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An exact solution of the 3-D-problem of an interface semi-infinite plane crack

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Abstract

The problem of a semi-infinite plane interface crack between three-dimensional (3-D) isotropic half-spaces is considered. Mathematically the problem is reduced to the analysis of the 3×3 matrix Wiener–Hopf problem. Explicit expressions for the factors of the 3×3 matrix are determined in quadratures. Exact-closed formulae for the stresses, discontinuities of the displacements, the stress intensity factors and the weight functions are found.

Keywords: A. Stress intensity factor; B. Crack mechanics; Inhomogeneous material; C. Integral transforms; Analytic functions

1. Introduction

The 3-D-problem of a semi-infinite plane crack in a uniform space was studied by Uflyand (1965). Kassir and Sih (1973) considered the case when the Uflyand technique had failed, namely, the case of two equal and opposite forces directed parallel to the edge of the crack. They found the Green's function and exact expressions for the distribution of stress and displacement in terms of elementary functions. Bueckner (1987) constructed the weight functions in the 3-D homogeneous case.

The 3-D problems of interface cracks were discussed systematically by Willis (1971a, b, 1972) who analysed the stresses around a crack on the plane interface between two dissimilar anisotropic half-spaces and considered the problems of an interface penny-shaped crack and of dissimilar half-spaces bonded together over a circular area in the case of arbitrary loading. The problems were reduced to an integral equation for the Radon transform of the displacement and were solved exactly. A particular case of the problem of an interface crack that is equivalent to the 2×2 matrix Wiener–Hopf problem and admits an exact solution was analysed by Tichomirov (1994, 1996).

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Duduchava and Wendland (1995) developed the Wiener–Hopf method for systems of boundary pseudo-differential equations which allowed them to manage without the factorization of corresponding matrix symbols and to investigate the asymptotic behaviour of the solution to the crack problem in an anisotropic medium. An explicit description of the exponents in the asymptotics for an interface crack in a 3-D anisotropic composite which is based on the same technique was presented by Duduchava et al. (1996).

Lazarus and Leblond (1998a, b) analysed a small coplanar perturbation of a semi-infinite plane interface crack in an infinite 3-D body and constructed an asymptotic expansion for the weight functions. The expansion involved the zeroth-order solution (the solution for a homogeneous material) and the first-order term in the 'bimaterial constant' ε solution. Lazarus and Leblond did not look for the complete solution of the elasticity problem but applied the so-called 'special' method for weight functions that needed some elementary properties of the interface crack weight functions and the explicit expressions for the weight functions in the 3-D homogeneous case.

A class of 2×2 matrices that admits constructive factorization based on the theory of functions on matrices was described by Chebotarev (1956). Explicit factors in this case were found by Khrapkov (1971). The generalization for a class of $N\times N$ matrices of a special structure was proposed by Jones (1984). He showed that any matrix of this class under a special restriction can be factorized commutatively and there are no other matrices with factors which commute if the factors possess distinct eigenvalues. Jones derived explicit formulae for the factors. Some cases of the 3×3 matrix Wiener–Hopf problem which arise in the elastodynamic diffraction theory and the theory of plates were analysed by Meister and Speck (1989), Câmara et al. (1993), Moiseyev and Popov (1990).

The present work is devoted to the construction of an exact solution in closed form of a 3×3 matrix Wiener–Hopf problem to which the 3-D-problem of a semi-infinite plane interface crack for the case of a general load is reduced. The problem leads to factorization of a 3×3 matrix $\mathbf{G}_0$ that does not satisfy the conditions imposed by Jones. The following representation for this matrix

$$\mathbf{G}_0 = \mathbf{N}_1\mathbf{G}_1\mathbf{N}_2$$

is found. Here $\mathbf{N}_1$ and $\mathbf{N}_2$ are rational matrices and the matrix $\mathbf{G}_1$ has a block diagonal structure

$$\begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \end{pmatrix} \tag{1.1}$$

with a scalar a and a 2×2 matrix $\mathbf{G}_2$ which admits the factorization by the Khrapkov method. Thus, the task of splitting the Wiener–Hopf kernel into two parts is achieved in terms of matrices which are analytic in the upper and lower half-planes, respectively, except at a finite number of poles. The subsequent elimination of these singularities leads to the exact solution of the problem by quadratures. The expressions for the weight functions are constructed in the explicit form. Additionally, a special non-homogeneous case of the elastic constants of the half-spaces in detail is analysed. By passing on to the limit $\lambda \to 0$, where λ is a parameter of the Fourier transform, a solution of the 2-D problem is found.

2. Formulation

Let us consider an elastic isotropic space consisting of two half-spaces $\mathbb{R}_3^+ = \{|x| < \infty, 0 < y < \infty, |z| < \infty\}$ and $\mathbb{R}_3^- = \{|x| < \infty, -\infty < y < 0, |z| < \infty\}$ with Poisson's ratio ν_+ and ν_- and shear modulus G_+ and G_-, respectively. Along the interface between the media there is a semi-infinite crack $\mathbb{R}_2^+ = \{0 < x < \infty, y = \pm 0, -\infty < z < \infty\}$ (see Fig. 1) that is acted on by normal and tangential loads

$$\sigma_y = p_1(x,z), \quad \tau_{xy} = p_2(x,z), \quad \tau_{yz} = p_3(x,z), \quad (x,y,z) \in \mathbb{R}_2^+. \tag{2.1}$$

The conditions of ideal contact take place on the other part $\mathbb{R}_2^- = \{-\infty < x < 0, y = \pm 0, -\infty < z < \infty\}$ of the interface. This is assured by requiring the traction and displacement components to be continuous across the interface

$$[\sigma_y] = [\tau_{xy}] = [\tau_{yz}] = 0, \quad [u] = [v] = [w] = 0,$$

where $[f]$ determines a discontinuity of a function while crossing the interface

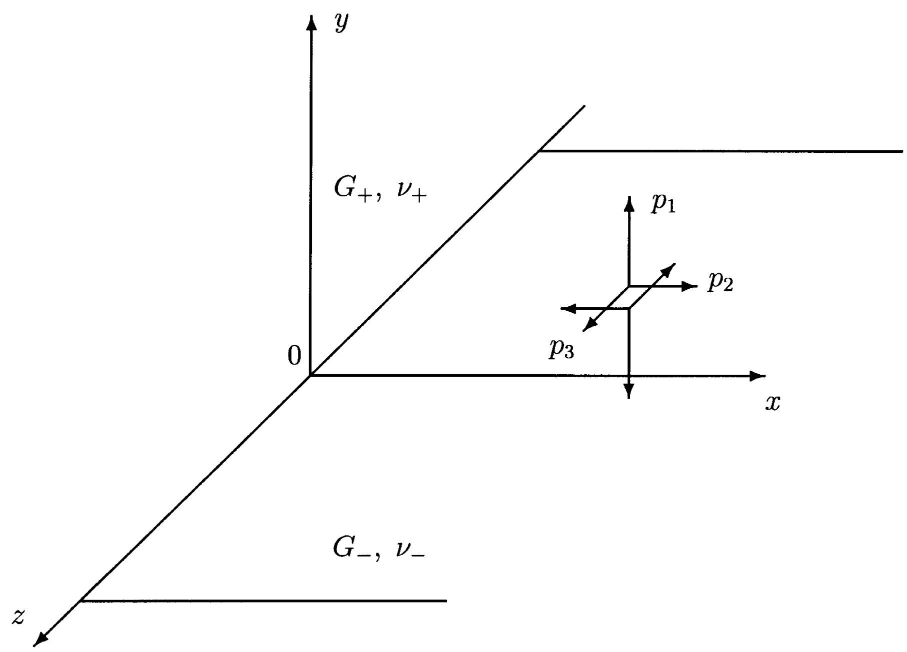

Fig. 1. The geometry of the problem.

$$[f] = f|_{x=+0} - f|_{x=-0}.$$

The equilibrium of an isotropic solid is governed by the Lamé displacement equations which in the absence of body forces appear as

$$\operatorname{grad}\operatorname{div}\mathbf{u} + (1-2\nu_{\pm})\Delta\mathbf{u} = 0, \quad (x,y,z) \in \mathbb{R}_3^{\pm},$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the displacement vector.

Let us introduce the Fourier transforms of the tractions on the crack faces and on the interface and of the jumps of the displacement

$$\mathbf{p}_{\beta\lambda} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{p}(x,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

$$[\mathbf{u}_{\beta\lambda}] = \int_{-\infty}^{\infty}\int_{0}^{\infty} [\mathbf{u}](x,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda z}\,\mathrm{d}x\,\mathrm{d}z.$$

Here

$$\mathbf{p}_{\beta\lambda} = \begin{pmatrix} p_{1\beta\lambda} \\ p_{2\beta\lambda} \\ p_{3\beta\lambda} \end{pmatrix}, \quad [\mathbf{u}_{\beta\lambda}] = \begin{pmatrix} [u_{\beta\lambda}] \\ [v_{\beta\lambda}] \\ [w_{\beta\lambda}] \end{pmatrix}, \quad \mathbf{p}(x,z) = \begin{pmatrix} \sigma_y(x, \pm 0, z) \\ \tau_{xy}(x, \pm 0, z) \\ \tau_{yz}(x, \pm 0, z) \end{pmatrix}. \tag{2.2}$$

The relationship that connects the jumps of the displacement transforms with the traction transforms on the interface has been obtained by Willis (1971b, 1972) and has the form

$$[\mathbf{u}_{\beta\lambda}] = \frac{1}{\rho}\mathbf{G}(\beta)\mathbf{p}_{\beta\lambda}, \quad \rho = (\lambda^2+\beta^2)^{1/2}, \tag{2.3}$$

where

$$\mathbf{G}(\beta) = -\frac{1}{\rho^2}\begin{pmatrix} -id\beta\rho & b\rho^2+e\lambda^2 & -e\beta\lambda \\ b\rho^2 & id\beta\rho & id\lambda\rho \\ -id\lambda\rho & -e\beta\lambda & b\rho^2+e\beta^2 \end{pmatrix},$$

$$b = \frac{1-\nu_+}{G_+} + \frac{1-\nu_-}{G_-}, \quad d = \frac{1-2\nu_+}{2G_+} - \frac{1-2\nu_-}{2G_-}, \quad e = \frac{\nu_+}{G_+} + \frac{\nu_-}{G_-}. \tag{2.4}$$

The matrix $\mathbf{G}(\beta)$ can also be obtained in another way (see Appendix A). It is convenient for the future to symmetrize the matrix $\mathbf{G}(\beta)$

$$\mathbf{G}(\beta) = \mathbf{J}_1\mathbf{G}_0(\beta)\mathbf{J}_2,$$

$$\mathbf{J}_1 = [\mathbf{J}_1^{-1}]^{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \mathbf{J}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{G}_0(\beta) = -\frac{1}{\rho^2}\begin{pmatrix} b\rho^2 & id\lambda\rho & id\beta\rho \\ -id\lambda\rho & b\rho^2+e\beta^2 & -e\beta\lambda \\ -id\beta\rho & -e\beta\lambda & b\rho^2+e\lambda^2 \end{pmatrix}, \tag{2.5}$$

where $\mathbf{J}^t$ is the transposed matrix. We split the vector $\mathbf{p}_{\beta\lambda}$ into two parts

$$\mathbf{p}_{\beta\lambda} = \mathbf{J}_2^{-1}[\mathbf{\Phi}^-(\beta)+\mathbf{F}^+(\beta)],$$

where

$$\mathbf{\Phi}^-(\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{0} \mathbf{J}_2\mathbf{p}(x,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

$$\mathbf{F}^+(\beta) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathbf{J}_2\mathbf{p}(x,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda z}\,\mathrm{d}x\,\mathrm{d}z \tag{2.6}$$

and introduce the vector

$$\mathbf{\Phi}^+(\beta) = \mathbf{J}_1^{-1}[\mathbf{u}_{\beta\lambda}]. \tag{2.7}$$

The unknown vector $\mathbf{\Phi}^+(\beta) = \{\Phi_j^+(\beta)\}$ and the known one $\mathbf{F}^+(\beta) = \{F_j^+(\beta)\}$ $(j = 1,2,3)$ are analytic in the upper half-plane $(\Im\beta > 0)$ and the unknown vector $\mathbf{\Phi}^-(\beta) = \{\Phi_j^-(\beta)\}$ is analytic in the lower one $(\Im\beta < 0)$. Thus, we arrive at the 3×3 matrix Wiener–Hopf problem

$$\mathbf{\Phi}^+(\beta) = \frac{1}{\rho}\mathbf{G}_0(\beta)[\mathbf{\Phi}^-(\beta)+\mathbf{F}^+(\beta)], \quad \beta\in\mathbb{R}_1 = \{-\infty,\infty\}. \tag{2.8}$$

The vectors $\mathbf{\Phi}^{\pm}(\beta)$, $\mathbf{F}^{\pm}(\beta)$ and the matrix $\mathbf{G}(\beta)$ depend on the parameter λ that we omit.

First, consider the more complicated case $d \neq 0$.

3. Factorization

The key step in the solution of problem (2.8) is the splitting of the matrix $\rho^{-1}\mathbf{G}_0(\beta)$ into a product of two factors which are analytic matrices in the upper and lower half-planes, respectively, with the exception maybe of a finite number of poles or the points at which the inverse matrices have poles. The matrix $\mathbf{G}_0(\beta)$ admits the following representation

$$\mathbf{G}_0(\beta) = \mathbf{U}(\beta)\mathbf{G}_1(\beta)\mathbf{T}^{-1}(\beta), \tag{3.1}$$

where the matrix $\mathbf{G}_1(\beta)$ has a block diagonal structure and $\mathbf{U}(\beta)$ and $\mathbf{T}^{-1}(\beta)$ are rational matrices

$$\mathbf{G}_1(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_1(\beta) & c_1(\beta) \\ 0 & -c(\beta)\rho^2 & b_1(\beta) \end{pmatrix}, \quad \mathbf{T}^{-1}(\beta) = \begin{pmatrix} 0 & \beta\rho^{-2} & -\lambda\rho^{-2} \\ 0 & \lambda\rho^{-2} & \beta\rho^{-2} \\ 1 & 0 & 0 \end{pmatrix},$$

$$b_1(\beta) = \frac{d^2\rho(b^2\rho+1)}{\delta_0}, \quad c_1(\beta) = -\frac{idb(d^2\rho+1)}{\delta_0}, \quad \delta_0 = b^2d^2\rho^2-1, \tag{3.2}$$

$$\mathbf{U}(\beta) = \begin{pmatrix} 0 & id^{-1} & -b \\ -(b+e)\beta & -b\lambda & -i\lambda d^{-1}\rho^{-2} \\ (b+e)\lambda & -b\beta & -i\beta d^{-1}\rho^{-2} \end{pmatrix}. \tag{3.3}$$

The derivation of splitting (3.1) is recorded in Appendix B. Let us construct the Wiener–Hopf factors of the matrix $\mathbf{G}_1(\beta)$. According to the Khrapkov notations (see Khrapkov, 1971) we have the following functions

$$l = 0, \quad m = 1, \quad n = -\rho^2, \quad f(\beta) = -\rho^2,$$

$$\Lambda_1(\beta) = b_1(\beta)+i\rho c_1(\beta), \quad \Lambda_2(\beta) = b_1(\beta)-i\rho c_1(\beta), \quad \Delta(\beta) = b_1^2(\beta)+\rho^2 c_1^2(\beta),$$

where $\Lambda_1(\beta)$, $\Lambda_2(\beta)$ are the characteristic functions of the matrix

$$\mathbf{G}_2(\beta) = \begin{pmatrix} b_1(\beta) & c_1(\beta) \\ -c_1(\beta)\rho^2 & b_1(\beta) \end{pmatrix}$$

and $\Delta(\beta) = \det \mathbf{G}_2(\beta)$. In order to fixed a branch of the function $f^{1/2}(\beta) = i\rho$, $\rho = (\lambda^2+\beta^2)^{1/2}$, we cut the β-plane by a straight line that joins the branch points $i|\lambda|$ and $-i|\lambda|$ and passes through infinity. Additionally, we stipulate that

$$-\frac{3\pi}{2} < \arg(\beta - i|\lambda|) < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \arg(\beta+i|\lambda|) < \frac{3\pi}{2}.$$

Without loss of generality we assume $d > 0$. The next step is to factorize the function

$$\Delta^{1/2}(\beta) = \frac{d\sqrt{b^2-d^2}\rho}{(b^2d^2\rho^2-1)^{1/2}} \quad (b > d).$$

