi\left(\frac{3}{2}\right)\right] \\ & -\frac{x^{-5/2}}{\pi^2\lambda^2}\left[\Phi\left(\frac{5}{2}\right)\left(2-\frac{3}{4}\log x\right)+\frac{3}{4}\Phi'\left(\frac{5}{2}\right)+\frac{1}{x}\Phi\left(\frac{7}{2}\right)\left(4-\frac{15}{4}\log x\right)+\frac{15}{4x}\Phi'\left(\frac{7}{2}\right)\right] \\ & +\frac{1}{2\pi i\lambda^3}\int\limits_{\Omega_2}(s-1)(s-2)(s-3)(s-4)\tan^3\pi s\,\Phi_+(s)x^{-s}ds. \end{aligned} \quad (5.3)$$

In the strip Π_2, the poles $s = \frac{9}{2}$ and $s = \frac{11}{2}$ are of the third order. By the periodicity property (4.2), finally, we obtain the following asymptotic expansion for $x \to \infty$

$$\begin{aligned} f(x) = & \frac{x^{-\frac{1}{2}}}{\pi\lambda}\left[\Phi\left(\frac{1}{2}\right)+\frac{1}{x}\Phi\left(\frac{3}{2}\right)\right]+\frac{x^{-5/2}}{\pi^2\lambda^2}\left[\Phi\left(\frac{1}{2}\right)\left(2-\frac{3}{4}\log x\right)+\frac{3}{4}\Phi'\left(\frac{1}{2}\right)\right. \\ & \left.+\frac{1}{x}\Phi\left(\frac{3}{2}\right)\left(4-\frac{15}{4}\log x\right)+\frac{15}{4x}\Phi'\left(\frac{3}{2}\right)\right]+\frac{x^{-9/2}}{\pi^3\lambda^3}\left[\Phi\left(\frac{1}{2}\right)\left(\frac{43}{2}-22\log x+\frac{105}{32}\log^2 x\right)\right. \\ & +\Phi'\left(\frac{1}{2}\right)\left(22-\frac{105}{16}\log x\right)+\frac{105}{32}\Phi''\left(\frac{1}{2}\right)+\frac{1}{x}\Phi\left(\frac{3}{2}\right)\left(\frac{103}{2}-93\log x+\frac{945}{32}\log^2 x\right) \\ & \left.+\frac{1}{x}\Phi'\left(\frac{3}{2}\right)\left(93-\frac{945}{16}\log x\right)+\frac{945}{32}\Phi''\left(\frac{3}{2}\right)\right]+V(x), \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} V(x) = & \frac{1}{2\pi i\lambda^4}\int\limits_{\Omega_3}(s-1)(s-2)(s-3)(s-4)(s-5)(s-6)\tan^4\pi s\,\Phi_+(s)x^{-s}ds \\ = & O(x^{-13/2}\log^3 x), \quad x \to +\infty. \end{aligned} \quad (5.5)$$

6. Analysis of the solution

First, we show that the left- and right-hand sides of equation (3.1) have the same asymptotic behaviour as $x \to 0$. From (4.13) we get

$$f(x) \sim 1+\alpha x+\frac{\lambda}{2\pi}x^2\log x, \quad f''(x) \sim \frac{\lambda}{\pi}\log x, \quad x \to +0. \quad (6.1)$$

On the other hand, the Cauchy integral in (3.1) with a bounded density at $x = 0$ has a logarithmic singularity:

$$\frac{\lambda}{\pi}\int\limits_0^\infty \frac{f(t)dt}{x-t} \sim \frac{\lambda}{\pi}\log x, \quad x \to +0. \quad (6.2)$$

Thus, the behaviour at $x = 0$ of the function $f''(x)$ is the same as that of the Cauchy integral in (3.1).

At infinity, from (5.4),

$$f''(x) \sim \frac{3x^{-5/2}}{4\pi\lambda}\Phi\left(\frac{1}{2}\right), \quad x \to +\infty. \tag{6.3}$$

To estimate the behaviour at infinity of the right-hand side in (3.1), note that

$$f(x) \sim \frac{x^{-\frac{1}{2}}}{\pi\lambda}\Phi\left(\frac{1}{2}\right) + \frac{x^{-3/2}}{\pi\lambda}\Phi\left(\frac{3}{2}\right) + \frac{3x^{-5/2}}{4\pi^2\lambda^2}\Phi\left(\frac{1}{2}\right)\log\frac{1}{x}, \quad x \to +\infty. \tag{6.4}$$

