IMA Journal of Applied Mathematics (2003) **68**, 299–311

On the uniformity of stresses inside an inhomogeneity of arbitrary shape

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[Received on 17 May 2002]

We consider an inhomogeneity of 'arbitrary' shape embedded within an infinite isotropic elastic medium (matrix) subjected to antiplane shear deformations under the assumption of uniform remote loading. The inhomogeneity–matrix interface is assumed to be imperfect, characterized by a single interface function. Under these assumptions, we present a novel method leading to the solution of the problem concerned with identifying the shape of the inhomogeneity and the form of the corresponding interface function which leads to a uniform interior stress field. The analysis is based on complex variable methods. Specific solutions are derived in closed form and verified by comparison with existing solutions. As a consequence of our analysis, we present an interesting result on the uniqueness (within a certain class of smooth curves) of the circular inhomogeneity as the only inhomogeneity which, under the given conditions, leads to a uniform interior stress field when the interface function is constant (homogeneously imperfect interface).

Keywords: elastic inhomogeneity; imperfect interface; uniform stress.

1. Introduction

Problems involving elastic inhomogeneities with imperfect bonding at the inhomogeneity/matrix interface (imperfect interface) are receiving an increasing amount of attention in the literature (see, for example, Ru & Schiavone, 1997 for an extensive literature review). Interest in these problems is motivated mainly by a desire to study interface damage in composites (for example, debonding, sliding and/or cracking across an interface) and its subsequent effect on the effective properties of composites.

One of the more widely used models of an imperfect interface is based on the assumption that tractions are continuous but displacements are discontinuous across the interface. More precisely, jumps in the displacement components are assumed to be proportional, in terms of 'spring-factor type' interface functions (which characterize the imperfect interface), to their respective interface traction components. Under these assumptions, Hashin (1991) has examined the case of a spherical inhomogeneity

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imperfectly bonded to a three-dimensional matrix. The analogous problem for plane deformations has been investigated by Gao (1995).

In this paper we use the same interface model to study the problem associated with an elastic inhomogeneity of 'arbitrary' shape embedded within an infinite matrix in antiplane shear under the assumption of uniform remote loading. Specifically, we aim to identify particular shapes of inhomogeneity and corresponding interface functions which guarantee *uniform* stresses inside the inhomogeneity. Using complex variable methods and conformal mapping techniques, we develop a novel and constructive method for identifying both the shape of the inhomogeneity and the corresponding interface function. We mention also that our method is sufficiently general to allow straightforward extension to the analogous problem in the more practical case of plane elastic deformations and in the study of neutral elastic inhomogeneities (see, for example, Ru, 1998, and Milton & Serkov, 2001)

This study was motivated by the result established in Ru & Schiavone (1997) which reports that the stress field inside the inhomogeneity is uniform when the inhomogeneity is circular and the interface is homogeneously imperfect (characterized by a *constant* interface function). This surprising result is in sharp contrast to the results obtained by Hashin (1991) and Gao (1995) for the corresponding problems in three-dimensional and plane elasticity, respectively, where, in each case, it is shown that the stress field inside the inhomogeneity is intrinsically non-uniform. The same has been shown for the corresponding antiplane problem in the case of an *elliptic* inhomogeneity (see, for example, Shen *et al.*, 2000). Consequently, it is of interest to investigate whether the circular inhomogeneity is unique in delivering a uniform interior stress field under these conditions and, further, which combination of inhomogeneity shape/interface functions will guarantee uniform interior stresses. These results are important in that they allow for the design of the interface (choice of interface function) to achieve a state of uniform stress inside an inhomogeneity, the practical significance of which lies in the fact that a uniform stress distribution is optimal in the sense that it eliminates stress peaks within the inhomogeneity. In many practical cases, it is the maximum stress (rather than the average stress) that dominates the mechanical failure of the inhomogeneity. Consequently, the results in this paper will provide valuable information for, say, the design of radically inhomogeneous functionally graded interfaces in composite materials, an area which has recently received considerable attention in the literature (see, for example, Suresh & Mortensen, 1997).

The formulation of the basic boundary value problem describing the antiplane deformation of an elastic inhomogeneity with imperfect interface is presented in Section 2. The case of an inhomogeneity of 'arbitrary' shape is discussed in Section 3. Here, we identify conditions (in terms of shape and interface function) which the smooth curve representing the boundary of the inhomogeneity must satisfy in order to achieve uniformity of interior stresses. In Section 4, we present examples of inhomogeneities together with corresponding interface functions which lead to uniform interior stress. Finally, in Section 5, we examine the circular inhomogeneity and show that, within a certain class of smooth curves, this shape of inhomogeneity is the only one, under the given conditions, which leads to a uniform interior stress field when the interface function is constant (i.e. when the interface is homogeneously imperfect, often referred to as an 'equal thickness' interphase layer).