There are two possible cases: the first one when $|\lambda| < (bd)^{-1}$ and the second one when $|\lambda| > (bd)^{-1}$. Let us consider the first case. The function $\delta_0 = b^2d^2\rho^2-1$ has two real roots: $\beta = \lambda_R$ and $\beta = -\lambda_R$, where

$$\lambda_R = \sqrt{|d^{-2}b^{-2}-\lambda^2|}. \tag{3.4}$$

We cut the β-plane by a semi-infinite section (λ_R, ∞) of the real axis and demand that $0 < \arg(\beta-\lambda_R) < 2\pi$. Thus, the function

$$\Lambda^+(\beta) = \frac{d_0(\beta+i|\lambda|)^{1/2}}{(\beta-\lambda_R)^{1/2}}, \quad d_0 = \left(1-\frac{d^2}{b^2}\right)^{1/4} \tag{3.5}$$

is analytic in the plane cut by the lines $\{\lambda_R < \Re\beta < \infty, \Im\beta = \pm 0\}$ and $\{\Re\beta = \pm 0, -\infty < \Im\beta < -|\lambda|\}$. Similarly, the function

$$\Lambda^-(\beta) = \frac{(\beta+\lambda_R)^{1/2}}{d_0(\beta - i|\lambda|)^{1/2}}, \quad -\pi < \arg(\beta+\lambda_R) < \pi \tag{3.6}$$

is analytic in the β-plane if we cut it by the lines $\{-\infty < \Re\beta < -\lambda_R, \Im\beta = \pm 0\}$ and $\{\Re\beta = \pm 0, |\lambda| < \Im\beta < \infty\}$. Let us introduce the following contour $\Gamma = \Gamma_- \bigcup \Gamma_0 \bigcup \Gamma_+$ (see Fig. 2), where

$$\Gamma_- = \{\beta = \tau - i0, -\infty < \tau < -\lambda_R\},$$

$$\Gamma_0 = \{\beta = \tau, -\lambda_R < \tau < \lambda_R\},$$

$$\Gamma_+ = \{\beta = \tau + i0, \lambda_R < \tau < \infty\},$$

that divides the β-plane into two half-planes: the upper half-plane $\mathbb{C}^+ \ni -\lambda_R$ and the lower one $\mathbb{C}^- \ni \lambda_R$. In this case the contour Γ specified above avoids the poles on the real axis. The functions $\Lambda^\pm(\beta)$ are analytic in $\mathbb{C}^\pm$, do not have zeros and poles there and give the factorization of the function

$$\Delta^{1/2}(\beta) = \frac{\Lambda^+(\beta)}{\Lambda^-(\beta)}, \quad \beta \in \Gamma. \tag{3.7}$$

Note, that in accordance with the results of Rapoport (1948) there is no need to have a common strip of analyticity for the factors that the technique of Wiener and Hopf

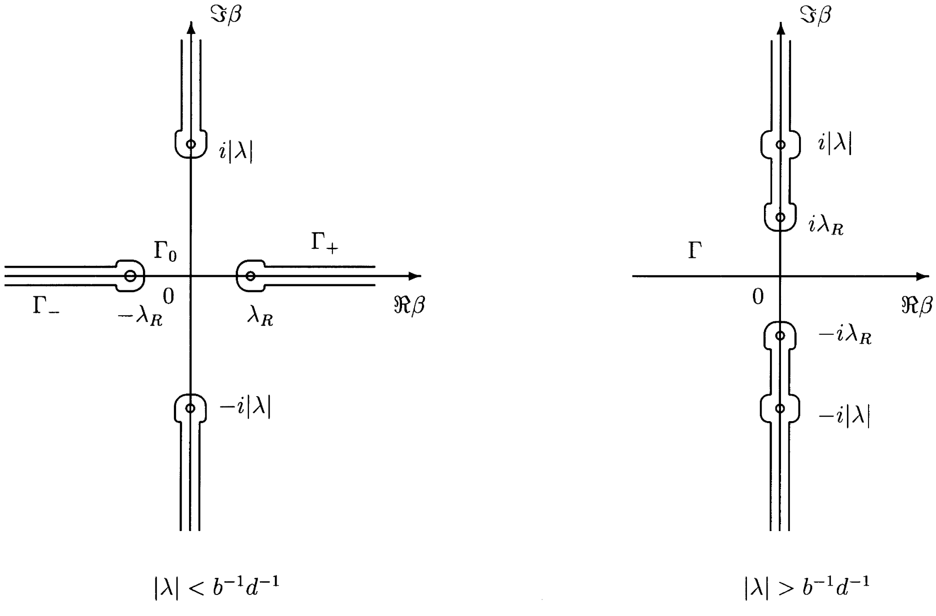

Fig. 2. Branch cuts and the contours Γ_-, Γ_0, Γ_+, Γ.

(1931) demands. The factors Λ^+ and Λ^- must be only analytically continued into the upper and lower half-planes, respectively, and on the contour Γ satisfy boundary condition (3.7).

In the case $|\lambda| > b^{-1}d^{-1}$ the function δ_0 has two complex-conjugate roots: $\beta = \pm i\lambda_R$, where λ_R is the same as that in (3.4). The function $\Delta^{1/2}(\beta)$ admits decomposition (3.7), where we assume $\Gamma = \mathbb{R}_1 = \{-\infty, \infty\}$ (see Fig. 2) and

$$\Lambda^+(\beta) = \frac{d_0(\beta+i|\lambda|)^{1/2}}{(\beta+i\lambda_R)^{1/2}}, \quad -\frac{\pi}{2} < \arg(\beta+i\lambda_R) < \frac{3\pi}{2},$$

$$\Lambda^-(\beta) = \frac{(\beta-i\lambda_R)^{1/2}}{d_0(\beta-i|\lambda|)^{1/2}}, \quad -\frac{3\pi}{2} < \arg(\beta-i\lambda_R) < \frac{\pi}{2}.$$

In both cases $|\lambda| < b^{-1}d^{-1}$ and $|\lambda| > b^{-1}d^{-1}$ the functions $\Lambda^+(\beta)$, $\Lambda^-(\beta)$ possess the following properties

$$\Lambda^+(\beta) = \frac{1}{\Lambda^-(-\beta)}, \quad \Lambda^\pm(\beta) = d_0^{\pm 1}\left[1+O\left(\frac{1}{\beta}\right)\right], \quad \beta\to\infty, \quad \beta\in\mathbb{C}^\pm. \tag{3.8}$$

Now we find the exponent $\Xi(\beta)$ of the matrix $\mathbf{G}_2(\beta)$

$$\Xi(\beta) = \frac{1}{2}\log\frac{\Lambda_1(\beta)}{\Lambda_2(\beta)} = \frac{1}{2}\log\left(\frac{b+d}{b-d}\frac{bd\rho+1}{bd\rho-1}\right).$$

According to the choice of the branches of the functions $(\beta\pm\lambda_R)^{1/2}$ in the case $|\lambda| < b^{-1}d^{-1}$ we fix the branch of the exponent $\Xi(\beta)$

$$\Xi(\beta) = \frac{d_1}{2} + \frac{1}{2}\log\left|\frac{bd\rho+1}{bd\rho-1}\right| - \begin{cases} 0, & \beta\in\Gamma_+ \\ \dfrac{i\pi}{2}, & \beta\in\Gamma_0 \\ 0, & \beta\in\Gamma_- \end{cases} \quad \left(|\lambda| < \frac{1}{bd}\right), \tag{3.9}$$

where

$$d_1 = \log\frac{b+d}{b-d}.$$

Obviously, in another case we have

$$\Xi(\beta) = \frac{d_1}{2} + \frac{1}{2}\log\left|\frac{bd\rho+1}{bd\rho-1}\right| \quad \left(|\lambda| > \frac{1}{bd}\right). \tag{3.10}$$

The factorization of the matrix $\mathbf{G}_1(\beta)$ is constructed by formulae

$$\mathbf{G}_1(\beta) = \mathbf{X}_1^+(\beta)[\mathbf{X}_1^-(\beta)]^{-1} = [\mathbf{X}_1^-(\beta)]^{-1}\mathbf{X}_1^+(\beta), \quad \beta\in\Gamma,$$

$$\mathbf{X}_1^{\pm}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda^{\pm}(\beta)\cos\{\rho B^{\pm}(\beta)\} & \rho^{-1}\Lambda^{\pm}(\beta)\sin\{\rho B^{\pm}(\beta)\} \\ 0 & -\rho\Lambda^{\pm}(\beta)\sin\{\rho B^{\pm}(\beta)\} & \Lambda^{\pm}(\beta)\cos\{\rho B^{\pm}(\beta)\} \end{pmatrix},$$

$$[\mathbf{X}_1^{\pm}(\beta)]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & [\Lambda^{\pm}(\beta)]^{-1}\cos\{\rho B^{\pm}(\beta)\} & -\rho^{-1}[\Lambda^{\pm}(\beta)]^{-1}\sin\{\rho B^{\pm}(\beta)\} \\ 0 & \rho[\Lambda^{\pm}(\beta)]^{-1}\sin\{\rho B^{\pm}(\beta)\} & [\Lambda^{\pm}(\beta)]^{-1}\cos\{\rho B^{\pm}(\beta)\} \end{pmatrix},$$

$$\beta \in \mathbb{C}^{\pm} \bigcup \Gamma \tag{3.11}$$

where $B(\beta) = B^{\pm}(\beta)$, $\beta \in \mathbb{C}^{\pm}$ is a sectionally analytic function that vanishes as $\beta \to \infty$ and should be determined in accordance with a given jump

$$B^{+}(\beta) - B^{-}(\beta) = \frac{1}{i\rho}\Xi(\beta), \quad \beta \in \Gamma.$$

Taking into account formulae (3.9) and (3.10) we represent the solution in the form

$$B^{\pm}(\beta) = \frac{d_1}{2i}\Psi^{\pm}(\beta) + B_1^{\pm}(\beta), \quad \beta \in \mathbb{C}^{\pm},$$

$$\Psi^{\pm}(\beta) = \frac{i}{\pi\rho}\log\frac{\beta+\rho}{\pm i|\lambda|}, \quad -\pi \leqslant \arg\frac{\beta+\rho}{\pm i|\lambda|} \leqslant \pi, \tag{3.12}$$

$$B_1^{\pm}(\beta) = -\frac{1}{4\pi}\int_{\Gamma}\log\frac{bd(t^2+\lambda^2)^{1/2}+1}{bd(t^2+\lambda^2)^{1/2}-1}\frac{\mathrm{d}t}{(t^2+\lambda^2)^{1/2}(t-\beta)}, \quad \beta \in \mathbb{C}^{\pm}. \tag{3.13}$$

Now we proceed to an examination of the behaviour of the function $B^{\pm}(\beta)$ as $\beta \to \infty$ and in the case $|\lambda| < b^{-1}d^{-1}$ as $\beta \to \pm\lambda_R$, in order to investigate the properties at these points of the factors $\mathbf{X}_1^{\pm}(\beta)$. The choice of the branch of the function ρ dictates the following behaviour

$$\rho \sim \beta\,\mathrm{sgn}\,(\Re\beta), \quad \beta \to \infty \quad \beta \in \mathbb{C}^{\pm}. \tag{3.14}$$

First, taking $\beta \in \mathbb{C}^{+}$ and $\arg\beta \in (0, \pi/2)$ we have

$$\Psi^{+}(\beta) = \frac{i}{\pi\beta}\log\frac{2\beta}{i|\lambda|} + O\left(\frac{1}{\beta^3}\right), \quad \beta \to \infty, \quad \beta \in \mathbb{C}^{+}. \tag{3.15}$$

If $\arg\beta \in (\pi/2, \pi)$ then the relation (3.12) can be written in the form

$$\Psi^{+}(\beta) = -\frac{i}{\pi\rho}\log\frac{\beta-\rho}{i|\lambda|}$$

and from (3.14) it follows the same asymptotic equality (3.15). In a similar manner, for the case $\beta \in \mathbb{C}^{-}$ we arrive at the relation

$$\Psi^-(\beta) = \frac{i}{\pi\beta}\log\frac{2\beta}{-i|\lambda|}+O\left(\frac{1}{\beta^3}\right), \quad \beta\to\infty.$$

Analysis of the Cauchy type integral (3.13) leads to the following asymptotics of the functions $B^{\pm}(\beta)$

$$B^{\pm}(\beta) = \frac{d_1}{2\pi\beta}\log\beta+\frac{1}{\beta}\left(\frac{d_1}{2\pi}\log\frac{2}{\pm i|\lambda|}+B_0\right)+O\left(\frac{1}{\beta^3}\right), \quad \beta\to\infty, \quad \beta\in\mathbb{C}^{\pm},$$

$$B_0 = \frac{1}{2\pi}\int_0^{\infty}\log\frac{bd(t^2+\lambda^2)^{1/2}+1}{bd(t^2+\lambda^2)^{1/2}-1}\frac{\mathrm{d}t}{(t^2+\lambda^2)^{1/2}}. \tag{3.16}$$

For the calculation of B_0 in the case $|\lambda| < b^{-1}d^{-1}$ it is necessary to take into account the relation (3.9).

Before proceeding to an investigation of the behaviour of the function $B(\beta)$ at the points $\beta = \pm\lambda_R$ where the density has singularities of a logarithmic type, let us consider the Cauchy type integral

$$\omega_1(\beta) = \frac{1}{2\pi i}\int_{\Gamma_-}\frac{\log(t+\lambda_R)}{t-\beta}\mathrm{d}t$$

as $\beta\to-\lambda_R$ and apply the Gakhov method (Gakhov, 1966). The contour Γ_- coincides with the lower side of the cut $\{\beta = \tau\pm i0, -\infty<\tau<-\lambda_R\}$ and thus,

$$\log(t+\lambda_R) = [\log(t+\lambda_R)]^-, \quad [\log(t+\lambda_R)]^+ = \log(t+\lambda_R)+2\pi i,$$

where $\log(t+\lambda_R)$, $[\log(t+\lambda_R)]^+$ and $[\log(t+\lambda_R)]^-$ are the values of the function $\log(\beta+\lambda_R)$ on the contour Γ_-, upper and lower side of the cut, respectively. For this case the Melnik formulae (see Gakhov, 1966) have the form

$$\omega_1(\beta) = \frac{1}{4\pi i}\log^2(\beta+\lambda_R)-\frac{1}{2}\log(\beta+\lambda_R)+\Omega_{1-}(\beta), \quad \beta\to-\lambda_R, \quad \beta\in\mathbb{C}^{\pm},$$

$$\omega_1(t) = \frac{1}{4\pi i}\log^2(t+\lambda_R)+\Omega_{1-}(t), \quad t\to-\lambda_R, \quad t\in\Gamma_-,$$

where $\Omega_{1-}(\beta)$ is a function analytic in the vicinity of the point $-\lambda_R$. According to the definition of the contour Γ_+ we put

$$\log(t-\lambda_R) = [\log(t-\lambda_R)]^+, \quad [\log(t-\lambda_R)]^- = \log(t-\lambda_R)+2\pi i.$$

In a similar manner, for the Cauchy type integral over the contour Γ_+

$$\omega_2(\beta) = \frac{1}{2\pi i}\int_{\Gamma_+}\frac{\log(t-\lambda_R)}{t-\beta}\mathrm{d}t$$

we obtain the following relations

$$\omega_2(\beta) = -\frac{1}{4\pi i}\log^2(\beta-\lambda_R)+\frac{1}{2}\log(\beta-\lambda_R)+\Omega_{1+}(\beta), \quad \beta\to\lambda_R, \quad \beta\in\mathbb{C}^{\pm},$$

$$\omega_2(t) = -\frac{1}{4\pi i}\log^2(t-\lambda_R)+\Omega_{1+}(t), \quad t\to\lambda_R, \quad t\in\Gamma_+, \tag{3.17}$$

where $\Omega_{1+}(\beta)$ is a function analytic in the neighbourhood of the point λ_R. We consider the integral

$$\omega_0(\beta) = \frac{1}{2\pi i}\int_{\Gamma_-\cup\Gamma_0}\frac{\log(t+\lambda_R)}{t-\beta}\mathrm{d}t.$$

In order to estimate this integral in the vicinity of the point $-\lambda_R$, we split the integral into two parts and study the integral over the section Γ_0

$$J(\beta) = \frac{1}{2\pi i}\int_{\Gamma_0}\frac{\log(t+\lambda_R)}{t-\beta}\mathrm{d}t. \tag{3.18}$$

As the contour Γ_0 is not the cut line then

$$[\log(t+\lambda_R)]^{\pm} = \log(t+\lambda_R), \quad \arg(t+\lambda_R) = 0, \quad t\in(-\lambda_R,\lambda_R). \tag{3.19}$$

According to the first Sokhotski–Plemelj formula

$$\omega_0^+(t)-\omega_0^-(t) = \log(t+\lambda_R), \quad t\in\Gamma_-\bigcup\Gamma_0. \tag{3.20}$$

We seek the representation for the function $J(\beta)$ in the vicinity of the point $-\lambda_R$ in the form

$$J(\beta) = C_{\pm}\log^2(\beta+\lambda_R)+D_{\pm}\log(\beta+\lambda_R)+\Omega_{2-}(\beta), \quad \beta\in\mathbb{C}^{\pm}$$

that is different on the upper and lower side of the contour $\Gamma_-\bigcup\Gamma_0$. The coefficients $C_{\pm}$, $D_{\pm}$ should be determined. Due to the relations (3.19) and (3.20), we get

$$C_+-C_- = 0, \quad D_+-D_- = 1. \tag{3.21}$$

Since the relations (3.17) do not depend on the choice of the branch of the function $\log(\beta-\lambda_R)$ it follows that

$$\frac{1}{2}[(C_++C_-)\log^2(t+\lambda_R)+(D_++D_-)\log(t+\lambda_R)]$$

$$= -\frac{1}{4\pi i}\log^2(t+\lambda_R), \quad t\in\Gamma_0.$$

This relation gives immediately

$$C_++C_- = -\frac{1}{2\pi i}, \quad D_++D_- = 0. \tag{3.22}$$

Equations (3.21) and (3.22) define the constants $C_{\pm}$, $D_{\pm}$ and thus,

$$J(\beta) = -\frac{1}{4\pi i}\log^2(\beta+\lambda_R) \pm \frac{1}{2}\log(\beta+\lambda_R) + \mathbf{\Omega}_{3-}(\beta), \quad \beta \to -\lambda_R, \quad \beta \in \mathbb{C}^{\pm},$$

$$\omega_0(\beta) = \begin{cases} \mathbf{\Omega}_{4-}(\beta), & \beta \to -\lambda_R, \quad \beta \in \mathbb{C}^+ \\ -\log(\beta+\lambda_R) + \mathbf{\Omega}_{4-}(\beta), & \beta \to -\lambda_R, \quad \beta \in \mathbb{C}^- \end{cases}, \tag{3.23}$$

where $\mathbf{\Omega}_{m-}(\beta)$ are functions analytic in the vicinity of the point $-\lambda_R$. The generalization of the representation (3.23) for the common case of a logarithmic singularity is found in the standard manner (Muskhelishvili, 1953)

$$B^+(\beta) = \mathbf{\Omega}_{0-}(\beta), \quad \beta \to -\lambda_R, \quad \beta \in \mathbb{C}^+,$$

$$B^-(\beta) = -\frac{i}{2}\frac{\log(\beta+\lambda_R)}{(\lambda_R^2+\lambda^2)^{1/2}} + \mathbf{\Omega}_{0-}(\beta), \quad \beta \to -\lambda_R, \quad \beta \in \mathbb{C}^-. \tag{3.24}$$

Similarly, in the case of the point $\beta = \lambda_R$ we arrive at the following relations

$$B^+(\beta) = \frac{i}{2}\frac{\log(\beta-\lambda_R)}{(\lambda_R^2+\lambda^2)^{1/2}} + \mathbf{\Omega}_{0+}(\beta), \quad \beta \to \lambda_R, \quad \beta \in \mathbb{C}^+,$$

$$B^-(\beta) = \mathbf{\Omega}_{0+}(\beta), \quad \beta \to \lambda_R, \quad \beta \in \mathbb{C}^-, \tag{3.25}$$

where $\mathbf{\Omega}_{0\pm}(\beta)$ are the functions analytic in the vicinity of the points $\pm\lambda_R$.