Therefore

$$\int_0^\infty \frac{f(t)dt}{x-t} = \int_0^1 \frac{f(t)dt}{x-t} + \frac{\Phi(\frac{1}{2})}{\pi\lambda x}\int_0^1 \frac{t^{-\frac{1}{2}}dt}{t-1/x} + \frac{\Phi(\frac{3}{2})}{\pi\lambda x}\int_0^1 \frac{t^{\frac{1}{2}}dt}{t-1/x} + \frac{3\Phi(\frac{1}{2})}{4\pi^2\lambda^2 x}\int_0^1 \frac{t^{3/2}\log t}{t-1/x}dt + \dots, \quad x \to +\infty. \tag{6.5}$$

Evaluating the integral

$$\int_0^1 \frac{t^{3/2}\log t}{x-t}dt = \frac{\pi^2}{x^{3/2}} - \sum_{j=0}^\infty \frac{x^{-j}}{(3/2-j)^2} \quad (x > 1) \tag{6.6}$$

enables us to obtain the following estimate:

$$\int_0^\infty \frac{f(t)dt}{x-t} = \frac{3x^{-5/2}}{4\lambda^2}\Phi\left(\frac{1}{2}\right) + \Upsilon(x), \quad x \to +\infty, \tag{6.7}$$

where the function $\Upsilon(x)$ decays not slower than x^{-1} as $x \to +\infty$. We show that, in fact, $|\Upsilon(x)| \leqslant Dx^{-3}$ (D=const). Indeed, assume that

$$\int_0^\infty \frac{f(t)dt}{x-t} = D_1 x^{-1} + D_2 x^{-2} + o(x^{-2}), \quad x \to +\infty, \tag{6.8}$$

with $D_1 \neq 0$ and $D_2 \neq 0$. By formally expanding the Cauchy integral at infinity we get $D_1 = F(1)$ and $D_2 = F(2)$, where $F(1)$ and $F(2)$ are the corresponding values of the analytical continuation of the integral (3.3) into the exterior of the strip $\operatorname{Re} s \in (0, \beta)$. By the relation (3.7), it is clear that $F(1) = F(2) = 0$. We emphasise that the values $F(1)$ and $F(2)$ have nothing in common with the total areas under the curves described by the functions $f(x)$ and $xf(x)$, respectively (which are obviously infinite for the function decaying at infinity as $D_0 x^{-\frac{1}{2}}$, D_0=const).

Therefore, the regular function $\Upsilon(x)$ vanishes at infinity at least as fast as x^{-3}. This means that the left- and right-hand sides of (3.1) have the same behaviour at infinity.

7. Solution by quadrature. Numerical results

In this section we aim to analyse the integral representation of the solution (4.1) which we rewrite as follows:

$$f(x) = \frac{1}{2\pi i}\int_{\Omega} F(\sigma)x^{-\sigma}\,d\sigma, \tag{7.1}$$

where

$$F(\sigma) = -\frac{(1+i)\Gamma(\sigma)\cos\{\frac{1}{2}\pi(\frac{1}{2}+\sigma-c)\}Q(\sigma)}{\sqrt{2}\lambda^{\sigma/2+1}\sin\frac{1}{4}\pi\sigma\cot\pi\sigma}(C_0+C_1\eta),$$

$$\eta = i\tan\tfrac{1}{2}\pi(\tfrac{1}{2}+\sigma-c), \quad \sigma\in\Omega, \quad \eta\in\gamma. \tag{7.2}$$

To evaluate $Q(\sigma)$, one needs to take into account that the contour Ω is transformed into γ^- and that the function $\Phi(\sigma)$ $(\sigma\in\Omega)$ corresponds to the limiting value $\varphi^-(\eta)$ $(\eta\in\gamma)$. Therefore by the Sokhotski–Plemelj formulae (Gakhov (**9**))

$$Q(\sigma) = [G(\eta)]^{-\frac{1}{2}}e^{Y(\eta)}, \quad \eta\in\gamma, \tag{7.3}$$

where $Y(\eta)$ is understood in the sense of the principal value:

$$Y(\eta) = \frac{1}{2\pi}\int_0^{\pi}\frac{\log G(e^{i\theta})d\theta}{1-e^{i(\psi-\theta)}}, \quad \psi = -i\log\eta\in(0,\pi). \tag{7.4}$$