2. Formulation

Consider a domain in $\mathbb{R}^2$, infinite in extent, containing a single internal elastic inhomogeneity, with elastic properties different from the surrounding matrix. The linearly elastic materials occupying the matrix and the inhomogeneity are assumed to be homogeneous and isotropic with associated shear moduli $\mu_1(>0)$ and $\mu_2(>0)$, respectively. At infinity, the prescribed deformation is such that the elastic antiplane deformation $u(x, y)$ in the matrix satisfies

$$u(x, y) = ax - by + O(1), \qquad x^2 + y^2 \to \infty,$$

where a and b are given real constants (remote stress parameters) and (x, y) is a generic point in $\mathbb{R}^2$. We represent the matrix by the domain S_1 and assume that the inhomogeneity occupies a region S_2. The inhomogeneity–matrix interface will be denoted by the curve Γ. In what follows, the subscripts 1 and 2 refer to the regions S_1 and S_2, respectively and $u_\alpha(x, y)$, $\alpha = 1, 2$ denotes the elastic (antiplane) deformation at the point (x, y) in S_α, respectively.

It is assumed that the inhomogeneity is imperfectly bonded to the matrix along Γ by the ‘spring-layer type’ interface referred to in Section 1. The interface condition on Γ is therefore given by

$$\beta(x, y)[u_1 - (u_2 + u^*)] = \mu_2 \frac{\partial u_2}{\partial n} = \mu_1 \frac{\partial u_1}{\partial n}, \quad \text{on} \quad \Gamma, \tag{2.1}$$

where n is the outward unit normal to Γ, $\beta(x, y) : \Gamma(\subset \mathbb{R}^2) \longrightarrow \mathbb{R}^+$ is the imperfect interface function and $u^*(x, y)$ represents the additional displacement induced within the inhomogeneity by a uniform (stress-free) eigenstrain specified below. In accordance with Hashin (1991), we note that, for a homogeneously imperfect interface (β is constant), if $\beta = 0$, the condition (2.1) reduces to the case of a traction-free interface while if β becomes infinite, (2.1) corresponds to a perfectly bonded interface. Consequently, the following boundary value problem describes the antiplane deformation of an inhomogeneity with imperfect interface of the form (2.1) (see Ru & Schiavone, 1997):

$$\begin{aligned} &\nabla^2 u_1 = 0 \qquad \text{in} \quad S_1, \\ &\nabla^2 u_2 = 0 \qquad \text{in} \quad S_2, \\ &\beta(x, y)(u_1 - u_2) = \mu_2 \frac{\partial u_2}{\partial n} + \beta(x, y) u^*(x, y), \qquad \mu_1 \frac{\partial u_1}{\partial n} = \mu_2 \frac{\partial u_2}{\partial n} \qquad \text{on} \quad \Gamma, \\ &u_1(x, y) = ax - by + O(1), \qquad x^2 + y^2 \to \infty. \end{aligned} \tag{2.2}$$

Denote by $\upsilon_i(x, y)$ the harmonic functions conjugate to $u_i(x, y)$. Since the external loading is self-equilibrated, $\upsilon_i(x, y)$ are single-valued and uniquely determined to within an integration constant and the corresponding complex potentials $\phi_1(z)$ and $\phi_2(z)$, with $z = x + \mathrm{i}y$, are analytic within S_1 and S_2, respectively. Thus,

$$2u_i(z) = \phi_i(z) + \overline{\phi_i(z)}, \quad \sigma_{13} - i\sigma_{23} = \mu_i \phi_i'(z), \quad z \in S_i \ (i = 1, 2). \tag{2.3}$$

Noting that

$$2\frac{\partial u_2}{\partial n} = \phi_2'(z)e^{in(z)} + \overline{\phi_2'(z)}e^{-in(z)}, \quad z \in \Gamma, \tag{2.4}$$

where $e^{in(z)}$ represents (in complex form) the outward normal to Γ at z, the boundary value problem (2.2) can be written in the following form:

$$\begin{aligned} &\phi_1(z) = \delta\phi_2(z) + (1-\delta)\overline{\phi_2(z)} + \alpha(z)[\phi_2'(z)e^{in(z)} + \overline{\phi_2'(z)}e^{-in(z)}] + u^*(z),\ z \in \Gamma, \\ &\phi_1(z) = Az + D + o(1), \quad |z| \to \infty. \end{aligned} \tag{2.5}$$

Here,

$$A = a + ib, \quad \alpha(z) \equiv \frac{\mu_2}{2\beta(z)} \geqslant 0, \quad \delta \equiv \frac{\mu_1+\mu_2}{2\mu_1} > \frac{1}{2}, \quad u^* = \omega z + \overline{\omega z}, \tag{2.6}$$

D is a (complex) constant and ω is a known (complex) constant determined by the uniform eigenstrain given inside the inhomogeneity. Without loss of generality, we have assumed that the origin of coordinates has been chosen such that the rigid-body displacement at infinity is zero.

We require that the stress field inside the inhomogeneity is uniform. Consequently, we take $\phi_2(z) = Bz + C,\ z \in S_2$, where B and C are complex constants. The conditions (2.5) now become

$$\begin{aligned} &\phi_1(z) = \delta(Bz + C) + (1-\delta)(\bar{B}\bar{z} + \bar{C}) + \alpha(z)[Be^{in(z)} + \overline{B}e^{-in(z)}] + u^*(z),\ z \in \Gamma, \\ &\phi_1(z) = Az + D + o(1), \quad |z| \to \infty. \end{aligned} \tag{2.7}$$

The problem now is to determine the shape of the inhomogeneity as well as $\alpha(z)$, B, C and $\phi_1(z)$ which are compatible with the interface condition (2.7).