Substituting the representation for the matrix $\mathbf{G}_1(\beta)$ (3.11) into (3.1) in accordance with the factorization of the function ρ

$$\rho = (\beta + i|\lambda|)^{1/2}(\beta - i|\lambda|)^{1/2}$$

we obtain the splitting of the matrix $\rho^{-1}\mathbf{G}_0(\beta)$

$$\rho^{-1}\mathbf{G}_0(t) = [\mathbf{Y}_+(t)]^{-1}\mathbf{Y}_-(t), \quad t \in \Gamma, \tag{3.26}$$

where

$$\mathbf{Y}_+(\beta) = (\beta + i|\lambda|)^{1/2}[\mathbf{X}_1^+(\beta)]^{-1}\mathbf{U}^{-1}(\beta), \quad \beta \in \mathbb{C}^+ \bigcup \Gamma,$$

$$\mathbf{Y}_-(\beta) = (\beta - i|\lambda|)^{-1/2}[\mathbf{X}_1^-(\beta)]^{-1}\mathbf{T}^{-1}(\beta), \quad \beta \in \mathbb{C}^- \bigcup \Gamma.$$

The matrices $[\mathbf{X}_1^{\pm}(\beta)]^{-1}$ are analytic and not singular in $\mathbb{C}^{\pm}$ except maybe for isolated singularities at the points of the real axis $-\lambda_R$, λ_R (in the case $|\lambda| < b^{-1}d^{-1}$) or the interior points $\pm i = \lambda_R \in \mathbb{C}^{\pm}$ (in the case $|\lambda| > b^{-1}d^{-1}$). The elements of the matrix $\mathbf{U}^{-1}(\beta)$

$$\mathbf{U}^{-1}(\beta) = \begin{pmatrix} 0 & -(b+e)^{-1}\beta\rho^{-2} & -(b+e)^{-1}\lambda\rho^{-2} \\ id\delta_0^{-1} & -d^2 b\lambda\delta_0^{-1} & -d^2 b\beta\delta_0^{-1} \\ -d^2 b\rho^2\delta_0^{-1} & -id\lambda\delta_0^{-1} & -id\beta\delta_0^{-1} \end{pmatrix}$$

have simple poles in $\mathbb{C}^+$ not only at singular points of the matrix $[\mathbf{X}_1^+(\beta)]^{-1}$ but at the point $i|\lambda|$. The matrix $\mathbf{T}^{-1}(\beta)$ is a rational matrix and it has in $\mathbb{C}^-$ only one singularity at the point $-i|\lambda|$. The matrices $\mathbf{Y}_{\pm}(\beta)$ are analytic in $\mathbb{C}^{\pm}$ except maybe for the isolated singularities of the matrices $[\mathbf{X}_1^+(\beta)]^{-1}$, $\mathbf{U}^{-1}(\beta)$ and $[\mathbf{X}_1^-(\beta)]^{-1}$, $\mathbf{T}^{-1}(\beta)$ in $\mathbb{C}^{\pm}$, respectively. Additionally, it should be verified that the orders of the columns of the matrices $\mathbf{Y}_{\pm}(\beta)$ at infinity are $n_j^{\pm}+1/2$, $j = 1, 2, 3$, where $n_j^{\pm}$ are integers.

Let $|\lambda| < b^{-1}d^{-1}$. Investigate the behaviour of the matrices

$$\mathbf{Y}_+(\beta) = (\beta+i|\lambda|)^{1/2}\begin{bmatrix} 0 & -\dfrac{\beta}{(b+e)\rho^2} & \dfrac{\lambda}{(b+e)\rho^2} \\ \dfrac{\Sigma_1^+}{\delta_0\Lambda^+} & \dfrac{\lambda\Sigma_2^+}{\delta_0\Lambda^+} & \dfrac{\beta\Sigma_2^+}{\delta_0\Lambda^+} \\ \dfrac{\rho^2\Sigma_2^+}{\delta_0\Lambda^+} & -\dfrac{\lambda\Sigma_1^+}{\delta_0\Lambda^+} & -\dfrac{\beta\Sigma_1^+}{\delta_0\Lambda^+} \end{bmatrix},$$

$$\mathbf{Y}_-(\beta) = (\beta-i|\lambda|)^{-1/2}\begin{bmatrix} 0 & \dfrac{\beta}{\rho^2} & -\dfrac{\lambda}{\rho^2} \\ -\dfrac{\sin(\rho B^-)}{\rho\Lambda^-} & \dfrac{\lambda\cos(\rho B^-)}{\rho^2\Lambda^-} & \dfrac{\beta\cos(\rho B^-)}{\rho^2\Lambda^-} \\ \dfrac{\cos(\rho B^-)}{\Lambda^-} & \dfrac{\lambda\sin(\rho B^-)}{\rho\Lambda^-} & \dfrac{\beta\sin(\rho B^-)}{\rho\Lambda^-} \end{bmatrix},$$

$$\Sigma_1^+ = di[\cos(\rho B^+) - ibd\rho\sin(\rho B^+)],$$

$$\Sigma_2^+ = di\left[\frac{1}{\rho}\sin(\rho B^+) + ibd\cos(\rho B^+)\right] \tag{3.27}$$

in the vicinities of the points $-\lambda_R$, λ_R. According to the relations (3.24) the functions $\cos(\rho B^+)$, $\sin(\rho B^+)$ are analytic as $\beta \to -\lambda_R$, $\beta \in \mathbb{C}^+$. At the other point $\beta = \lambda_R$ taking into account that

$$bd\rho = 1 + O(\beta - \lambda_R), \quad \beta \to \lambda_R \tag{3.28}$$

we arrive at the following asymptotics

$$\cos(\rho B^+) = \frac{1}{2}(\beta-\lambda_R)^{-1/2} + O\{(\beta-\lambda_R)^{1/2}\},$$

$$\sin(\rho B^+) = -\frac{i}{2}(\beta-\lambda_R)^{-1/2} + O\{(\beta-\lambda_R)^{1/2}\},$$

$$\Sigma_1^+ = O\{(\beta-\lambda_R)^{1/2}\}, \quad \Sigma_2^+ = O\{(\beta-\lambda_R)^{1/2}\}, \quad \beta \to \lambda_R, \quad \beta \in \mathbb{C}^+. \tag{3.29}$$

These asymptotics, together with (3.5) and

$$\delta_0 = b^2 d^2 (\beta-\lambda_R)(\beta+\lambda_R)$$

give immediately

$$\frac{\Sigma_j^+}{\delta_0\Lambda^+} = O(1) \quad (j=1,2), \quad \beta\to\lambda_R, \quad \beta\in\mathbb{C}^+,$$

$$\frac{\Sigma_j^+}{\delta_0\Lambda^+} = O\left(\frac{1}{\beta+\lambda_R}\right) \quad (j=1,2), \quad \beta\to-\lambda_R, \quad \beta\in\mathbb{C}^+.$$

Analysis of the elements of the matrix $\mathbf{Y}_-(\beta)$ yields the same asymptotics in $\mathbb{C}^-$

$$\frac{\cos(\rho B^-)}{\Lambda^-} = O(1), \quad \frac{\sin(\rho B^-)}{\Lambda^-} = O(1), \quad \beta\to\lambda_R, \quad \beta\in\mathbb{C}^-,$$

$$\frac{\cos(\rho B^-)}{\Lambda^-} = O\left(\frac{1}{\beta+\lambda_R}\right), \quad \frac{\sin(\rho B^-)}{\Lambda^-} = O\left(\frac{1}{\beta+\lambda_R}\right), \quad \beta\to-\lambda_R, \quad \beta\in\mathbb{C}^-.$$

Consequently, for the case of the real roots $\pm\lambda_R$ of the function δ_0 both matrices $\mathbf{Y}_\pm(\beta)$ are bounded in the vicinity of the point λ_R and have a pole at another point $-\lambda_R$

$$\mathbf{Y}_\pm(\beta) = O(\mathbf{C}), \quad \beta\to\lambda_R, \quad \beta\in\mathbb{C}^\pm,$$

$$\mathbf{Y}_\pm(\beta) = \begin{pmatrix} 0 & O(1) & O(1) \\ O\{(\beta+\lambda_R)^{-1}\} & O\{(\beta+\lambda_R)^{-1}\} & O\{(\beta+\lambda_R)^{-1}\} \\ O\{(\beta+\lambda_R)^{-1}\} & O\{(\beta+\lambda_R)^{-1}\} & O\{(\beta+\lambda_R)^{-1}\} \end{pmatrix},$$

$$\beta\to-\lambda_R, \quad \beta\in\mathbb{C}^\pm, \quad (3.30)$$

where $\mathbf{C}$ is a bounded matrix.

In the case of complex roots

$$\delta_0 = b^2d^2(\beta+i\lambda_R)(\beta-i\lambda_R), \quad |\lambda| > b^{-1}d^{-1},$$

it is clear that

$$\mathbf{Y}_+(\beta) = \begin{pmatrix} 0 & O(1) & O(1) \\ O\{(\beta-i\lambda_R)^{-1}\} & O\{(\beta-i\lambda_R)^{-1}\} & O\{(\beta-i\lambda_R)^{-1}\} \\ O\{(\beta-i\lambda_R)^{-1}\} & O\{(\beta-i\lambda_R)^{-1}\} & O\{(\beta-i\lambda_R)^{-1}\} \end{pmatrix}, \quad \beta\to i\lambda_R(\in\mathbb{C}^+),$$

$$\mathbf{Y}_-(\beta) = O(\mathbf{C}), \quad \beta\to-i\lambda_R(\in\mathbb{C}^-). \quad (3.31)$$

Analysis of the behaviour of the matrices $\mathbf{U}^{-1}(\beta)$ as $\beta\to i|\lambda|$ and $\mathbf{T}^{-1}(\beta)$ as $\beta\to-i|\lambda|$ gives us

$$\mathbf{Y}_+(\beta) = \begin{bmatrix} 0 & -\dfrac{(\beta-i|\lambda|)^{-1}}{2(b+e)}+O(1) & \dfrac{\operatorname{sgn}\lambda(\beta-i|\lambda|)^{-1}}{2i(b+e)}+O(1) \\ O(1) & O(1) & O(1) \\ O(\beta-i|\lambda|) & O(1) & O(1) \end{bmatrix},$$

$$\beta\to i|\lambda|,$$

$$\mathbf{Y}_-(\beta) = \begin{bmatrix} 0 & \dfrac{(\beta+i|\lambda|)^{-1}}{2}+O(1) & \dfrac{\operatorname{sgn}\lambda(\beta+i|\lambda|)^{-1}}{2i}+O(1) \\ O(1) & -\dfrac{\operatorname{sgn}\lambda(\beta+i|\lambda|)^{-1}}{2iM_-}+O(1) & \dfrac{(\beta+i|\lambda|)^{-1}}{2M_-}+O(1) \\ O(1) & O(1) & O(1) \end{bmatrix},$$

$$\beta \to -\mathrm{i}|\lambda|, \quad (3.32)$$

where

$$M_- = \frac{1}{M_+},$$

$$M_+ = d_0(2|\lambda|)^{1/2}\begin{cases}(|\lambda|+i\lambda_R)^{-1/2}, & |\lambda| < b^{-1}d^{-1} \\ (|\lambda|+\lambda_R)^{-1/2}, & |\lambda| > b^{-1}d^{-1}\end{cases},$$

$$0 < \arg(|\lambda|+i\lambda_R) < \frac{\pi}{2}. \quad (3.33)$$

Thus, some of the elements of the matrices $\mathbf{Y}_\pm(\beta)$ have a pole at the points $\pm i\lambda$, others are bounded or have a zero.

To establish the behaviour of the matrices $\mathbf{Y}_\pm(\beta)$ at infinity we take into account the relations (3.8) and (3.16) and introduce the notations

$$c_\beta^\pm = \cos\left(\varepsilon\log\beta \mp \frac{\pi\varepsilon}{2}i + \varepsilon\log\frac{2}{|\lambda|} + B_0\right),$$

$$s_\beta^\pm = \sin\left(\varepsilon\log\beta \mp \frac{\pi\varepsilon}{2}i + \varepsilon\log\frac{2}{|\lambda|} + B_0\right),$$

$$c_\beta^\pm = O(1), \quad s_\beta^\pm = O(1), \quad \beta\to\infty, \quad \beta\in\mathbb{C}^\pm,$$

$$\varepsilon = \frac{d_1}{2\pi} = \frac{1}{2\pi}\log\frac{b+d}{b-d} = \frac{1}{2\pi}\log\frac{\kappa_+ + \sigma}{1+\kappa_-\sigma}, \quad \sigma = \frac{G_+}{G_-}. \quad (3.34)$$

$\kappa_\pm = 3-4\nu_\pm$ are Muskhelishvili's constants. As a result, we obtain that the elements of the matrices $\beta^{\mp 1/2}\mathbf{Y}_\pm(\beta)$ have the following behaviour at infinity

$$\mathbf{Y}_+(\beta) = \beta^{1/2}\begin{bmatrix} 0 & O\left(\dfrac{1}{\beta}\right) & \left(\dfrac{1}{\beta^2}\right) \\ O\left(\dfrac{1}{\beta}\right) & O\left(\dfrac{1}{\beta^2}\right) & O\left(\dfrac{1}{\beta}\right) \\ -\dfrac{c_\beta^+}{d_0 b}+O\left(\dfrac{1}{\beta}\right) & O\left(\dfrac{1}{\beta}\right) & -\dfrac{s_\beta^+}{d_0 b}+O\left(\dfrac{1}{\beta}\right) \end{bmatrix},$$

$$\beta\to\infty, \quad \beta\in\mathbb{C}^+,$$

$$\mathbf{Y}_-(\beta) = \beta^{-1/2} \begin{bmatrix} 0 & O\left(\frac{1}{\beta}\right) & \left(\frac{1}{\beta^2}\right) \\ O\left(\frac{1}{\beta}\right) & O\left(\frac{1}{\beta^2}\right) & O\left(\frac{1}{\beta}\right) \\ d_0 c_\beta^- + O\left(\frac{1}{\beta}\right) & O\left(\frac{1}{\beta}\right) & d_0 s_\beta^- + O\left(\frac{1}{\beta}\right) \end{bmatrix}, \quad (3.35)$$

$$\beta \to \infty, \beta \in \mathbb{C}^-.$$

Substituting into the homogeneous boundary condition (2.8) the matrices $\mathbf{Y}_\pm(\beta)$ we arrive after simplification at the identity

$$[\mathbf{Y}_+(t)]^{-1} = \frac{1}{\rho}\mathbf{G}_0(t)[\mathbf{Y}_-(t)]^{-1}, \quad t \in \Gamma.$$

On the other hand, the matrices $[\mathbf{Y}_\pm(\beta)]^{-1}$ are analytic and not singular in $\mathbb{C}^\pm$ except for a finite number of isolated singularities and have the asymptotics (3.35) at infinity. There are two ways to solve the non-homogeneous Wiener–Hopf problem (2.8). The first one is use of the Gakhov algorithm (Gakhov, 1952) of the transformation of the matrices $[\mathbf{Y}_\pm(\beta)]^{-1}$ to the canonical form. The second way admits a finite number of poles or points where the matrices are singular. This approach requires the matrices $\mathbf{Y}_+(\beta)$, $\mathbf{Y}_-(\beta)$ only, it is based on the direct application of the generalized Liouville's theorem and leads to determination of a finite number of constants. The steps in the first algorithm are as follows:

(1) Write the boundary condition in the form

$$\mathbf{Y}_+(t)\mathbf{X}^+(t) = \mathbf{Y}_-(t)\mathbf{X}^-(t), \quad t \in \Gamma.$$

(2) Multiply it on the left by a rational matrix $\mathbf{Z}(t)$ so that $\mathbf{Z}(\beta)\mathbf{Y}_+(\beta)$ and $\mathbf{Z}(\beta)\mathbf{Y}_-(\beta)$ regular and not singular in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. The matrix $\mathbf{Z}(\beta)$ can be constructed with the help of the Gakhov procedure (Gakhov, 1952).
(3) Find the canonical matrices of solutions

$$\mathbf{X}_+(\beta) = [\mathbf{Z}(\beta)\mathbf{Y}_+(\beta)]^{-1}, \quad \mathbf{X}_-(\beta) = [\mathbf{Z}(\beta)\mathbf{Y}_-(\beta)]^{-1}.$$

Note, that the Chebotarev approach (Chebotarev, 1956) cannot be applied in our case because more than one eigenvalue of the matrix $(\beta - \alpha)\mathbf{Y}_\pm(\beta)$ (α is a pole that should be eliminated) is equal to zero.