For computational purposes, it is convenient to represent the Cauchy-type integral (7.4) as follows:

$$Y(\eta) = \frac{1}{2\pi}\int_0^{\pi}h(\theta,\psi)d\theta + \frac{1}{2\pi i}\log G(e^{i\psi})\log\frac{\pi-\psi}{\psi}, \tag{7.5}$$

where

$$\log G(e^{i\psi}) = O(\{\pi-\psi\}^{\frac{1}{2}}), \quad \psi\to\pi-0,$$
$$\log G(e^{i\psi}) = O(\psi^{\frac{1}{2}}), \quad \psi\to+0,$$

$$h(\theta,\psi) = \frac{\log G(e^{i\theta})}{1-e^{i(\psi-\theta)}} + \frac{\log G(e^{i\psi})}{i(\psi-\theta)} \sim \frac{1}{2}\log G(e^{i\psi}) + \frac{G'(e^{i\psi})e^{i\psi}}{G(e^{i\psi})}, \quad \theta\to\psi. \tag{7.6}$$

We emphasize that the function $h(\theta,\psi)$ is bounded as $\theta\to\psi$.

Let us now show that the function $F(\sigma)$ decays exponentially as $\sigma = c+it$ and $|t|\to\infty$. Using the asymptotic expansion for the Γ-function (Jahnke *et al.* (**10**))

$$\Gamma(c+it) = e^{-\frac{1}{2}\pi(t-ic)}\chi(t-ic), \quad t>0,$$

$$\chi(z) = \sqrt{\frac{2\pi}{z}}\exp\left\{i\left(-\frac{\pi}{4}+z(\log z-1)-\frac{1}{12z}-\frac{1}{360z^3}-\frac{1}{1260z^5}-\frac{1}{1680z^7}-\frac{1}{1188z^9}-\dots\right)\right\}, \quad |\arg z|<\pi, \tag{7.7}$$

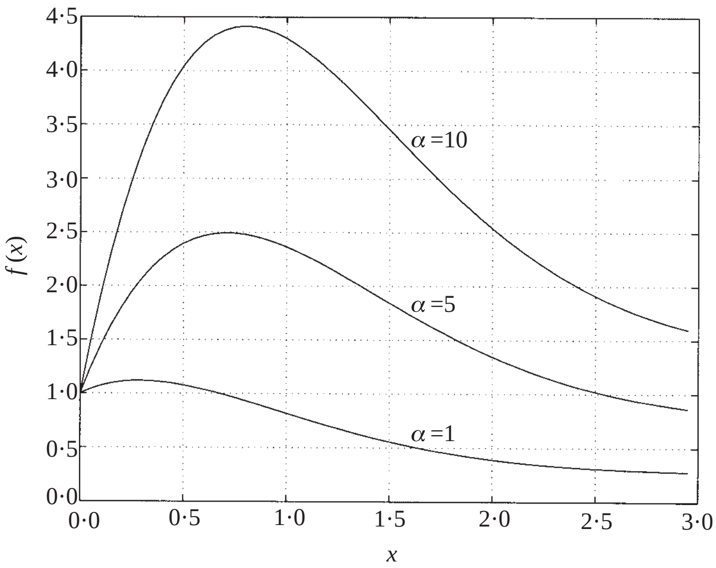

Fig. 2 The stress distribution function $f(x)$ (the integral representation (7.1)) along the interface for $\lambda = \pi$ and $\alpha = 1$, $\alpha = 5$ and $\alpha = 10$

we can establish that for $\sigma = c + it$, $t \gg 1$,

$$F(c+it) = -\frac{i\chi(t-ic)e^{\pi/4(-t+ic)}(1+ie^{-\pi t})\tan\{\pi(c+it)\}}{\lambda^{(c+it)/2+1}(e^{-\pi t/2} - e^{-\pi ci/2})}(C_0 + C_1\eta)Q(\sigma). \tag{7.8}$$

Also, the identity

$$\Gamma(1-\sigma) = \frac{\pi}{\sin\pi\sigma\Gamma(\sigma)}, \tag{7.9}$$

together with formulae (7.2) and (7.7), reveals that for $\sigma = c - it$, $t \gg 1$,

$$F(c-it) = -\frac{2\pi i e^{-\pi/4(t+ic)}(i+e^{-\pi t})\tan\{\pi(c+it)\}}{\lambda^{(c-it)/2+1}\chi(t-i+ic)(e^{i\pi c/2} - e^{-\pi t/2})(1-e^{-2\pi(t+ic)})}(C_0 + C_1\eta)Q(\sigma). \tag{7.10}$$

Thus, for large $|t|$, the function $F(\sigma)$ decays exponentially: $F(c+it) = O(e^{-\pi|t|/4})$, $t \to \pm\infty$. The asymptotic expansions (7.8), (7.10) and (7.7) are helpful for numerical computations.