We choose the second direct method of the solution and do not claim advantages in comparison with those involving the construction of the canonical matrix in the general case. It should be said that both methods give the same results. Nevertheless, in our case the second procedure is more straightforward and leads to the exact solution of the boundary value problem directly.

4. Solution of the matrix Wiener–Hopf problem

Substituting the representation for the matrix $\mathbf{G}(\beta)$ (3.26) into the boundary condition (2.8) we obtain

$$\mathbf{Y}_+(t)\mathbf{\Phi}^+(t)-\mathbf{H}^+(t) = \mathbf{Y}_-(t)\mathbf{\Phi}^-(t)-\mathbf{H}^-(t), \quad t\in\Gamma, \tag{4.1}$$

where

$$\mathbf{H}^\pm(\beta) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{Y}_-(t)\mathbf{F}^+(t)\,\mathrm{d}t}{t-\beta}, \quad \beta\in\mathbb{C}^\pm. \tag{4.2}$$

As before, in the case $|\lambda| < b^{-1}d^{-1}$ the contour Γ is defined in such a way

$$\Gamma = \{-\infty < \Re\beta < -\lambda_R, \Im\beta = -0\} \bigcup \{-\lambda_R < \Re\beta < \lambda_R, \Im\beta = 0\} \bigcup \{\lambda_R < \Re\beta < \infty, \Im\beta = +0\}.$$

If $|\lambda| > b^{-1}d^{-1}$ the contour Γ coincides with the real axis

$$\Gamma = \{-\infty < \Re\beta < \infty, \Im\beta = 0\}.$$

Let us introduce the notations for the left- and right-hand-sides of the equality (4.1)

$$\mathbf{E}_+(\beta) = \mathbf{Y}_+(\beta)\mathbf{\Phi}^+(\beta)-\mathbf{H}^+(\beta),$$
$$\mathbf{E}_-(\beta) = \mathbf{Y}_-(\beta)\mathbf{\Phi}^-(\beta)-\mathbf{H}^-(\beta). \tag{4.3}$$

It follows immediately from the asymptotics of the matrices $\mathbf{Y}_\pm(\beta)$ (3.30)–(3.32) and (3.35) that the vectors $\mathbf{E}_\pm(\beta)$ admit the following estimates

$$\mathbf{E}_\pm(\beta) = O(\mathbf{1}), \quad \beta\to\lambda_R, \quad \beta\in\mathbb{C}^\pm,$$
$$\mathbf{E}_\pm(\beta) = \begin{pmatrix} O(1) \\ O\{(\beta+\lambda_R)^{-1}\} \\ O\{(\beta+\lambda_R)^{-1}\} \end{pmatrix}, \quad \beta\to-\lambda_R, \quad \beta\in\mathbb{C}^\pm, \tag{4.4}$$

where $\mathbf{1}$ is the unit vector. Formulae (4.4) are valid for the case of the real roots of the function δ_0. If the roots are complex then we have

$$\mathbf{E}_+(\beta) = \begin{pmatrix} O(1) \\ O\{(\beta-i\lambda_R)^{-1}\} \\ O\{(\beta-i\lambda_R)^{-1}\} \end{pmatrix}, \quad \beta\to i\lambda_R(\in\mathbb{C}^+),$$
$$\mathbf{E}_-(\beta) = O(\mathbf{1}), \quad \beta\to -i\lambda_R(\in\mathbb{C}^-). \tag{4.5}$$

At the points $\pm i|\lambda|$ the vectors $\mathbf{E}_\pm(\beta)$ have simple poles

$$\mathbf{E}_+(\beta) = \begin{pmatrix} O\{(\beta - i|\lambda|)^{-1}\} \\ O(1) \\ O(1) \end{pmatrix}, \quad \beta \to i|\lambda|,$$

$$\mathbf{E}_-(\beta) = \begin{pmatrix} O\{(\beta + i|\lambda|)^{-1}\} \\ O\{(\beta + i|\lambda|)^{-1}\} \\ O(1) \end{pmatrix}, \quad \beta \to -i|\lambda|. \tag{4.6}$$

Using the estimates for the vectors $\mathbf{H}^\pm(\beta)$

$$|\mathbf{H}^\pm(\beta)| < \frac{1}{|\beta|}\mathbf{C}_0, \quad \beta \to \infty, \quad \beta \in \mathbb{C}^\pm$$

where $\mathbf{C}_0$ is a vector with positive constant components and taking into account that in view of the Abelian theorems (Noble, 1988) as $\beta \to \infty$

$$\mathbf{\Phi}^+(\beta) = O(\beta^{-3/2}\mathbf{1}), \quad \beta \in \mathbb{C}^+, \quad \mathbf{\Phi}^-(\beta) = O(\beta^{-1/2}\mathbf{1}), \quad \beta \in \mathbb{C}^-, \tag{4.7}$$

we find that the vectors $\mathbf{E}_+(\beta)$, $\mathbf{E}_-(\beta)$ vanish at infinity

$$\mathbf{E}_\pm(\beta) = O(\beta^{-1}\mathbf{1}), \quad \beta \to \infty, \quad \beta \in \mathbb{C}^\pm. \tag{4.8}$$

It follows from the asymptotics of $\mathbf{E}_\pm(\beta)$ (4.4) or (4.5) as $\beta \to \pm\lambda_R$ for the case $|\lambda| < b^{-1}d^{-1}$ or as $\beta \to \pm i\lambda_R$, if $|\lambda| > b^{-1}d^{-1}$, as well as (4.6) as $\beta \to \pm i|\lambda|$ and the behaviour at infinity (4.8) that according to the generalized Liouville's theorem the vectors $\mathbf{E}_+(\beta)$, $\mathbf{E}_-(\beta)$ constitute the analytic continuation of one another and are a rational vector

$$\mathbf{E}_+(\beta) = \mathbf{E}_-(\beta) = \begin{Bmatrix} \dfrac{C_1}{\beta - i|\lambda|} + \dfrac{C_2}{\beta + i|\lambda|} \\ \dfrac{C_3}{\beta + i|\lambda|} + \dfrac{C_4}{\beta - a} \\ \dfrac{C_5}{\beta - a} \end{Bmatrix}, \quad \beta \in \mathbb{C},$$

where $\mathbb{C}$ is a complex plane, $C_1, \ldots, C_5$ are arbitrary constants; $a = -\lambda_R$ in the case $|\lambda| < b^{-1}d^{-1}$ and $a = i\lambda_R$, $|\lambda| > b^{-1}d^{-1}$. Hence, from (4.3) we have the solution of the matrix Wiener–Hopf problem

$$\mathbf{\Phi}^\pm(\beta) = \mathbf{Y}_\pm^{-1}(\beta) \begin{Bmatrix} \dfrac{C_1}{\beta - i|\lambda|} + \dfrac{C_2}{\beta + i|\lambda|} + H_1^\pm(\beta) \\ \dfrac{C_3}{\beta + i|\lambda|} + \dfrac{C_4}{\beta - a} + H_2^\pm(\beta) \\ \dfrac{C_5}{\beta - a} + H_3^\pm(\beta) \end{Bmatrix}, \quad \beta \in \mathbb{C}^\pm. \tag{4.9}$$

In order to find the constants, we check the properties of the vectors $\mathbf{\Phi}^{+}(\beta)$, $\mathbf{\Phi}^{-}(\beta)$ at the points $\beta = \pm i|\lambda|$, $\pm\lambda_R$, $\pm i\lambda_R$ and at infinity. Let us write down the expressions of the inverse matrices

$$[\mathbf{Y}_{+}(\beta)]^{-1} = (\beta+i|\lambda|)^{-1/2} \begin{bmatrix} 0 & \frac{1}{d^2}\Lambda^{+}\Sigma_1^{+} & \frac{1}{d^2}\Lambda^{+}\Sigma_2^{+} \\ -(b+e)\beta & \frac{\lambda}{d^2}\Lambda^{+}\Sigma_2^{+} & -\frac{\lambda}{d^2\rho^2}\Lambda^{+}\Sigma_1^{+} \\ (b+e)\lambda & \frac{\beta}{d^2}\Lambda^{+}\Sigma_2^{+} & -\frac{\beta}{d^2\rho^2}\Lambda^{+}\Sigma_1^{+} \end{bmatrix}, \quad \beta\in\mathbb{C}^{+}, \qquad (4.10)$$

$$[\mathbf{Y}_{-}(\beta)]^{-1} = (\beta-i|\lambda|)^{1/2} \begin{bmatrix} 0 & -\rho\Lambda^{-}\sin(\rho B^{-}) & \Lambda^{-}\cos(\rho B^{-}) \\ \beta & \lambda\Lambda^{-}\cos(\rho B^{-}) & \frac{\lambda}{\rho}\Lambda^{-}\sin(\rho B^{-}) \\ -\lambda & \beta\Lambda^{-}\cos(\rho B^{-}) & \frac{\beta}{\rho}\Lambda^{-}\sin(\rho B^{-}) \end{bmatrix}, \quad \beta\in\mathbb{C}^{-}. \qquad (4.11)$$

Analysing the behaviour of these matrices at the points $\beta = \pm i|\lambda|$ we get

$$[\mathbf{Y}_{+}(\beta)]^{-1} \sim (2i|\lambda|)^{-1/2} \begin{bmatrix} 0 & \frac{i}{d}M_{+} & \lambda_1 M_{+} \\ -(b+e)i|\lambda| & \lambda\lambda_1 M_{+} & -\frac{\operatorname{sgn}\lambda}{2d}M_{+}(\beta-i|\lambda|)^{-1} \\ (b+e)\lambda & i|\lambda|\lambda_1 M_{+} & -\frac{i}{2d}M_{+}(\beta-i|\lambda|)^{-1} \end{bmatrix}, \quad \beta\to i|\lambda|,$$

$$[\mathbf{Y}_{-}(\beta)]^{-1} \sim (-2i|\lambda|)^{1/2} \begin{pmatrix} 0 & -2i|\lambda|\lambda_2 M_{-}(\beta+i|\lambda|) & M_{-} \\ -i|\lambda| & \lambda M_{-} & -\lambda\lambda_2 M_{-} \\ -\lambda & -i|\lambda|M_{-} & i|\lambda|\lambda_2 M_{-} \end{pmatrix}, \quad \beta\to -i|\lambda|, \qquad (4.12)$$

where $M_{\pm}$ are defined in (3.33). The following formulae will not contain the constants λ_1 and λ_2 and we omit the expressions for them. The asymptotics of the solution as $\beta\to\pm i|\lambda|$ may be found from formulae (4.9) and (4.12)

$$\mathbf{\Phi}^+(\beta) \sim (2i|\lambda|)^{-1/2} \begin{pmatrix} O(1) \\ -\dfrac{i\,\mathrm{sgn}\,\lambda}{\beta - i|\lambda|} L_\lambda + O(1) \\ \dfrac{1}{\beta - i|\lambda|} L_\lambda + O(1) \end{pmatrix}, \quad \beta \to i|\lambda|,$$

$$\mathbf{\Phi}^-(\beta) \sim (-2i|\lambda|)^{1/2} \begin{pmatrix} O(1) \\ -\dfrac{i|\lambda|}{\beta + i|\lambda|}(C_2 + i\,\mathrm{sgn}\,\lambda M_- C_3) + O(1) \\ -\dfrac{\lambda}{\beta + i|\lambda|}(C_2 + i\,\mathrm{sgn}\,\lambda M_- C_3) + O(1) \end{pmatrix}, \quad \beta \to -i|\lambda|,$$

where

$$L_\lambda = (b+e)\lambda C_1 - \frac{iM_+}{2d}\left[\frac{C_5}{i|\lambda| - a} + H_3^+(i|\lambda|)\right].$$

In order for the vectors $\mathbf{\Phi}^\pm(\beta)$ to be analytic at the points $\pm i|\lambda|$, it is necessary and sufficient that the following two conditions be satisfied

$$L_\lambda = 0, \quad C_2 + i\,\mathrm{sgn}\,\lambda M_- C_3 = 0.$$

There is a need for the asymptotics of the matrices $[\mathbf{Y}_\pm(\beta)]^{-1}$ at infinity, not only in order to check the behaviour of the vectors $\mathbf{\Phi}^\pm(\beta)$ but for the subsequent computation of the stress intensity factors

$$[\mathbf{Y}_+(\beta)]^{-1} \sim \beta^{-1/2} \begin{pmatrix} 0 & bd_0 s_\beta^+ \beta & -bd_0 c_\beta^+ \\ -(b+e)\beta & -bd_0 c_\beta^+ \lambda & -bd_0 s_\beta^+ \lambda\beta^{-1} \\ (b+e)\lambda & -bd_0 c_\beta^+ \beta & -bd_0 s_\beta^+ \end{pmatrix}, \quad \beta \to \infty, \quad \beta \in \mathbb{C}^+,$$

$$[\mathbf{Y}_-(\beta)]^{-1} \sim \beta^{1/2} \begin{pmatrix} 0 & -\dfrac{\beta}{d_0} s_\beta^- & \dfrac{1}{d_0} c_\beta^- \\ \beta & \dfrac{\lambda}{d_0} c_\beta^- & \dfrac{\lambda}{\beta d_0} s_\beta^- \\ -\lambda & \dfrac{\beta}{d_0} c_\beta^- & \dfrac{1}{d_0} s_\beta^- \end{pmatrix}, \quad \beta \to \infty, \quad \beta \in \mathbb{C}^-. \quad (4.13)$$

On account of the following representation

$$\mathbf{H}^\pm(\beta) = \beta^{-1}\mathbf{H}^0 + o(\beta^{-1}\mathbf{1}), \quad \beta \to \infty, \quad \beta \in \mathbb{C}^\pm,$$

$$\mathbf{H}^0 = \begin{pmatrix} H_1^0 \\ H_2^0 \\ H_3^0 \end{pmatrix} = -\frac{1}{2\pi i}\int_\Gamma \mathbf{Y}_-(\beta)\mathbf{F}^+(\beta)\,\mathrm{d}\beta \quad (4.14)$$

that follows from (4.2), comparing formulae (4.9) for the solution and (4.13) for the matrices $[\mathbf{Y}_\pm(\beta)]^{-1}$ at infinity we find that the vectors $\mathbf{\Phi}^\pm(\beta)$ possess the asymptotics at infinity

$$\mathbf{\Phi}^+(\beta) = \beta^{-1/2}\begin{pmatrix} bd_0 s_\beta^+(C_3+C_4+H_2^0) \\ -(b+e)(C_1+C_2+H_1^0) \\ -bd_0 s_\beta^+(C_3+C_4+H_2^0)\end{pmatrix} + O(\beta^{-3/2}\mathbf{1}), \quad \beta\to\infty, \quad \beta\in\mathbb{C}^+,$$

$$\mathbf{\Phi}^-(\beta) = \beta^{1/2}\begin{pmatrix} -d_0^{-1}s_\beta^-(C_3+C_4+H_2^0) \\ C_1+C_2+H_1^0 \\ d_0^{-1}s_\beta^-(C_3+C_4+H_2^0)\end{pmatrix} + O(\beta^{-1/2}\mathbf{1}), \quad \beta\to\infty, \quad \beta\in\mathbb{C}^-.$$

The conditions (4.7) at infinity will apply if and only if the conditions

$$C_1+C_2+H_1^0 = 0,$$
$$C_3+C_4+H_2^0 = 0 \tag{4.15}$$

hold. The final step in the analysis is to study the behaviour of the solution at the point a. If $|\lambda| < b^{-1}d^{-1}$ then $a = -\lambda_R$. Formulae (4.9) and (4.10) give immediately

$$\mathbf{\Phi}^+(\beta) = \frac{(-\lambda_R+i|\lambda|)^{-1/2}\Lambda^+}{(\beta+\lambda_R)d^2}\begin{pmatrix} C_4\Sigma_1^+ + C_5\Sigma_2^+ \\ \lambda(C_4\Sigma_2^+ - \rho^{-2}C_5\Sigma_1^+) \\ \beta(C_4\Sigma_2^+ - \rho^{-2}C_5\Sigma_1^+)\end{pmatrix} + O(\mathbf{1}),$$

$$\beta\to -\lambda_R, \quad \beta\in\mathbb{C}^+,$$

$$\mathbf{\Phi}^-(\beta) = \frac{(-\lambda_R-i|\lambda|)^{-1/2}\Lambda^-}{\beta+\lambda_R}\begin{pmatrix} -C_4\rho\sin(\rho B^-)+C_5\cos(\rho B^-) \\ \lambda[C_4\cos(\rho B^-)+\rho^{-1}C_5\sin(\rho B^-)] \\ \beta[C_4\cos(\rho B^-)+\rho^{-1}C_5\sin(\rho B^-)]\end{pmatrix} + O(\mathbf{1}),$$

$$\beta\to -\lambda_R, \quad \beta\in\mathbb{C}^-.$$

The vectors $\mathbf{\Phi}^+(\beta)$, $\mathbf{\Phi}^-(\beta)$ are analytic at the point $\beta = -\lambda_R$ if the following four conditions are satisfied

$$C_4\Sigma_1^+ + C_5\Sigma_2^+ = 0, \quad \beta = -\lambda_R,$$

$$C_4\Sigma_2^+ - \frac{1}{\rho^2}C_5\Sigma_1^+ = 0, \quad \beta = -\lambda_R,$$

$$\lim_{\beta\to-\lambda_R}\Lambda^-[-C_4\rho\sin(\rho B^-)+C_5\cos(\rho B^-)] = 0,$$

$$\lim_{\beta\to-\lambda_R}\Lambda^-\left[C_4\cos(\rho B^-)+\frac{1}{\rho}C_5\sin(\rho B^-)\right] = 0. \tag{4.16}$$

Let us prove that the rank of this system is equal to one. According to the relations (3.6) and (3.24) the last two equations lead to the following equalities

$$-i\rho C_4+C_5=0,$$

$$C_4+\frac{i}{\rho}C_5=0,\quad \rho=\frac{1}{bd}\quad \text{as}\,\beta=-\lambda_R,$$

which are linearly dependent. Since

$$(\Sigma_1^+)^2+\rho^2(\Sigma_2^+)^2=d^2\delta_0$$

and $\delta_0=0$ as $\beta=-\lambda_R$, then the first two equations from (4.16) are also linearly dependent. On the other hand, since

$$\Sigma_1^+ + i\rho\Sigma_2^+ = id[\cos(\rho B^+)+i\sin(\rho B^+)](1-bd\rho)$$

and $1-bd\rho=0$ as $\beta=-\lambda_R$ then these equations give

$$C_5=\frac{i}{bd}C_4. \tag{4.17}$$

Observe that the vectors $\mathbf{\Phi}^{\pm}(\beta)$ are analytic at the point $\beta=\lambda_R$ without additional conditions. In the case $|\lambda|>b^{-1}d^{-1}$, $a=i\lambda_R$ and obviously, it suffices to study the vector $\mathbf{\Phi}^+(\beta)$ at this point. The asymptotics of the vector $\mathbf{\Phi}^+(\beta)$

$$\mathbf{\Phi}^+(\beta)=\frac{(i\lambda_R+i|\lambda|)^{-1/2}\Lambda^+}{(\beta-i\lambda_R)d^2}\begin{pmatrix} C_4\Sigma_1^+ +\rho^{-1}C_5\Sigma_2^+ \\ \lambda\rho^{-1}(C_4\Sigma_2^+-\rho^{-1}C_5\Sigma_1^+) \\ \beta\rho^{-1}(C_4\Sigma_2^+-\rho^{-1}C_5\Sigma_1^+)\end{pmatrix}+O(\mathbf{1}),\quad \beta\to i\lambda_R$$

lead to two conditions under which this vector will be analytic in the vicinity of the point $\beta=i\lambda_R$

$$C_4\Sigma_1^+ +C_5\Sigma_2^+ =0,\quad \beta=i\lambda_R,$$

$$C_4\Sigma_2^- -\frac{1}{\rho^2}C_5\Sigma_1^+ =0,\quad \rho=\frac{1}{bd},\quad \beta=i\lambda_R.$$

These equations are linearly dependent and give the same condition (4.17). Clearly, the vector $\mathbf{\Phi}^-(\beta)$ is analytic at the point $\beta=-i\lambda_R$.

Thus, the vectors $\mathbf{\Phi}^{\pm}(\beta)$ are analytic in $\mathbb{C}^{\pm}$ if and only if the constants C_j $(j=1,\ldots,5)$ satisfy the following system of algebraic equations

$$\begin{aligned} 2d(b+e)\lambda C_1-iM_+[(i|\lambda|-a)^{-1}C_5+H_3^+(i|\lambda|)]&=0\\ C_2+i\,\mathrm{sgn}\,\lambda M_-C_3&=0\\ C_1+C_2+H_1^0&=0\\ C_3+C_4+H_2^0&=0\\ iC_4-dbC_5&=0.\end{aligned}$$

Solution of this system leads to the expressions for the constants

$$C_1 = \frac{-i\,\mathrm{sgn}\,\lambda M_+ H_1^0 + H_2^0 - bd(|\lambda| + ia)H_3^+(i|\lambda|)}{2ibd^2(b+e)\lambda(|\lambda|+ia)M_+^{-1} + iM_+\,\mathrm{sgn}\,\lambda},$$

$$C_2 = -C_1 - H_1^0,$$

$$C_3 = -i\,\mathrm{sgn}\,\lambda M_+(C_1 + H_1^0),$$

$$C_4 = i\,\mathrm{sgn}\,\lambda M_+(C_1 + H_1^0) - H_2^0,$$

$$C_5 = \frac{i}{bd}[i\,\mathrm{sgn}\,\lambda M_+(C_1 + H_1^0) - H_2^0]. \tag{4.18}$$

The definition of these constants completes the exact solution of the 3×3 matrix Wiener–Hopf problem (2.8).

5. Physical quantities

By using the inverse Fourier transform we find the tractions across the interface surface and discontinuities of the displacements on the crack

$$\begin{pmatrix} \sigma_y \\ \tau_{yz} \\ \tau_{xy} \end{pmatrix}(x,0,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i(\beta x + \lambda z)}\mathbf{\Phi}^-(\beta)\,\mathrm{d}\lambda\,\mathrm{d}\beta, \quad x<0, \quad |z|<\infty, \tag{5.1}$$

$$\begin{pmatrix} [v] \\ [w] \\ [u] \end{pmatrix}(x,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i(\beta x + \lambda z)}\mathbf{\Phi}^+(\beta)\,\mathrm{d}\lambda\,\mathrm{d}\beta, \quad x>0, \quad |z|<\infty. \tag{5.2}$$

In order to compute the stress intensity factors, we study the behaviour of the vector $\mathbf{\Phi}^-(\beta)$ at infinity with great care. Let

$$H_j^1 = \lim_{\beta\to\infty}[\beta^2 H_j^-(\beta) - \beta H_j^0] \quad (j = 1,2),$$

where H_j^0 has been written in (4.14). Then we have

$$H_j^-(\beta) = \frac{1}{\beta}H_j^0 + \frac{1}{\beta^2}H_j^1 + o\left(\frac{1}{\beta^2}\right), \quad (j=1,2) \quad \beta\to\infty, \quad \beta\in\mathbb{C}^-,$$

where

$$H_j^1 = -\frac{1}{2\pi i}\int_\Gamma h_j(\beta)\,\mathrm{d}\beta,$$

$$h_1(\beta) = \frac{\beta[\beta F_2^+(\beta) - \lambda F_3^+(\beta)]}{(\beta - i|\lambda|)^{1/2}\rho^2},$$

$$h_2(\beta) = \frac{\beta\{-\rho \sin(\rho B^-)F_1^+(\beta) + \cos(\rho B^-)[\lambda F_2^+(\beta) + \beta F_3^+(\beta)]\}}{(\beta - i|\lambda|)^{1/2}\rho^2\Lambda^-(\beta)}. \tag{5.3}$$

Let $p_j(x,z) \in L_1(0,\infty)$ as a function on x and $|p_j(x,z)| \leqslant C^{(0)}(z)x^{\alpha-1/2}$, $x \to +0$, $|z| < \infty, \alpha > 0$, $C^{(0)}(z) \in L_1(-\infty,\infty)$. Then the Fourier transforms of the loads $p_j(x,z)$ vanish at infinity as $|F_j^+(\beta)| \leqslant C^{(1)}\beta^{-\alpha-1/2}$, $\beta \to \infty$, $\beta \in \Gamma$, $C^{(1)} = \text{const}$ and $h_j(\beta) = O(\beta^{-\alpha-1})$, $\beta \to \infty$. Therefore, the improper integrals H_1^1 and H_2^1 exist.

Taking into account formula (4.9) and the conditions (4.15) we arrive at the asymptotics

$$\mathbf{\Phi}^-(\beta) = \begin{pmatrix} (N_1 s_\beta^- + N_2 c_\beta^-)\beta^{-1/2} \\ N_3\beta^{-1/2} \\ (-N_1 c_\beta^- + N_2 s_\beta^-)\beta^{-1/2} \end{pmatrix} + o(\beta^{-1/2}\mathbf{1}), \quad \beta \to \infty, \quad \beta \in \mathbb{C}^-, \tag{5.4}$$

where

$$N_1 = \frac{1}{d_0}(i|\lambda|C_3 - aC_4 - H_2^1),$$

$$N_2 = \frac{1}{d_0}(C_5 + H_3^0),$$

$$N_3 = i|\lambda|(C_1 - C_2) + H_1^1. \tag{5.5}$$

According to the relations (3.34) the functions $c_\beta^\pm$, $s_\beta^\pm$ admit the following representation

$$c_\beta^\pm = D_1^\pm \cos(\varepsilon \log \beta) - D_2^\pm \sin(\varepsilon \log \beta),$$

$$s_\beta^\pm = D_1^\pm \sin(\varepsilon \log \beta) + D_2^\pm \cos(\varepsilon \log \beta), \tag{5.6}$$

where

$$D_1^\pm = \cos\left(\mp\frac{\pi\varepsilon}{2}i + \varepsilon \log\frac{2}{|\lambda|} + B_0\right), \quad D_2^\pm = \sin\left(\mp\frac{\pi\varepsilon}{2}i + \varepsilon \log\frac{2}{|\lambda|} + B_0\right).$$

Formulae (5.6) can be rewritten in the form

$$c_\beta^\pm = \frac{1}{2}(e_0^{\pm 1}D\beta^{i\varepsilon} + e_0^{\mp 1}D^{-1}\beta^{-i\varepsilon}),$$

$$s_\beta^\pm = \frac{1}{2i}(e_0^{\pm 1}D\beta^{i\varepsilon} - e_0^{\mp 1}D^{-1}\beta^{-i\varepsilon}),$$

where

$$e_0 = \left(\frac{b+d}{b-d}\right)^{1/4}, \quad D = \exp = \left\{ i\left(\varepsilon \log \frac{2}{|\lambda|} + B_0\right)\right\}. \tag{5.7}$$

Imposing these relations on the asymptotic equality (5.4) results in a more convenient form

$$\mathbf{\Phi}^-(\beta) \sim \begin{bmatrix} \frac{D}{2e_0}(-iN_1+N_2)\beta^{-1/2+i\varepsilon} + \frac{e_0}{2D}(iN_1+N_2)\beta^{-1/2-i\varepsilon} \\ N_3\beta^{-1/2} \\ -i\left[\frac{D}{2e_0}(-iN_1+N_2)\beta^{-1/2+i\varepsilon} - \frac{e_0}{2D}(iN_1+N_2)\beta^{-1/2-i\varepsilon}\right] \end{bmatrix},$$

$$\beta \to \infty, \quad \beta \in \mathbb{C}^-. \tag{5.8}$$

On applying the Tauberian theorem and the inverse Fourier transform

$$\mathfrak{F}_z^{-1}[f_\lambda] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda z} f_\lambda \, \mathrm{d}\lambda$$

we find asymptotic formulae of the stresses ahead of the crack

$$\begin{pmatrix} \sigma_y \\ \tau_{xy} \\ \tau_{yz} \end{pmatrix}(x,0,z) \sim \frac{1}{\sqrt{2\pi}}\begin{pmatrix} K(z)(-x)^{-1/2-i\varepsilon} + \bar{K}(z)(-x)^{-1/2+i\varepsilon} \\ -i[K(z)(-x)^{-1/2-i\varepsilon} - \bar{K}(z)(-x)^{-1/2+i\varepsilon}] \\ K_{\mathrm{III}}(z)(-x)^{-1/2} \end{pmatrix}, \quad x \to -0, \tag{5.9}$$

where $K(z) = K_{\mathrm{I}}(z) - iK_{\mathrm{II}}(z)$ and $K_{\mathrm{III}}(z)$ are the stress intensity factors for which we have the exact representation

$$K(z) = \frac{(1+i)\sqrt{\pi}}{2\Gamma(1/2-i\varepsilon)}\mathfrak{F}_z^{-1}[D(-iN_1+N_2)], \tag{5.10}$$

$$K_{\mathrm{III}}(z) = (1+i)\mathfrak{F}_z^{-1}[N_3],$$

where D, N_1, N_2 and N_3 are the functions on λ which were defined in (5.7) and (5.5). Analysis of the formulae (5.8) and (5.9) leads to the equality

$$\frac{(1+i)}{\Gamma(1/2-i\varepsilon)}\mathfrak{F}_z^{-1}[D(-iN_1+N_2)] = \overline{\frac{(1+i)}{\Gamma(1/2+i\varepsilon)}\mathfrak{F}_z^{-1}\left[\frac{iN_1+N_2}{D}\right]}$$

that should be satisfied identically and may be useful as a test for numerical computations.

Let us prove that the inverse transforms (5.10) exist. As $\lambda \to \pm\infty$ we have the following asymptotics

$$\mathbf{Y}_-(\beta) = \begin{pmatrix} 0 & O(\lambda^{-2}) & O(\lambda^{-1}) \\ O(\lambda^{-1}) & O(\lambda^{-1}) & O(\lambda^{-2}) \\ O(1) & O(1) & O(\lambda^{-1}) \end{pmatrix}$$

and from the definitions (4.14) and (5.3) of the vectors $\mathbf{H}^0$, $\mathbf{H}^1$ it follows that

$$\mathbf{H}^m = \begin{pmatrix} O(\lambda^{-3/2}) \\ O(\lambda^{-3/2}) \\ O(\lambda^{-1/2}) \end{pmatrix} \quad (m = 0, 1),$$

$$H_3^+(i\lambda) = O(\lambda^{-3/2}), \quad \lambda \to \pm\infty.$$

The values C_j are $O(\lambda^{-3/2})$, $(j = 1, \ldots, 5)$ for $\lambda \to \pm\infty$. Taking into account formulae (5.5) we arrive at

$$N_1 = O(\lambda^{-1/2}), \quad N_2 = O(\lambda^{-3/2}), \quad N_3 = O(\lambda^{-1/2}), \quad \lambda \to \pm\infty.$$

It is clear that the integrals (5.10) exist.

Let us now consider the case when

$$p_j(x, z) = P_j \delta(z)\, \mathrm{e}^{-\gamma x}, \quad j = 1, 2, 3,$$

where P_j and γ are constants, $\gamma > 0$ and $\delta(z)$ is the δ-function. In this case

$$F_j^+(\beta) = \frac{P_j}{\gamma - i\beta}$$

and there is no need for the Cauchy type integral (4.2) for the definition of the vectors $\mathbf{H}^{\pm}(\beta)$. The solution of the matrix Wiener–Hopf problem (2.8) is obtained from (4.9) by replacing the vector $\mathbf{\Phi}^-(\beta)$ by $\mathbf{\Phi}^-(\beta) + (\gamma - i\beta)^{-1}\mathbf{P}_*$ and the vectors $\mathbf{H}^{\pm}(\beta)$ by $(\gamma - i\beta)^{-1}\mathbf{W}_\gamma \mathbf{P}_*$ where

$$\mathbf{P}_* = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \quad \mathbf{W}_\gamma = \mathbf{Y}_-(-i\gamma).$$

The expressions for the vectors $\mathbf{H}^0$ and $\mathbf{H}^1 = (H_1^1, H_2^1, H_3^1)$ are given by

$$\mathbf{H}^0 = i\mathbf{W}_\gamma \mathbf{P}_*, \quad \mathbf{H}^1 = \gamma \mathbf{W}_\gamma \mathbf{P}_*.$$

In the case of concentrated normal and shear forces

$$p_j(x, z) = P_j \delta(x - \gamma)\delta(z), \quad \gamma > 0,$$

we get $F_j^+(\beta) = P_j\, \mathrm{e}^{i\beta\gamma}$ and the improper integrals H_1^1, H_2^1 as well as H_1^0, H_2^0, H_3^0 exist.