Let us summarize the numerical procedure for evaluating the function $f(x)$. The exact solution is given in terms of the two integrals (7.1) and (7.5). Since the function $F(\sigma)$ decays exponentially as $\sigma \to c \pm \infty$, for the integral (7.1), it is possible to use, for instance, Laguerre integration or a non-uniform mesh for the trapezoidal rule. This integral is computed by the second technique. The integral (7.5) is not singular. The function $h(\theta, \psi)$ is continuous everywhere on the segment $[0, \pi]$

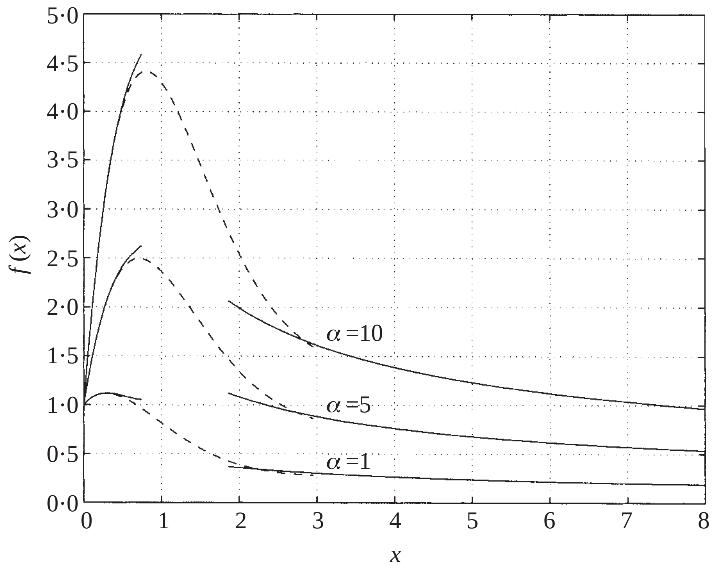

Fig. 3 The stress function $f(x)$: the asymptotic expansions for small x $(0 < x < 0{\cdot}75)$ and for large x $(1{\cdot}8 < x < 8)$—the solid curves; the exact integral representation—the dashed curves ($\lambda = \pi$ and $\alpha = 1, \alpha = 5, \alpha = 10$)

and decays at the ends. To evaluate this integral we use the standard trapezoidal rule. Alternatively, one can apply Gaussian integration using Chebyshev abscissas and weights.

In Fig. 2, we plot the graphs of the stress distribution function $f(x)$ for $\lambda = \pi$ for some values of α: $\alpha = 1$, $\alpha = 5$ and $\alpha = 10$. These graphs are in good agreement with the numerical solution by Chuang (**4**).

Figure 3 illustrates the asymptotic expansions of the function $f(x)$ for small x $(0 < x < 0{\cdot}75)$ and large x $(x > 1{\cdot}8)$ for the same values of λ and α.

The asymptotic expansions based on formulae (4.13) and (5.4) (the functions $S(x)$ and $V(x)$ were taken to be zero) give reasonable results for $0 \leqslant x < 0{\cdot}6$ and $x > 5$, respectively. To evaluate the function $\Phi(s)$ and its derivatives at the points $s = 1, s = 2$ and $s = \frac{1}{2}, s = \frac{3}{2}$ we use formulae (3.28) and (3.23). It is important to notice that in contrast with the integral (7.4), the integral (3.23) is not singular since neither of the points $s = 1, s = 2$ and $s = \frac{1}{2}, s = \frac{3}{2}$ belongs to the contour Ω. Therefore it is not surprising that direct application of the trapezoidal rule is effective for computing this integral.

We emphasize that the integral representation (7.1) yields excellent results for all $x \in (\varepsilon, A)$ at least within the range $\varepsilon \geqslant 10^{-3}$ and $A \leqslant 1000$.