6. Weight functions

The stress intensity factors can be represented in the following form

$$\mathbf{K}(z) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathfrak{W}(\xi, \zeta - z)\mathbf{p}(\xi, \zeta)\, \mathrm{d}\xi\, \mathrm{d}\zeta, \tag{6.1}$$

where

$$\mathbf{K}(z) = \begin{pmatrix} K_{\mathrm{I}}(z) \\ K_{\mathrm{II}}(z) \\ K_{\mathrm{III}}(z) \end{pmatrix}, \quad \mathbf{p}(\xi, \zeta) = \begin{pmatrix} p_1(\xi, \zeta) \\ p_2(\xi, \zeta) \\ p_3(\xi, \zeta) \end{pmatrix},$$

$$\mathfrak{w}(\xi, \zeta) = \begin{pmatrix} \mathfrak{W}_{11}(\xi, \zeta) & \mathfrak{W}_{12}(\xi, \zeta) & \mathfrak{W}_{13}(\xi, \zeta) \\ \mathfrak{W}_{21}(\xi, \zeta) & \mathfrak{W}_{22}(\xi, \zeta) & \mathfrak{W}_{23}(\xi, \zeta) \\ \mathfrak{W}_{31}(\xi, \zeta) & \mathfrak{W}_{32}(\xi, \zeta) & \mathfrak{W}_{33}(\xi, \zeta) \end{pmatrix}.$$

The components of the matrix $\mathfrak{W}(\xi, \zeta)$ are the weight functions for the semi-infinite interface plane crack $\mathbb{R}_2^+$ which correspond to the modes of surface loading (2.1). At first, we find the function $\mathfrak{W}_{3j}(\xi, \zeta)$ $(j = 1, 2, 3)$. By inserting the value (5.5) of N_3 into formula (5.10) and taking into account the relations in (4.18), the stress intensity factor K_{III} in terms of $H_3^+(i|\lambda|)$, H_1^0, H_2^0 and H_1^1 can be written as follows

$$K_{\mathrm{III}}(z) = (1+i)\mathfrak{F}_z^{-1}\{H_1^1 + |\lambda| W_{1\lambda}^{-1}[iW_{2\lambda}H_1^0 + H_2^0 - bd(|\lambda| + ia)H_3^+(i|\lambda|)]\},$$

$$W_{j\lambda} = bd^2(b+e)\lambda(|\lambda| + ia)M_+^{-1} - \frac{(-1)^j}{2}M_+ \operatorname{sgn}\lambda \quad (j = 1, 2). \tag{6.2}$$

By taking into account the boundary condition (3.11) we continue analytically the matrix $[\mathbf{X}_1^-(\beta)]^{-1}$ given in the domain $\mathbb{C}^-$ into the half-plane that is above the contour Γ and is cut by the line $(i|\lambda|, +i\infty)$, assuming that in this domain

$$[\mathbf{X}_1^-(\beta)]^{-1} = [\mathbf{X}_1^+(\beta)]^{-1}\mathbf{G}_1(\beta), \quad \beta \in \mathbb{C}^+ \backslash \{\Re\beta = \pm 0, i|\lambda| < \Im\beta < \infty\}. \tag{6.3}$$

According to formulae (4.14), (4.2) and (3.11) for the components of the vector $\mathbf{H}^0$ as well as values of H_1^1, H_2^1 and $H_3^+(i|\lambda|)$ we obtain the representations which are of use for applying the contour integration procedure

$$H_1^j = -\frac{1}{2\pi i}\int_\Gamma \frac{(\beta - i|\lambda|)^{-1/2}}{\rho^2}[\beta F_2^+(\beta) - \lambda F_3^+(\beta)]\beta^j\, \mathrm{d}\beta \quad (j = 0, 1),$$

$$H_2^j = -\frac{1}{2\pi i}\int_\Gamma \frac{(\beta - i|\lambda|)^{-1/2}}{\Lambda^+(\beta)}\{\Theta_1(\beta)F_1^+(\beta) + \frac{1}{\rho^2}\Theta_2(\beta)[\lambda F_2^+(\beta)$$

$$+\beta F_3^+(\beta)]\}\beta^j\, \mathrm{d}\beta \quad (j = 0, 1),$$

$$H_3^0 = -\frac{1}{2\pi i}\int_\Gamma \frac{(\beta - i|\lambda|)^{-1/2}}{\Lambda^+(\beta)}\{\Theta_2(\beta)F_1^+(\beta) - \Theta_1(\beta)[\lambda F_2^+(\beta) + \beta F_3^+(\beta)]\}\,\mathrm{d}\beta,$$

$$H_3^+(i|\lambda|) = \frac{1}{2\pi i}\int_\Gamma \frac{(\beta - i|\lambda|)^{-3/2}}{\Lambda^+(\beta)}\{\Theta_2(\beta)F_1^+(\beta) - \Theta_1(\beta)[\lambda F_2^+(\beta) + \beta F_3^+(\beta)]\}\,\mathrm{d}\beta, \tag{6.4}$$

where

$$\Theta_1(\beta) = c_1(\beta)\cos\{\rho B^+(\beta)\} - \rho^{-1}b_1(\beta)\sin\{\rho B^+(\beta)\},$$

$$\Theta_2(\beta) = b_1(\beta)\cos\{\rho B^+(\beta)\} + \rho c_1(\beta)\sin\{\rho B^+(\beta)\}.$$

In order to calculate the functions $\mathfrak{W}_{3j}$ we need only the values of H_1^0, H_1^1, H_2^0 and $H_3^+(i|\lambda|)$. The other two expressions are necessary for the functions $\mathfrak{W}_{1j}$, $\mathfrak{W}_{2j}$. By inserting formulae (6.4) into (6.2) we change the order of integration and arrive at the integral presentation of the weight functions $\mathfrak{W}_{3j}$

$$\mathfrak{W}_{3j}(\xi,\zeta) = -\frac{1+i}{4\pi^2 i}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda\zeta} I_{3j}(\lambda)\,\mathrm{d}\lambda, \quad j = 1,2,3, \tag{6.5}$$

where

$$I_{31}(\lambda) = \frac{|\lambda|}{W_{1\lambda}}\int_\Gamma \left[\Theta_1(\beta) + \frac{bd(|\lambda| + ia)}{\beta - i|\lambda|}\Theta_2(\beta)\right]\frac{\mathrm{e}^{i\beta\xi}\,\mathrm{d}\beta}{(\beta - i|\lambda|)^{1/2}\Lambda^+(\beta)},$$

$$I_{32}(\lambda) = \int_\Gamma \left[\beta\Psi(\beta) - \frac{\lambda}{\rho^2}\left(\beta + \frac{i|\lambda|W_{2\lambda}}{W_{1\lambda}}\right)\right]\frac{\mathrm{e}^{i\beta\xi}\,\mathrm{d}\beta}{(\beta - i|\lambda|)^{1/2}},$$

$$I_{33}(\lambda) = \int_\Gamma \left[\lambda\Psi(\beta) + \frac{\beta}{\rho^2}\left(\beta + \frac{i|\lambda|W_{2\lambda}}{W_{1\lambda}}\right)\right]\frac{\mathrm{e}^{i\beta\xi}\,\mathrm{d}\beta}{(\beta - i|\lambda|)^{1/2}},$$

$$\Psi(\beta) = \frac{|\lambda|}{W_{1\lambda}\Lambda^+(\beta)}\left[\frac{1}{\rho^2}\Theta_2(\beta) - \frac{bd(|\lambda| + ia)}{\beta - i|\lambda|}\Theta_1(\beta)\right].$$

Let us transform these formulae. We start from the weight function $\mathfrak{W}_{31}(\xi,\zeta)$. In order to calculate the first integral $I_{31}(\lambda)$, we analyse the behaviour of the integrand in the vicinity of the points $\pm\lambda_R$ (in the case $|\lambda| < b^{-1}d^{-1}$), $i\lambda_R$ (if $|\lambda| > b^{-1}d^{-1}$) and $i|\lambda|$. Bearing in mind the asymptotics (3.28) and (3.29) as well as the fact that the functions $\cos(\rho B^+\}$ and $\sin\{\rho B^+\}$ are analytic as $\beta \to -\lambda_R$, $\beta \in \mathbb{C}^+$ we have the following results

$$\frac{1}{\Lambda^+(\beta)}\Theta_j(\beta) = O(1), \quad \beta \to \lambda_R,$$

$$\frac{1}{\Lambda^+(\beta)}\,\Theta_j(\beta) = O\left(\frac{1}{\beta+\lambda_R}\right), \quad \beta\to -\lambda_R; \quad (j=1,2)$$

but nevertheless

$$\frac{1}{\Lambda^+(\beta)}\left[\Theta_1(\beta)+\frac{bd(|\lambda|+ia)}{\beta-i|\lambda|}\Theta_2(\beta)\right] = O(1), \quad \beta\to a,$$

$$\frac{1}{\Lambda^+(\beta)}\left[\frac{1}{\rho^2}\Theta_2(\beta)-\frac{bd(|\lambda|+ia)}{\beta-i|\lambda|}\Theta_1(\beta)\right] = O(1), \quad \beta\to a,$$

$$a = \begin{cases} -\lambda_R, & |\lambda| < b^{-1}d^{-1} \\ i\lambda_R, & |\lambda| > b^{-1}d^{-1} \end{cases}. \tag{6.6}$$

Applying the Cauchy theorem we reduce the integral $I_{31}(\lambda)$ to an integral along the two sides of the branch cut from $i|\lambda|$ to $+i\infty$ and take into account the identity

$$\int_\Gamma \frac{e^{i\beta\xi}f_\lambda(\beta)}{(\beta-i|\lambda|)^{1/2}}\,d\beta = \frac{2i}{i+1}\,e^{-|\lambda|\xi}\mathfrak{G}_\xi[f_\lambda(\beta)], \quad \arg(\beta-i|\lambda|)\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right) \tag{6.7}$$

that is true for a function $f_\lambda(\beta)$ admitting the analytical continuation in the upper half-plane $\mathbb{C}^+$ and satisfying the natural conditions of integrability on Γ and of decrease as $\beta\to\infty$, $\beta\in\mathbb{C}^+$, $\mathfrak{G}_\xi$ is the following integral operator

$$\mathfrak{G}_\xi[f_\lambda(\beta)] = \sqrt{2}\int_0^\infty t^{-1/2}\,e^{-\xi t}f_\lambda(it+i|\lambda|)\,dt. \tag{6.8}$$

It follows that

$$I_{31}(\lambda) = \frac{2i}{i+1}\frac{|\lambda|}{W_{1\lambda}}\,e^{-|\lambda|\xi}\mathfrak{G}_\xi[f_\lambda^{3,1}(\beta)],$$

$$f_\lambda^{3,1}(\beta) = \frac{db}{\Lambda^+(\beta)\delta_0(\beta)}\{[-i+b^2d^2(|\lambda|+ia)(\beta+i|\lambda|)]\cos(\rho B^+)$$
$$-\frac{bd\rho(\beta-a)}{\beta-i|\lambda|}\sin(\rho B^+)\},$$

where the function $B^+(\beta)$ is defined in (3.13) and

$$B_1^+(it+i|\lambda|) = \frac{t+|\lambda|}{2\pi i}\int_0^\infty \log\frac{bd(\tau^2+\lambda^2)^{1/2}+1}{bd(\tau^2+\lambda^2)^{1/2}-1}\,\frac{dt}{(\tau^2+\lambda^2)^{1/2}[\tau^2+(t+|\lambda|)^2]}. \tag{6.9}$$

For $\beta = it+i|\lambda|$ we have

$$\frac{\rho \sin(\rho B^+)}{\beta - i|\lambda|} = O(1), \quad t \to 0.$$

The functions $W_{1\lambda}$ and $I_{31}(\lambda)$ are odd on λ and in view of (6.5) we find

$$\mathfrak{W}_{31}(\xi, \zeta) = \frac{1}{\pi^2 i} \int_0^\infty \frac{\lambda \sin \lambda \zeta}{W_{1\lambda}} e^{-\lambda\xi} \mathfrak{G}_\xi [f_\lambda^{3,1}(\beta)] \, d\lambda. \tag{6.10}$$

Both integrals (6.8) and (6.10) are improper ones that are convenient for numerical calculations.

The integrals $I_{32}(\lambda)$ and $I_{33}(\lambda)$ can be found by analogy with $I_{31}(\lambda)$ if we take into account the following formulae

$$\int_\Gamma \frac{e^{i\beta\xi}}{(\beta - i|\lambda|)^{1/2}} d\beta = \frac{2i}{i+1} \sqrt{\frac{2\pi}{\xi}} e^{-|\lambda|\xi},$$

$$\int_\Gamma \frac{e^{i\beta\xi}}{(\beta - i|\lambda|)^{3/2}} d\beta = -\frac{4}{i+1} \sqrt{2\pi\xi} \, e^{-|\lambda|\xi},$$

$$\int_\Gamma \frac{e^{i\beta\xi}}{(\beta - i|\lambda|)^{3/2}} d\beta = \frac{2i}{i+1} e^{-|\lambda|\xi} \left\{ 2i\sqrt{2\pi\xi} f_\lambda(i|\lambda|) + \mathfrak{G}_\xi \left[\frac{f_\lambda(\beta) - f_\lambda(i|\lambda|)}{\beta - i|\lambda|} \right] \right\},$$

$$\arg(\beta - i|\lambda|) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2} \right) \tag{6.11}$$

and that some of the summands in the representation of the integrands for the integrals $I_{32}(\lambda)$ and $I_{33}(\lambda)$ have a simple pole at the point $\beta = i|\lambda|$. Note, the integrands for $I_{32}(\lambda)$ and $I_{33}(\lambda)$ have removable singularity at the point a. Finally, substituting the expressions for $I_{32}(\lambda)$ and $I_{33}(\lambda)$ into (6.5) we obtain

$$\mathfrak{W}_{3k}(\xi, \zeta) = \frac{(-1)^{k-1}}{\pi^2} \int_0^\infty \lambda s_k(\lambda\zeta) \, e^{-\lambda\xi} \{ U_{0\lambda} + i^{k-1} \mathfrak{G}_\xi [f_\lambda^{3,k}(\beta)] \} \, d\lambda,$$

$$U_{0\lambda} = \sqrt{2\pi\xi} \left[1 + \frac{W_{2\lambda}}{W_{1\lambda}} - 2\frac{b^2 d^2 \lambda(\lambda + ia)}{W_{1\lambda}\Lambda^+(i\lambda)} \right]$$

$$+ \frac{\pi\lambda^{1/2} d^2}{W_{1\lambda}\Lambda^+(i\lambda)} [1 + 2bd\lambda(\lambda + ia)(iB^+(i\lambda) - bd)],$$

$$f_\lambda^{3,k}(\beta) = \frac{(2\beta/\lambda)\delta_{k3} + i^{-k}[1 - (-1)^k W_{1\lambda}^{-1} W_{2\lambda}]}{2(\beta + i\lambda)} + \frac{b\lambda(\beta/\lambda)^{3-k}}{W_{1\lambda}\delta_0(\beta)\Lambda^+(\beta)}$$

$$\times \left\{ -\Sigma_2^+(\beta) + \frac{\lambda + ia}{\beta - i\lambda} bd \left[\Sigma_1^+(\beta) + \frac{id\Lambda^+(\beta)\delta_0(\beta)}{\Lambda^+(i\lambda)} \left(\frac{i\lambda}{\beta} \right)^{3-k} \right] \right\},$$

$$k = 2, 3; \quad s_2(t) = \sin t, \quad s_3(t) = \cos t, \tag{6.12}$$

where δ_{kn} is Kronecker's symbol and the functions $\Sigma_1^+(\beta)$, $\Sigma_2^+(\beta)$ are written in (3.27).

To determine the weight functions $\mathfrak{W}_{1j}$, $\mathfrak{W}_{2j}$ $(j = 1, 2, 3)$ we find the complex functions

$$\mathfrak{W}_j(\xi,\zeta) = \mathfrak{W}_{1j}(\xi,\zeta) - i\mathfrak{W}_{2j}(\xi,\zeta) \quad (j = 1, 2, 3).$$

In view of formulae (5.10), (5.5) and (4.18) we have

$$K(z) = \frac{(1+i)\pi^{1/2}}{2d_0\Gamma(1/2-i\varepsilon)}\mathfrak{F}_z^{-1}\{D(\lambda)[V_{1\lambda}H_1^0 + V_{2\lambda}H_2^0 + iH_2^1 + H_3^0 + V_{3\lambda}H_3^+(i|\lambda|)]\}, \tag{6.13}$$

where

$$V_{1\lambda} = -\frac{bd^2(b+e)}{W_{1\lambda}}|\lambda|(|\lambda|+ia)\left(i|\lambda|+a+\frac{1}{db}\right),$$

$$V_{2\lambda} = -\frac{\lambda}{W_{1\lambda}}\left[\frac{M_+}{2}+id(abd+1)(b+e)(|\lambda|+ia)M_+^{-1}\right],$$

$$V_{3\lambda} = -\frac{bdi}{2W_{1\lambda}}(|\lambda|+ia)M_+\,\mathrm{sgn}\,\lambda\left(i|\lambda|+a\frac{1}{db}\right). \tag{6.14}$$

Substituting the expressions (6.4) into (6.13) and take into account (5.1) we get

$$\mathfrak{W}_j(\xi,\zeta) = \frac{1+i}{4i}\Pi\int_{-\infty}^{\infty}\mathrm{e}^{i\lambda\zeta}D(\lambda)I_j(\lambda)\,\mathrm{d}\lambda \quad (j = 1, 2, 3),$$

where

$$I_1(\lambda) = \int_\Gamma\left[\Theta_1(\beta)(V_{2\lambda}+i\beta)+\Theta_2(\beta)\left(1-\frac{V_{3\lambda}}{\beta-i|\lambda|}\right)\right]\frac{\mathrm{e}^{i\beta\xi}\,\mathrm{d}\beta}{(\beta-i|\lambda|)^{1/2}\Lambda^+(\beta)},$$

$$I_k(\lambda) = \int_\Gamma\frac{\mathrm{e}^{i\beta\xi}}{(\beta-i|\lambda|)^{1/2}}\left\{-\frac{(-1)^k\lambda(\beta/\lambda)^{k-2}}{\rho^2}V_{1\lambda}\right.$$
$$\left.+\frac{\beta(\lambda/\beta)^{k-2}}{\Lambda^+(\beta)}\left[\Theta_2(\beta)\frac{V_{2\lambda}+i\beta}{\rho^2}-\Theta_1(\beta)\left(1-\frac{V_{3\lambda}}{\beta-i|\lambda|}\right)\right]\right\}\mathrm{d}\beta,$$

$k = 2, 3;$

$$\Pi = -\frac{1}{2\pi^{3/2}d_0\Gamma(1/2-i\varepsilon)}. \tag{6.15}$$

The integrands in the integrals (6.15) are analytic in the vicinity of the points λ_R. Let us convince ourselves that the integrands in (6.15) have a removable singularity at the point a ($a = -\lambda_R$ if $|\lambda| < b^{-1}d^{-1}$ and $a = i\lambda_R$ if $|\lambda| > b^{-1}d^{-1}$). According to (6.6) for the first integral we have as $\beta \to a$

$$\Theta_1(\beta)(V_{2\lambda}+i\beta)+\Theta_2(\beta)\left(1-\frac{V_{3\lambda}}{\beta-i|\lambda|}\right)$$
$$=\frac{b+d}{b}\left[1-\frac{V_{3\lambda}}{a-i|\lambda|}-idb(V_{2\lambda}+ia)\right]+O(\beta-a). \quad (6.16)$$

Inserting formulae (6.14) for $V_{j\lambda}$ into the expression in square brackets in (6.16) gives

$$1-\frac{V_{3\lambda}}{a-i|\lambda|}-idb(V_{2\lambda}+ia)=0.$$

This identity leads to the boundedness at the point a not only of the integrand for the integral $I_1(\lambda)$ but for two other integrals $I_2(\lambda)$, $I_3(\lambda)$. Applying as before formulae (6.7) and (6.11) we obtain

$$\mathfrak{W}_1(\xi,\zeta)=\Pi\int_0^\infty D(\lambda)\cos\lambda\zeta\,\mathrm{e}^{-\lambda\xi}\left\{\frac{2\pi id^2\sqrt{\lambda}}{\Lambda^+(i\lambda)}V_{3\lambda}+\mathfrak{G}_\xi[f_\lambda^1(\beta)]\right\}\mathrm{d}\lambda,$$

$$\mathfrak{W}_k(\xi,\zeta)=\Pi\int_0^\infty D(\lambda)s_{5-k}(\lambda\zeta)\,\mathrm{e}^{-\lambda\xi}\{U_{1\lambda}+i^{k-2}\mathfrak{G}_\xi[f_\lambda^k(\beta)]\}\,\mathrm{d}\lambda \quad (k=2,3), \quad (6.17)$$

where

$$f_\lambda^1(\beta)=-\frac{b}{\Lambda^+(\beta)\delta_0(\beta)}\left[\Sigma_1^+(\beta)(V_{2\lambda}+i\beta)+\rho^2\left(1-\frac{V_{3\lambda}}{\beta-i\lambda}\right)\Sigma_2(\beta)\right],$$

$$U_{1\lambda}=-\sqrt{2\pi\xi}\left[V_{1\lambda}+\frac{2i\lambda bdV_{3\lambda}}{\Lambda^+(i\lambda)}\right]-\frac{\pi id^2\sqrt{\lambda}}{\Lambda^+(i\lambda)}[V_{2\lambda}-\lambda-2i\lambda(idb+B^+(i\lambda))V_{3\lambda}],$$

$$f_\lambda^k(\beta)=\frac{V_{1\lambda}}{2i^{3-k}(\beta+i\lambda)}+\frac{db\lambda V_{3\lambda}}{i^{k-2}(\beta-i\lambda)\Lambda^+(i\lambda)}$$
$$+\frac{b\beta}{\delta_0(\beta)\Lambda^+(\beta)}\left(\frac{\lambda}{\beta}\right)^{k-2}\left[-\Sigma_2(\beta)(V_{2\lambda}+i\beta)+\Sigma_1(\beta)\left(1-\frac{V_{3\lambda}}{\beta-i\lambda}\right)\right]$$
$$(k=2,3).$$

Thus, we have found expressions for the weight functions in the form of double integrals. Both integrands decay exponentially at infinity and do not have singularities anywhere in the semi-infinite interval $(0,\infty)$.

7. Vanishing of the parameter d

Up to here the parameter d is assumed to be strictly positive. In the case $d=0$ there is no need for the factorization of the matrix $\mathbf{G}_0(\beta)$

$$\mathbf{G}_0(\beta) = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b-e\beta^2\rho^{-2} & e\beta\lambda\rho^{-2} \\ 0 & e\beta\lambda\rho^{-2} & -b-e\lambda^2\rho^{-2} \end{pmatrix}$$

since this matrix is a rational one. We introduce the vectors $\mathbf{H}^{\pm}(\beta)$ as follows

$$\mathbf{H}^{\pm}(\beta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(t-i|\lambda|)^{-1/2}\mathbf{G}_0(t)\mathbf{F}^+(t)\,\mathrm{d}t}{t-\beta}, \quad \beta\in\mathbb{C}^{\pm},$$

where $\mathbb{C}^+ = \{\beta\colon \Im\beta > 0\}$ and $\mathbb{C}^- = \{\beta\colon \Im\beta < 0\}$ and apply to the boundary condition

$$(t+i\,|\,\lambda)^{1/2}\mathbf{\Phi}^+(t) - H^+(t) = (t-i\,|\,\lambda)^{-1/2}\mathbf{G}_0(t)\mathbf{\Phi}^-(t) - \mathbf{H}^-(t), \quad t\in\mathbb{R}_1 \tag{7.1}$$

the theorem of analytic continuation and the generalized Liouville theorem taking into account that the only possible singularity of the right-hand-side of eqn (7.1) is the point $-i|\lambda|$. In the selected class of solutions (4.7) we have

$$\mathbf{\Phi}^+(\beta) = (\beta+i|\lambda|)^{-1/2}\left[\mathbf{H}^+(\beta) + \frac{1}{\beta+i|\lambda|}\begin{pmatrix} 0 \\ C_2 \\ C_3 \end{pmatrix}\right], \quad \beta\in\mathbb{C}^+,$$

$$\mathbf{\Phi}^-(\beta) = (\beta-i|\lambda|)^{1/2}\mathbf{G}_0^{-1}(\beta)\left[\mathbf{H}^-(\beta) + \frac{1}{\beta+i|\lambda|}\begin{pmatrix} 0 \\ C_2 \\ C_3 \end{pmatrix}\right], \quad \beta\in\mathbb{C}^-,$$

where C_2, C_3 are arbitrary constants. The components of the vector $\mathbf{\Phi}^-(\beta)$ can be rewritten in the form

$$\Phi_1^-(\beta) = -\frac{1}{b}(\beta-i|\lambda|)^{1/2}H_1^-(\beta),$$

$$\Phi_k^-(\beta) = -\frac{1}{b(b+e)}(\beta-i|\lambda|)^{1/2}\Psi_k(\beta) \quad (k=2,3),$$

$$\Psi_2(\beta) = \left(b+e\frac{\lambda^2}{\rho^2}\right)\left[H_2^-(\beta) + \frac{C_2}{\beta+i|\lambda|}\right] + \frac{e\beta\lambda}{\rho^2}\left[H_3^-(\beta) + \frac{C_3}{\beta+i|\lambda|}\right],$$

$$\Psi_3(\beta) = \frac{e\beta\lambda}{\rho^2}\left[H_2^-(\beta) + \frac{C_2}{\beta+i|\lambda|}\right] + \left(b+e\frac{\beta^2}{\rho^2}\right)\left[H_3^-(\beta) + \frac{C_3}{\beta+i|\lambda|}\right].$$

Thus, the functions $\Phi_j^+(\beta)$ $(j=1,2,3)$ are analytic in $\mathbb{C}^+$, the function $\Phi_1^-(\beta)$ is analytic in $\mathbb{C}^-$. Only the functions $\Phi_2^-(\beta)$, $\Phi_3^-(\beta)$ which should be analytic in $\mathbb{C}^-$ admit poles at the point $\beta = -i|\lambda|$. Let us eliminate these poles and find the constants C_2, C_3. For the first function $\Psi_2(\beta)$ we can obtain the following expansion in the vicinity of the point $\beta = -i|\lambda|$

$$\Psi_2(\beta) = \frac{e\lambda}{2(\beta+i|\lambda|)^2}(C_3 + i\,\mathrm{sgn}\,\lambda C_2) + \frac{1}{\beta+i|\lambda|}$$
$$\times\left\{\left(b+\frac{e}{4}\right)C_2 + \frac{e}{4}i\,\mathrm{sgn}\,\lambda C_3 + \frac{e\lambda}{2}[H_3^-(-i|\lambda|) + i\,\mathrm{sgn}\,\lambda H_2^-(-i|\lambda|)]\right\} + O(1),$$
$$\beta \to -i|\lambda|.$$

In order for the function $\Phi_2^-(\beta)$ to be analytic in the vicinity of the point $\beta = -i|\lambda|$, it is necessary and sufficient that the two conditions

$$C_3 = -i\,\mathrm{sgn}\,\lambda C_2,$$
$$C_2 = -\frac{e\lambda}{2b+e}[H_3^-(-i|\lambda|) + i\,\mathrm{sgn}\,\lambda H_2^-(-i|\lambda|)] \tag{7.2}$$

be satisfied. The analysis of the behaviour of the function $\Psi_3(\beta)$ in the neighbourhood of the point $\beta = -i|\lambda|$ leads to the same conditions (7.2). Therefore, for the functions $\Psi_2(\beta)$ and $\Psi_3(\beta)$ to be analytic in $\mathbb{C}^-$ we have to choose the constants C_2, C_3 as (7.2).

Let us find the stress intensity factors and the weight functions. It is easy to obtain that

$$\begin{pmatrix}\sigma_y\\ \tau_{xy}\\ \tau_{yz}\end{pmatrix}(x,0,z) \sim -\frac{1+i}{\sqrt{2\pi}b(b+e)}\mathfrak{F}_z^{-1}\begin{pmatrix}(b+e)H_1^0\\ (b+e)(H_3^0+C_3)\\ b(H_2^0+C_2)\end{pmatrix}(-x)^{-1/2}, \quad x\to -0,$$

where

$$\mathbf{H}^0 = \begin{pmatrix}H_1^0\\ H_2^0\\ H_3^0\end{pmatrix} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(t-i|\lambda|)^{-1/2}\mathbf{G}_0(t)\mathbf{F}^+(t)\,\mathrm{d}t.$$

Hence, in view of the formula

$$(\sigma_y, \tau_{xy}, \tau_{yz})(x,0,z) \sim \frac{(-x)^{-1/2}}{\sqrt{2\pi}}(K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}})(z), \quad x\to -0,$$

we arrive at the relations

$$K_{\mathrm{I}}(z) = -\frac{1+i}{b}\mathfrak{F}_z^{-1}[H_1^0],$$

$$K_{\mathrm{II}}(z) = -\frac{1+i}{b}\mathfrak{F}_z^{-1}[H_3^0 + C_3],$$

$$K_{\mathrm{III}}(z) = -\frac{1+i}{b+e}\mathfrak{F}_z^{-1}[H_2^0 + C_2].$$

For the weight functions which were introduced in (6.1) we find the integral representations

$$\mathfrak{W}_{11}(\xi,\zeta) = -\frac{1+i}{4\pi^2 i}\int_{-\infty}^{\infty} e^{i\lambda\zeta}\,d\lambda \int_{-\infty}^{\infty} \frac{e^{i\beta\xi}\,d\beta}{(\beta - i|\lambda|)^{1/2}}\,d\beta,$$

$$\mathfrak{W}_{kk}(\xi,\zeta) = \mathfrak{W}_{11}(\xi,\zeta) + \frac{(-1)^k e}{2b+e}\frac{1+i}{4\pi^2}\int_{-\infty}^{\infty} e^{i\lambda\zeta}|\lambda|\,d\lambda \int_{-\infty}^{\infty} \frac{e^{i\beta\xi}\,d\beta}{(\beta - i|\lambda|)^{3/2}} \quad (k=2,3),$$

$$\mathfrak{W}_{23}(\xi,\zeta) = \mathfrak{W}_{32}(\xi,\zeta) = \frac{3}{2b+e}\frac{1+i}{4\pi^2 i}\int_{-\infty}^{\infty} e^{i\lambda\zeta}\lambda\,d\lambda \int_{-\infty}^{\infty} \frac{e^{i\beta\xi}\,d\beta}{(\beta - i|\lambda|)^{3/2}}.$$

All these integrals are calculated in elementary functions

$$\mathfrak{W}_{11}(\xi,\zeta) = -\frac{\sqrt{2\xi}}{\pi^{3/2}(\xi^2+\zeta^2)},$$

$$\mathfrak{W}_{12}(\xi,\zeta) = \mathfrak{W}_{21}(\xi,\zeta) = 0,$$

$$\mathfrak{W}_{kk}(\xi,\zeta) = -\frac{\sqrt{2\xi}}{\pi^{3/2}(\xi^2+\zeta^2)}\left[1 + (-1)^k \frac{2e}{2b+e}\frac{\xi^2-\zeta^2}{\xi^2+\zeta^2}\right] \quad (k=2,3),$$

$$\mathfrak{W}_{23}(\xi,\zeta) = \mathfrak{W}_{32}(\xi,\zeta) = -\frac{e}{2b+e}\frac{4\sqrt{2\xi}}{\pi^{3/2}}\frac{\xi\zeta}{(\xi^2+\zeta^2)^2}. \quad (7.3)$$

In the particular case $v_+ = v_-$, $G_+ = G_-$ these expressions coincide with the weight functions which were found by Movchan et al. (1998) for the homogeneous case.

8. 2-D case

Passing to the limit $\lambda \to 0$ in formula (2.8) we arrive at two independent Wiener–Hopf problems

$$\begin{pmatrix}\Phi_1^+(\beta)\\ \Phi_3^+(\beta)\end{pmatrix} = \begin{bmatrix} -\dfrac{b}{|\beta|} & -\dfrac{id}{\beta} \\ \dfrac{id}{\beta} & -\dfrac{b}{|\beta|}\end{bmatrix} \begin{pmatrix}\Phi_1^-(\beta)+F_1^+(\beta)\\ \Phi_3^-(\beta)+F_3^+(\beta)\end{pmatrix}, \quad (8.1)$$

$$\Phi_2^+(\beta) = -\frac{b+e}{|\beta|}[\Phi_2^-(\beta)+F_2^+(\beta)]. \quad (8.2)$$

The factorization of the matrix in the first problem is defined by the formulae

$$\mathbf{X}_2^{\pm}(\beta) = d_0^{\pm 1}\begin{pmatrix}\cos B^{\pm}(\beta) & \sin B^{\pm}(\beta)\\ -\sin B^{\pm}(\beta) & \cos B^{\pm}(\beta)\end{pmatrix},$$

where

$$B^{\pm}(\beta) = -\varepsilon\log(\mp i\beta), \quad \beta\in\mathbb{C}^{\pm}, -\frac{\pi}{2}\leqslant \arg(\mp i\beta)\leqslant\frac{\pi}{2}.$$

The solution of the problem (8.1) in the case

$$p_j(x) = P_j\,\mathrm{e}^{-\gamma x}, \quad j=1,2,3 \tag{8.3}$$

has the form

$$\begin{pmatrix}\Phi_1^+(\beta)\\ \Phi_3^+(\beta)\end{pmatrix} = -\frac{b(1+i)}{(\gamma-i\beta)\sqrt{2\gamma}\sqrt{\beta}^+}\mathbf{X}_2^+(\beta)[\mathbf{X}_2^-(-i\gamma)]^{-1}\begin{pmatrix}P_1\\ P_3\end{pmatrix}, \quad \beta\in\mathbb{C}^+,$$

$$\begin{pmatrix}\Phi_1^-(\beta)\\ \Phi_3^-(\beta)\end{pmatrix} = \frac{1}{\gamma-i\beta}\left\{-\mathbf{I}_2+\frac{1+i}{\sqrt{2\gamma}}\sqrt{\beta}^-\mathbf{X}_2^-(\beta)[\mathbf{X}_2^-(-i\gamma)]^{-1}\right\}\begin{pmatrix}P_1\\ P_3\end{pmatrix}, \quad \beta\in\mathbb{C}^- \tag{8.4}$$

where $\mathbf{I}_2$ is the 1×2 unit vector and $\sqrt{\beta}^+$, $\sqrt{\beta}^-$ are functions which are analytic in $\mathbb{C}^+$, $\mathbb{C}^-$, respectively and give the factorization of the function $|\beta|$

$$|\beta| = \sqrt{\beta}^+\sqrt{\beta}^-, \quad \beta\in\mathbb{R}_1,$$

$$\sqrt{\beta}^+ = \begin{cases}\sqrt{\beta}, & \beta>0\\ i\sqrt{-\beta}, & \beta<0\end{cases}, \quad \sqrt{\beta}^- = \begin{cases}\sqrt{\beta}, & \beta>0\\ -i\sqrt{-\beta}, & \beta<0\end{cases}.$$

The solution of the problem (8.2) is obtained immediately. In the case (8.3) we have

$$\Phi_2^+(\beta) = -\frac{(b+e)(1+i)}{\sqrt{2\gamma}\sqrt{\beta}^+}\frac{P_2}{\gamma-i\beta}, \quad \beta\in\mathbb{C}^+,$$

$$\Phi_2^-(\beta) = -\left(1-\frac{1+i}{\sqrt{2\gamma}}\sqrt{\beta}^-\right)\frac{P_2}{\gamma-i\beta}, \quad \beta\in\mathbb{C}^-. \tag{8.5}$$

Analysis of the function (8.4) and (8.5) at infinity and the Abelian theorems give us the following representations at the tip of an interface crack in the 2-D case

$$\sqrt{2\pi}\sigma_y(x,0)\sim K(-x)^{-1/2+i\varepsilon}+\bar{K}(-x)^{-1/2-i\varepsilon}, \quad x\to -0,$$

$$\sqrt{2\pi}\tau_{xy}(x,0)\sim i[K(-x)^{-1/2+i\varepsilon}-\bar{K}(-x)^{-1/2-i\varepsilon}], \quad x\to -0,$$

$$\sqrt{2\pi}\tau_{yz}(x,o)\sim K_{\mathrm{III}}(-x)^{-1/2}, \quad x\to -0,$$

where

$$K = -\sqrt{\frac{\pi}{2\gamma}}\frac{\gamma^{i\varepsilon}(P_1-iP_3)}{\Gamma(1/2+i\varepsilon)}, \quad K_{\mathrm{III}} = -\sqrt{\frac{2}{\gamma}}P_2.$$

It is possible to pass to the limit $\lambda \to 0$ also in the representation (3.1) and to follow further the method of the solution in the case $|\lambda| < b^{-1}d^{-1}$. This limit case is more delicate.