The principal term of the asymptotic expansion (5.4) yields the exact formula for the stress-intensity factor at infinity

$$k_I(\alpha) = \lim_{x \to +\infty} \sqrt{2\pi x}\, f(x) = \sqrt{\frac{2}{\pi}}\frac{\Phi(\frac{1}{2})}{\lambda}. \tag{7.11}$$

By regularizing the original integro-differential equation, it is shown by Chuang (**4**) that $k_I(\alpha)$ is a linear function of the parameter α: $k_I(\alpha) = a\alpha + b$ with

$$a = 0{\cdot}24\sqrt{2\pi} = 0{\cdot}6016, \quad b = 0{\cdot}30\sqrt{2\pi} = 0{\cdot}7520. \tag{7.12}$$

The linear dependence of the factor $k_I(\alpha)$ on the parameter α also follows directly from the analysis of equation (3.1). Indeed, let $f(x) = f_0(x) + \alpha f_1(x)$, where the functions $f_j(x)$, $j = 0, 1$, provide the unique solutions of equation (3.1) subject to the conditions $f_j^{(m)}(0) = \delta_{jm}$, $j = 0, 1$; $m = 0, 1$ (δ_{jm} is the Kronecker symbol). These functions are independent of α, and therefore $k_I(\alpha) = a\alpha + b$. Computing $k_I(\alpha)$ by formula (7.11) for $\alpha = 1$, 5 and 10 confirms this conclusion and also reveals $a = 0{\cdot}5993$, $b = 0{\cdot}7511$. These constants are in good agreement with the numerical solution (7.12).

Thus we have $k_I = 0{\cdot}5993\alpha + 0{\cdot}7511$. It is straightforward to convert this non-dimensional form back to its original counterpart so as to establish the functional relationship between $K_I = k_I\sigma(0)\sqrt{L}$ and V. To do this, we use the definition of α from (2.17). Finally, we get

$$K_I = 0{\cdot}5993\sigma'(0)L^{3/2} + 0{\cdot}7511\sigma(0)L^{\frac{1}{2}}. \tag{7.13}$$

Substituting L from (2.10), $\sigma(0)$ from (2.11) and $\sigma'(0)$ from (2.12), we finally arrive at the following expression for K_I as a function of V:

$$K_I = AV^{1/12} + BV^{-1/12}, \tag{7.14}$$

where A and B are temperature-dependent materials constants:

$$A = \sqrt{\frac{\gamma_1 + \gamma_2 - \gamma_b}{R_1 + R_2}}\left[(kT)^{\frac{1}{3}}\left\langle\left\langle\frac{E}{1-\nu^2}\right\rangle\right\rangle D_b\delta_b\langle\Omega\rangle\right]^{\frac{1}{4}} \tag{7.15}$$

and

$$B = \sqrt{\frac{(\gamma_1 + \gamma_2 - \gamma_b)(R_1 + R_2)}{2}}\left[\frac{(kT)^{\frac{1}{3}} D_b\delta_b\langle\Omega\rangle}{\left\langle\left\langle\frac{E}{1-\nu^2}\right\rangle\right\rangle^3}\right]^{-\frac{1}{4}}. \tag{7.16}$$

Equation (7.14) reduces to a quadratic equation with respect to $U = V^{1/12}$: $AU^2 - K_I U + B = 0$ that has two real positive solutions. However, only one root has a physical meaning. The other solution is meaningless since it yields higher V for lower K_I and must therefore be discarded; see (**4**, **5**).

8. Conclusion

We have presented a viable mathematical procedure to provide analytical solutions of a class of singular integro-differential equations that have appeared as governing equations in the diffusional crack growth theory. The theory was first briefly introduced in terms of its physical background, mathematical modelling and derivations of the controlling equations leading to this Cauchy-type integro-differential equation for the unknown stress distribution on the grain boundary ahead of the moving crack tip. The procedure of solving this type of integral equation analytically involved the following steps. (1) Using the Mellin transformation, the equation was first converted into a functional-difference equation; (2) reducing this difference equation to a Riemann–Hilbert boundary-value problem on an arc; and (3) finally, analysing the coefficient of the problem, solve the Riemann–Hilbert problem by quadratures. Also included in the solution scheme were the asymptotic solutions at small x (that is, near the crack tip) as well as the far field (that is, as $x \to \infty$). Since the near-tip field contains the information on the crack velocity V, and the far field is related to the applied stress intensity K_I, this stress solution makes a connection of these two fields to yield a functional relationship between K_I and V.

It should be emphasized that the analytical solutions obtained herein are exact. They are in excellent agreement with the existing numerical solutions which overestimated the K_I values by approximately 0·3 per cent. This closed-form solution which pertains to steady-state creep crack growth is useful as a limiting asymptotic value when time approaches infinity in a more general time-dependent crack-growth problem in the transient creep regime. We leave this interesting problem for future research.

Acknowledgement

The support of the first author by EPSRC through contract GR/R23381/01 is gratefully acknowledged. Part of the work was implemented during a research visit he made to the Max Planck Institute for Metals Research (Stuttgart).

References

1. H. Riedel, *Fracture at High Temperatures* (Springer, Berlin 1987).
2. M. F. Ashby and L. M. Brown, *Perspectives in Creep Fracture* (Pergamon Press, Oxford 1983).
3. T.-J. Chuang, K. I. Kagawa, J. R. Rice and L. B. Sills, Non-equilibrium models for diffusive cavitation of graininterfaces, *Acta Metallurgica* **27** (1979) 265–284.
4. ——, A diffusive crack-growth model for creep fracture, *J. American Ceramic Soc.* **65** (1982) 93–103.
5. ——, J.-L. Chu and S. Lee, Diffusive crack growth at a bimaterial interface, *J. Appl. Mech.* **63** (1996) 796–803.
6. Y. A. Antipov and H. Gao, Exact solution of integro-differential equations of diffusion along a grain boundary, *Quart. Jl Mech. Appl. Math.* **53** (2000) 545–674.
7. A. H. England, A crack between dissimilar media, *J. Appl. Mech.* **32** (1965) 400–402.
8. T.-J. Chuang and J. R. Rice, The shape of intergranular creep cracks growing by surface diffusion, *Acta Metallurgica* **21** (1973) 1625–1628.
9. F. D. Gakhov, *Boundary Value Problems* (Pergamon Press, Oxford 1966).
10. Jahnke, Emde and Lösch, *Tables of Higher Functions* (Teubner, Stuttgart 1966).

APPENDIX

General case

In the text of this paper, we have discussed the solution to the integral equation (3.1) for a range of physically relevant parameters. However, the method of analysis we presented has no limitations with respect to the parameters chosen. Here we give a general discussion of the solutions to (3.1) for different parameter ranges in case similar problems might arise in other applications.

We have solved the integro-differential equation (3.1) for positive λ and $\beta = \frac{1}{2}$. It becomes evident that if $|f(x)| \leqslant Cx^{-\beta}$, $x \to \infty$ for all $\beta \in (0, \frac{1}{2}]$, then $|f(x)| \leqslant Cx^{-\frac{1}{2}}$, $x \to \infty$. Section 3 implies that in the class $0 < \beta \leqslant \frac{1}{2}$, for $\lambda > 0$, equation (3.1) has two linearly independent solutions. The general solution possesses two arbitrary constants fixed by the conditions (3.2). For $x \to 0$ and $x \to \infty$ we have obtained

$$f(x) = 1 + \alpha x + O(x^2 \log x), \quad x \to 0,$$
$$f(x) = \frac{\Phi(\frac{1}{2})}{\lambda\pi} x^{-\frac{1}{2}} + O(x^{-3/2}), \quad x \to \infty. \tag{A.1}$$

Consider now the next class $\frac{1}{2} < \beta \leqslant 1$ and as before $\lambda > 0$. In this class, the parameter $c = \mathrm{Re}(s)$ in formula (3.3) is defined in the range $\frac{1}{2} < c < 1$. We factorize $\tan \pi\sigma$ as follows:

$$\tan \pi\sigma = -\cot \frac{\pi\sigma}{4} K_0(\sigma), \quad \operatorname{ind} K_0(\sigma) = \frac{1}{2\pi}[\arg K_0(\sigma)]_\Omega = 0, \tag{A.2}$$

and introduce the new function

$$\Phi_0(s) = \frac{\lambda^{s/2}}{\Gamma(s)} \cos \frac{\pi s}{4} \Phi(s). \tag{A.3}$$

By using the identity

$$-\cot \frac{\pi\sigma}{4} = \frac{\cos \frac{1}{4}\pi\sigma}{\cos \frac{1}{4}\pi(\sigma + 2)} \tag{A.4}$$

we arrive at the problem (3.13). In contrast with the function $\Phi_0(s)$ from section 3, the function (A.3) must satisfy the additional condition $\Phi_0(2) = 0$. This is because $s = 2 \in \Pi$ and $\Phi(s)$ is analytic in the strip Π. By the approach of section 3 we get

$$\Phi(s) = -\frac{(1+i)\Gamma(s) \cos\{\frac{1}{2}\pi(\frac{1}{2} + s - c)\} Q(s)}{\sqrt{2}\lambda^{s/2} \cos \frac{1}{4}\pi s} (C_0 + C_1 w). \tag{A.5}$$