9. Conclusion

The 3-D problem of an interface crack whose surface are deformed by the application to them of a load with three arbitrary components is reduced to the 3×3 matrix Wiener–Hopf problem for which an exact solution in closed form by using matrix factorization is constructed.

The tractions across the interface and discontinuities of the displacements are found in terms of the Cauchy type integrals (3.13) and (4.2), the improper integral (4.14) and inverse double Fourier transforms (5.1) and (5.2).

The weight functions are constructed in the closed form that does not contain Cauchy type integrals but only the improper integrals (6.10), (6.12), (6.17) and (6.8) with an exponent-law decrease of the integrands and the improper integral (6.9).

In the particular but non-homogeneous case $d = 0$ the weight functions are written in the explicit form (7.3) in elementary functions.

There are two possibilities to pass on to the limit $\beta \to 0$ and to find an exact solution of the 2-D problem of an interface crack, which agrees with the results of Williams (1959) and Sih and Rice (1964).

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Appendix A: An alternative derivation of the matrix Wiener–Hopf problem

We use the Papkovich–Neuber potentials

$$\Delta\psi_j = 0 \quad (j = 1, 2, 3), \quad \Delta\chi = 2\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right)$$

in terms of which the displacement field has the form

$$2G_\pm(u, v, w) = 4(1 - v_\pm)(\psi_1, \psi_2, \psi_3) - \left(\frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y}, \frac{\partial\chi}{\partial z}\right). \tag{A.1}$$

In order to solve the problem, we need the expressions of the following stresses

$$\sigma_y = \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial z^2} + 2(1-\nu_\pm)\left(-\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} - \frac{\partial\psi_3}{\partial z}\right),$$

$$\tau_{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y} + 2(1-\nu_\pm)\left(\frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial x}\right),$$

$$\tau_{yz} = -\frac{\partial^2\chi}{\partial y\,\partial z} + 2(1-\nu_\pm)\left(\frac{\partial\psi_2}{\partial z} + \frac{\partial\psi_3}{\partial y}\right). \tag{A.2}$$

By applying the Fourier transform

$$\psi_{j\beta\lambda}(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_j(x,y,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda z}\,\mathrm{d}x\,\mathrm{d}z \quad (j = 1,2,3),$$

$$\chi_{\beta\lambda}(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi(x,y,z)\,\mathrm{e}^{i\beta x}\,\mathrm{e}^{i\lambda x}\,\mathrm{d}x\,\mathrm{d}z \tag{A.3}$$

the 3-D equations are reduced to the 1-D differential equations

$$\left(\frac{d^2}{\mathrm{d}y^2} - \rho^2\right)\psi_{j\beta\lambda} = 0,$$

$$\left(\frac{d^2}{\mathrm{d}y^2} - \rho^2\right)\chi_{\beta\lambda} = 2\left(\frac{d}{\mathrm{d}y}\psi_{2\beta\lambda} - \beta\psi_{1\beta\lambda} - i\lambda\psi_{3\beta\lambda}\right), \tag{A.4}$$

where $j = 1, 2, 3$; $\rho = (\lambda^2+\beta^2)^{1/2}$.

First, we consider, the upper half-space $\mathbb{R}_3^+$, extrapolate to the whole interface the boundary conditions (2.1) and assume the functions $p_j(x,z)$ $(j = 1,2,3)$ to be unknown as $x < 0$. By taking into account the relationships (A.2) and (A.3) the boundary conditions (2.1) can be written in the form

$$-\rho^2\chi_{\beta\lambda}(0) + 2(1-\nu_+)\left[\frac{d}{\mathrm{d}y}\psi_{2\beta\lambda}(0) + i\beta\psi_{1\beta\lambda}(0) + i\lambda\psi_{3\beta\lambda}(0)\right] = p_{1\beta\lambda},$$

$$i\beta\frac{d}{\mathrm{d}y}\chi_{\beta\lambda}(0) + 2(1-\nu_+)\left[\frac{d}{\mathrm{d}y}\psi_{1\beta\lambda}(0) - i\beta\psi_{2\beta\lambda}(0)\right] = p_{2\beta\lambda},$$

$$i\lambda\frac{d}{\mathrm{d}y}\chi_{\beta\lambda}(0) + 2(1-\nu_+)\left[-i\lambda\psi_{2\beta\lambda}(0) + \frac{d}{\mathrm{d}y}\psi_{3\beta\lambda}(0)\right] = p_{3\beta\lambda}, \tag{A.5}$$

where $p_{j\beta\lambda}$ are defined in (2.2). The following functions satisfy the boundary value problems (A.4), (A.5)

$$\psi_{j\beta\lambda}(y) = c_j\,\mathrm{e}^{-\rho y}, \quad \chi_{\beta\lambda}(y) = \frac{1+\rho y}{\rho^2}\mathrm{e}^{-\rho y}(i\beta c_1 + \rho c_2 + i\lambda c_3),$$

$$c_2 = -\frac{1}{2\rho}p_{1\beta\lambda} - \frac{i\kappa_+^-}{2\kappa_+^+\rho^2}(\beta p_{2\beta\lambda} + \lambda p_{3\beta\lambda}),$$

$$c_1 = -\frac{i\beta}{\rho}c_2 - \frac{p_{2\beta\lambda}}{\kappa_+^+\rho}, \quad c_3 = -\frac{i\lambda}{\rho}c_2 - \frac{p_{3\beta\lambda}}{\kappa_+^+\rho},$$

$$\kappa_\pm^+ = 2(1-\nu_\pm), \quad \kappa_\pm^- = 1-2\nu_\pm.$$

Substituting these formulae into the relationships (A.1) we obtain the expressions for the Fourier transforms of the displacements at the point $x = +0$

$$u_{\beta\lambda}(+0) = \frac{1}{2G_+}\left[\frac{i\kappa_+^-\beta}{\rho^2}p_{1\beta\lambda} + 2\left(-\frac{1}{\rho} + \frac{\nu_+\beta^2}{\rho^3}\right)p_{2\beta\lambda} + \frac{2\nu_+\beta\lambda}{\rho^3}p_{3\beta\lambda}\right],$$

$$v_{\beta\lambda}(+0) = \frac{1}{2G_+}\left[-\frac{\kappa_+^+}{\rho}p_{1\beta\lambda} - \frac{i\kappa_+^-\beta}{\rho^2}p_{2\beta\lambda} - \frac{i\kappa_+^-\lambda}{\rho^2}p_{3\beta\lambda}\right],$$

$$w_{\beta\lambda}(+0) = \frac{1}{2G_+}\left[\frac{i\kappa_+^-\lambda}{\rho^2}p_{1\beta\lambda} + \frac{2\nu_+\beta\lambda}{\rho^3}p_{2\beta\lambda} + 2\left(-\frac{1}{\rho} + \frac{\nu_+\lambda^2}{\rho^3}\right)p_{3\beta\lambda}\right]. \quad \text{(A.6)}$$

The corresponding expressions for the quantities $u_{\beta\lambda}(-0)$, $v_{\beta\lambda}(-0)$, $w_{\beta\lambda}(-0)$ referred to the half-space $x < 0$, are obtained from (A.6) by replacing the set of quantities $(\rho, \kappa_+^-, \kappa_+^+, \nu_+, G_+)$ by $(-\rho, \kappa_-^-, \kappa_-^+, \nu_-, G_-)$. Next, we subtract the expressions for $u_{\beta\lambda}(-0)$, $v_{\beta\lambda}(-0)$, $w_{\beta\lambda}(-0)$ from the corresponding ones (A.6) and arrive at the relationships (2.3) with the matrix $\mathbf{G}(\beta)$ (2.4). the consequent introduction (2.6), (2.7) of the vectors $\mathbf{\Phi}^\pm(\beta)$ and $\mathbf{F}^+(\beta)$ leads to the matrix Wiener–Hopf problem (2.8).

Appendix B: The splitting of the matrix coefficient into a product of a block diagonal matrix and rational ones

At first, we find a splitting of our matrix (2.5) into two factors

$$-\rho^2\mathbf{G}_0(\beta) = \mathbf{N}(\beta)\mathbf{C}(\beta), \quad \text{(B.1)}$$

where

$$\mathbf{C}(\beta) = b_0(\beta)\mathbf{I} + c_0(\beta)\mathbf{Q}(\beta),$$

$$\mathbf{N}(\beta) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}, \quad \mathbf{Q}(\beta) = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}. \quad \text{(B.2)}$$

The elements of the matrices $\mathbf{N}(\beta)$ and $\mathbf{Q}(\beta)$ are rational functions on β and the functions $b_0(\beta)$, $c_0(\beta)$ can have poles on $\mathbb{R}_1$. According to the form of the matrix $\mathbf{G}_0(\beta)$ (2.5) we should look for the functions $b_0(\beta)$ and $c_0(\beta)$ as follows

$$b_0(\beta) = r_0(\beta)\rho + r_1(\beta), \quad c_0(\beta) = r_2(\beta)\rho + r_3(\beta),$$

where $r_m(\beta)$ $(m = 0, 1, 2, 3)$ are rational functions. On substituting the matrices (B.2) in the eqn (B.1) it is not difficult to verify that

$$\begin{aligned} &n_{11}\delta_1 = b\rho^2, \quad n_{21}\delta_2 = id\lambda, \quad n_{31}\delta_2 = id\beta, \\ &n_{12}\delta_2 = -id\lambda, \quad n_{22}\delta_1 = b\rho^2 + e\beta^2, \quad n_{32}\delta_1 = -e\beta\lambda, \\ &n_{13}\delta_2 = -id\beta, \quad n_{23}\delta_1 = -e\beta\lambda, \quad n_{33}\delta_1 = b\rho^2 + e\lambda^2, \end{aligned} \tag{B.3}$$

$$\begin{aligned} &n_{j1}q_{11} + n_{j2}q_{21} + n_{j3}q_{31} = -n_{j1}\eta_j, \\ &n_{j1}q_{12} + n_{j2}q_{22} + n_{j3}q_{32} = -n_{j2}\zeta_j, \\ &n_{j1}q_{13} + n_{j2}q_{23} + n_{j3}q_{33} = -n_{j3}\zeta_j, \quad j = 1, 2, 3, \end{aligned} \tag{B.4}$$

where

$$\begin{aligned} &\eta_1 = \zeta_2 = \zeta_3 = \delta_3, \quad \eta_2 = \eta_3 = \zeta_1 = \delta_4, \\ &\delta_1 = \frac{r_1 r_2 - r_0 r_3}{r_2}, \quad \delta_2 = \frac{r_1 r_2 - r_0 r_3}{r_3}, \quad \delta_3 = \frac{r_0}{r_2}, \quad \delta_4 = \frac{r_1}{r_3}. \end{aligned} \tag{B.5}$$

We insert relations (B.3) into eqns (B.4) and determine the functions q_{jk} from the system

$$\begin{aligned} &\frac{\delta_2}{\delta_1} b\rho^2 q_{1j} - id\lambda q_{2j} - id\beta q_{3j} = p_{1j}, \\ &\frac{\delta_1}{\delta_2} id\lambda q_{1j} + (b\rho^2 + e\beta^2) q_{2j} - e\beta\lambda q_{3j} = p_{2j}, \\ &\frac{\delta_1}{\delta_2} id\beta q_{1j} - e\beta\lambda q_{2j} + (b\rho^2 + e\lambda^2) q_{3j} = p_{3j} \quad (j = 1, 2, 3), \end{aligned} \tag{B.6}$$

where

$$\begin{aligned} &p_{11} = -\frac{\delta_2\delta_3}{\delta_1} b\rho^2, \quad p_{21} = -\frac{\delta_1\delta_4}{\delta_2} id\lambda, \quad p_{31} = -\frac{\delta_1\delta_4}{\delta_2} id\beta, \\ &p_{12} = \delta_4 id\lambda, \quad p_{22} = -\delta_3(b\rho^2 + e\beta^2), \quad p_{32} = \delta_3 e\beta\lambda, \\ &p_{13} = \delta_4 id\beta, \quad p_{23} = \delta_3 e\beta\lambda, \quad p_{33} = -\delta_3(b\rho^2 + e\lambda^2). \end{aligned}$$

We possess 13 unknown parameters q_{kj} $(k, j = 1, 2, 3)$, δ_1, δ_2, δ_3 and δ_4 and only nine equations for their definition. Therefore, we may choose values for some of them

$$\delta_1 = 1, \quad \delta_2 = d^2, \quad \delta_3 = 0, \quad \delta_4 = 1.$$

The relations (B.2), (B.3), (B.5) and the system (B.6) give immediately

$$r_0 = 0, \quad r_1 = 1, \quad r_2 = d^2, \quad r_3 = 1,$$

$$b_0 = 1, \quad c_0 = d^2\rho + 1,$$

$$\mathbf{N}(\beta) = \begin{pmatrix} b\rho^2 & -i\lambda d^{-1} & -i\beta d^{-1} \\ i\lambda d^{-1} & b\rho^2 + e\beta^2 & -e\beta\lambda \\ i\beta d^{-1} & -e\beta\lambda & b\rho^2 + e\lambda^2 \end{pmatrix},$$

$$\mathbf{C}(\beta) = \mathbf{I} + c(\beta)\mathbf{A}(\beta), \quad c(\beta) = \frac{d^2\rho + 1}{\rho^2\delta_0}, \quad \delta_0 = d^2 b^2 \rho^2 - 1,$$

$$\mathbf{A}(\beta) = \begin{pmatrix} \rho^2 & idb\lambda\rho^2 & idb\beta\rho^2 \\ -idb\lambda\rho^2 & \lambda^2 & \lambda\beta \\ -idb\beta\rho^2 & \lambda\beta & \beta^2 \end{pmatrix}.$$

In order to find the splitting of the matrix $\mathbf{C}$

$$\mathbf{C}(\beta) = \mathbf{T}(\beta)\mathbf{G}_1(\beta)\mathbf{T}^{-1}(\beta),$$

where $\mathbf{T}(\beta)$ is a polynomial matrix and $\mathbf{G}_1(\beta)$ has the structure (1.1), we calculate (see Moiseyev and Popov, 1990) the characteristic polynomial of the matrix $\mathbf{A}(\beta)$

$$\det(\omega\mathbf{I} - \mathbf{A}(\beta)) = \omega\phi_2(\beta, \omega),$$

where

$$\phi_2(\beta, \omega) = [\omega - \rho^2(1 + \rho bd)][\omega - \rho^2(1 - \rho pd)].$$

We find the eigenvector of the matrix $\mathbf{A}$ that corresponds to the eigenvalue $\omega = 0$ and define the first column-vector $t^{(1)}$ of the matrix $\mathbf{T}$ as this eigenvector

$$t^{(1)} = \begin{pmatrix} 0 \\ \beta \\ -\lambda \end{pmatrix}.$$

Next, we consider the matrix operator $\phi_2(\beta, \mathbf{A}(\beta))$

$$\phi_2(\beta, \mathbf{A}(\beta)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta^2\rho^2\delta_0 & \lambda\beta\rho^2\delta_0 \\ 0 & \lambda\beta\rho^2\delta_0 & -\lambda^2\rho^2\delta_0 \end{pmatrix}$$

and find the basis of the kernel of this operator

$$t^{(2)} = \begin{pmatrix} 0 \\ \lambda \\ \beta \end{pmatrix}, \quad t^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we form the matrix $\mathbf{T}$ from the vectors $t^{(1)}$, $t^{(2)}$ and $t^{(3)}$

$$\mathbf{T}(\beta) = \begin{pmatrix} 0 & 0 & 1 \\ \beta & \lambda & 0 \\ -\lambda & \beta & 0 \end{pmatrix}, \quad \det \mathbf{T}(\beta) = \rho^2$$

such that

$$\mathbf{C}(\beta) = \mathbf{T}(\beta)\mathbf{G}_1(\beta)\mathbf{T}^{-1}(\beta), \tag{B.7}$$

where the block diagonal matrix $\mathbf{G}_1(\beta)$ and the rational matrix $\mathbf{T}^{-1}(\beta)$ are presented in (3.2). Inserting the relation (B.7) into formula (B.1) we arrive at the following splitting of the matrix $\mathbf{G}_0(\beta)$

$$\mathbf{G}_0(\beta) = \mathbf{U}(\beta)\mathbf{G}_1(\beta)\mathbf{T}^{-1}(\beta),$$

where $\mathbf{U}(\beta) = \mathbf{N}(\beta)\mathbf{T}(\beta)$ is a rational matrix (3.3).

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