However, the constants C_0, C_1 are not arbitrary and are linked by

$$C_0 = -iC_1 \tan\{\tfrac{1}{2}\pi(\tfrac{1}{2} - c)\}. \tag{A.6}$$

Thus, equation (3.1) has only one non-trivial solution. Since $s = \frac{1}{2} \in \Pi$, and the first pole of the integrand in (4.1) in the strip Π_1 is $s = \frac{3}{2}$, the function $f(x)$ decays at infinity as

$$f(x) \sim \frac{\Phi(\frac{3}{2})}{\pi\lambda} x^{-3/2}, \quad x \to \infty. \tag{A.7}$$

This means that the solution meets the requirement $|f(x)| \leqslant Cx^{-\beta}$, $x \to \infty$, $\beta \in (\frac{1}{2}, 1]$. For $x \to 0$, $f(x) \sim -\Phi(2) + x\Phi(1)$.

Next, analyse the class $1 < \beta \leqslant \frac{3}{2}$ ($1 < c < \frac{3}{2}$). We use the factorization (3.12), (3.14). The solution with two arbitrary constants is defined by (3.28). At infinity, the asymptotics of the function $f(x)$ is given by (A.7). However, at the point $x = 0$, its first derivative has the logarithmic singularity

$$f(x) = -\Phi(2) + O(x \log x), \quad x \to 0. \tag{A.8}$$

This occurs because now the point $s = -1 \in \Pi_{-2}$, and it is a pole of the second order for the first integrand in (4.7). The fact that $f'(x) = O(\log x)$, $x \to 0$, makes it impossible to employ the second condition in (3.2).

If $\frac{3}{2} < \beta \leqslant 2$ ($\frac{3}{2} < c < 2$), the factorization (A.2) is applied and the solution is described by (A.5). Again, since $s = 2 \in \Pi$ is a zero of the function $\Phi_0(s)$, by the same argument as in the case $\frac{1}{2} < \beta \leqslant 1$, there is only one solution to equation (3.1). Its behaviour at $x = 0$ is given by (A.8). At infinity,

$$f(x) \sim \frac{\Phi(\frac{5}{2})}{\pi\lambda} x^{-\frac{5}{2}}, \quad x \to \infty. \tag{A.9}$$

For $2 < \beta \leqslant \frac{5}{2}$ $(2 < c < \frac{5}{2})$, the factorization of the function $K(\sigma)$ is given by (A.2), (A.4). However, now, the point $s = 2$ is outside the strip Π (the next zero $s = 6$ does not belong to the strip Π either). The general solution possesses two arbitrary constants and the previous formula (A.9) is still valid for this case. At $x = 0$, the solution has the logarithmic singularity: $f(x) = O(\log x)$.

In the class $\frac{5}{2} < \beta \leqslant 3$, we return to the original factorization (3.12), (3.14). Since $s = 4 \in \Pi$ $(\frac{5}{2} < c < 3)$ we should not expect two arbitrary constants. By the condition $\Phi_0(4) = 0$ one of the constants C_0, C_1 is fixed, and by the same argument as in the previous case

$$f(x) = O(\log x), \quad x \to 0, \quad \text{and} \quad f(x) = O(x^{-\frac{7}{2}}), \quad x \to \infty. \tag{A.10}$$

We note that for $\beta > 3$, in general, the solution $f(x)$ has a non-integrable singularity at $x = 0$. For instance, if $3 < \beta \leqslant \frac{7}{2}$, then $f(x) = O(1/x)$. However, in this case, by an appropriate choice of one of the two arbitrary constants, it is possible to remove this singularity.

So far we studied the case $\lambda > 0$. Let now $\lambda < 0$. If $0 < \beta \leqslant \frac{1}{2}$, then we need to factorize the function $-\tan\pi\sigma$:

$$-\tan\pi\sigma = \frac{\cos\frac{1}{4}\pi(\sigma+2)}{\cos\frac{1}{4}\pi\sigma} K_0(\sigma), \quad \operatorname{ind} K_0(\sigma) = 0. \tag{A.11}$$

Therefore, instead of dealing with the function (3.14) we get

$$\Phi_0(s) = \frac{\lambda^{s/2}\Phi(s)}{\Gamma(s)\cos\frac{1}{4}\pi s}. \tag{A.12}$$

The function $\Phi_0(s)$ admits a pole at the point $s = 2 \in \Pi$. The scheme of section 3 yields the following relations for the integers p, q and κ:

$$-\frac{3}{4} < p < -\frac{1}{4} - \kappa, \quad q = -1 - p - \kappa. \tag{A.13}$$

The above inequality implies $\kappa = -1$ and $p = q = 0$. Because of the pole $s = 2$ of the function $\varphi(w)$, the solution to the Riemann–Hilbert problem (3.19) is not trivial and has one arbitrary constant:

$$\varphi(w) = \frac{C_* e^{Y(w)}}{w - w_0}, \tag{A.14}$$

where $w_0 = i\tan\{\frac{1}{2}\pi(\frac{1}{2} - c)\}$. Therefore, the integro-differential equation (3.1) has only one solution. As $x \to 0$, $f(x) \sim -\Phi(2) + x\Phi(1)$, and as $x \to \infty$, the function $f(x)$ has the asymptotics (A.1).

In the next class $\frac{1}{2} < \beta \leqslant 1$, we get

$$-\tan\pi\sigma = \frac{\sin\frac{1}{4}\pi(\sigma+2)}{\sin\frac{1}{4}\pi\sigma} K_0(\sigma), \quad \operatorname{ind} K_0(\sigma) = 0, \tag{A.15}$$

and

$$\Phi_0(s) = \frac{\lambda^{s/2}\Phi(s)}{\Gamma(s)\sin\frac{1}{4}\pi s}. \tag{A.16}$$

It is obvious that the function $\Phi_0(s)$ is free of poles in the strip Π and by the above argument, the corresponding equation (3.13) has the trivial solution only. Thus, $f(x) \equiv 0$.

Let us analyse the other cases. For $1 < \beta \leqslant \frac{3}{2}$ $(1 < c < \frac{3}{2})$, we use formulae (A.11), (A.12). The function $\Phi_0(s)$ has a simple pole at $s = 2 \in \Pi$. The solution is defined uniquely up to an arbitrary constant factor and displays the following behaviour at the singular points:

$$f(x) = C' + C'' x\log x, \quad x \to 0; \quad f(x) = O(x^{-3/2}), \quad x \to \infty, \tag{A.17}$$

where C', C'' are constants.

If $\frac{3}{2} < \beta \leqslant 2$ ($\frac{3}{2} < c < 2$), we introduce the auxiliary function (A.16) and, as in the case $\frac{1}{2} < \beta \leqslant 1$, $f(x) \equiv 0$.

For $2 < \beta \leqslant \frac{5}{2}$ ($2 < c < \frac{5}{2}$), we use the factorization (A.15). Clearly, the function (A.16) has a simple pole at the point $s = 4$. This circumstance remedies the situation, the solution has one arbitrary constant and it is given by (A.14). At the singular points, its behaviour is described by (A.10).

We now state the final result.

THEOREM 1. *Let $f(x) \in \mathcal{H}[b, B]$ for all b, B such that $0 < b < B < \infty$ and, in addition, $|f(x)| \leqslant Cx^{-\beta}$ as $x \to \infty$, where C, β are positive constants. Let $\lambda > 0$ and $k = 0, 1, 2, \dots$. Then the integro-differential equation*

$$f''(x) = \frac{\lambda}{\pi} \int_0^\infty \frac{f(t)dt}{x - t}, \quad 0 < x < \infty, \tag{A.18}$$

has two linearly independent solutions for $k < \beta \leqslant k + \frac{1}{2}$, and at infinity, $f(x) = O(x^{-k-\frac{1}{2}})$.

If $k + \frac{1}{2} < \beta \leqslant k + 1$, equation (A.18) *is solvable uniquely (up to an arbitrary constant factor), and $f(x) = O(x^{-k-\frac{3}{2}})$, $x \to \infty$. For $0 < \beta \leqslant 3$ as $x \to 0$, the function $f(x)$ behaves as follows:*

$$f(x) \sim \begin{cases} C_0' + C_0'' x, & 0 < \beta \leqslant 1, \\ C_1' + C_1'' x \log x, & 1 < \beta \leqslant 2, \\ C_2' \log x + C_2'', & 2 < \beta \leqslant 3, \end{cases} \quad x \to 0, \tag{A.19}$$

where C_k', C_k'' ($k = 0, 1, 2$) are constants. For $\beta > 3$, the solution is non-integrable at $x = 0$.

Let $\lambda < 0$. For $k < \beta \leqslant k + \frac{1}{2}$ equation (A.18) *has only one non-trivial solution which has the same asymptotic properties at $x = 0$ and at infinity as in the case $\lambda > 0$.*

For $k + \frac{1}{2} < \beta \leqslant k + 1$, equation (A.18) *has the trivial solution only.*