3. Inhomogeneity of arbitrary shape

We suppose that the region S_1 (in the *z-plane*) is mapped onto the region $\sigma = \{|\xi| \geqslant 1\}$ (in the ξ*-plane*) by the function (see, for example, England, 1971)

$$z = w(\xi) = \sum_{n=-N}^{1} P_n\xi^n,\ N > 2,\ N \in \mathbb{N}, \tag{3.1}$$

where $w'(\xi) = \sum_{n=-N}^{1} nP_n\xi^{n-1} \neq 0$ for $|\xi| \geqslant 1$. Let

$$\phi_1(z) = \phi_1(w(\xi)) = \Phi_1(\xi),$$

where $\Phi_1(\xi)$ is an analytic function in the region $\mathcal{D}_1 = \{|\xi| > 1\}$ with a first-order pole at infinity. Therefore,

$$\Phi_1(\xi) = \sum_{n=-\infty}^{1} Q_n\xi^n, \quad Q_0 = AP_0 + D, \quad Q_1 = AP_1. \tag{3.2}$$

The interface condition in (2.7) now becomes

$$\sum_{n=-\infty}^{1} Q_n\xi^n = \delta\left(B\sum_{n=-N}^{1} P_n\xi^n + C\right) + (1-\delta)\left(\bar{B}\sum_{n=-N}^{1}\bar{P}_n\,\xi^{-n} + \bar{C}\right)$$
$$+\,\alpha(w(\xi))[B\mathrm{e}^{\mathrm{i}n(w(\xi))} + \overline{B}\mathrm{e}^{-\mathrm{i}n(w(\xi))}]$$
$$+\,\omega\sum_{n=-N}^{1} P_n\xi^n + \overline{\omega}\sum_{n=-N}^{1}\bar{P}_n\,\xi^{-n}, \tag{3.3}$$

where $\xi \in \partial\sigma = \{|\xi| = 1\}$. Next, we expand $\alpha(w(\xi))[B\mathrm{e}^{\mathrm{i}n(w(\xi))} + \overline{B}\mathrm{e}^{-\mathrm{i}n(w(\xi))}]$ in Laurent's series to obtain

$$\alpha(w(\xi)) = [B\mathrm{e}^{\mathrm{i}n(w(\xi))} + \overline{B}\mathrm{e}^{-\mathrm{i}n(w(\xi))}]^{-1}\sum_{n=-\infty}^{\infty} E_n\xi^n, \tag{3.4}$$

where

$$E_n = \frac{1}{2\pi\mathrm{i}}\int_{\partial\sigma}\alpha(w(\xi))[B\mathrm{e}^{\mathrm{i}n(w(\xi))} + \overline{B}\mathrm{e}^{-\mathrm{i}n(w(\xi))}]\frac{\mathrm{d}\xi}{\xi^{n+1}}$$

are fixed. The interface condition (3.3) now becomes

$$\sum_{n=-\infty}^{-1} Q_n\xi^n - \delta B\sum_{n=-N}^{-1} P_n\xi^n - (1-\delta)\bar{B}\bar{P}_1\xi^{-1} - \omega\sum_{n=-N}^{-1} P_n\xi^n - \overline{\omega}\bar{P}_1\,\xi^{-1} - \sum_{n=-\infty}^{-1} E_n\xi^n$$
$$= -Q_0 - Q_1\xi + \delta BP_0 + \delta BP_1\xi + \delta C + (1-\delta)\bar{C} + (1-\delta)\bar{B}\bar{P}_0 + (1-\delta)\bar{B}\sum_{n=1}^{N}\bar{P}_{-n}\xi^n$$
$$+\,E_0 + \sum_{n=1}^{\infty} E_n\xi^n + \omega P_1\xi + \omega P_0 + \overline{\omega}\bar{P}_0 + \overline{\omega}\sum_{n=1}^{N}\bar{P}_{-n}\,\xi^n \equiv 0,$$

by Liouville's theorem. Equating coefficients of ξ^{-n}, $n = -1, 0, 1, 2, \ldots, N, \ldots$, we obtain

$$Q_1 = \delta BP_1 + (1-\delta)\bar{B}\bar{P}_{-1} + E_1 + \omega P_1 + \overline{\omega}\bar{P}_{-1}, \tag{3.5}$$
$$Q_0 - \delta BP_0 = \delta C + (1-\delta)[\bar{C} + \bar{B}\bar{P}_0] + E_0 + \omega P_0 + \overline{\omega}\bar{P}_0, \tag{3.6}$$
$$Q_{-1} = \delta BP_{-1} + (1-\delta)\bar{B}\bar{P}_1 + \omega P_{-1} + \overline{\omega}\bar{P}_1 + E_{-1}, \tag{3.7}$$
$$Q_n - \delta BP_n - \omega P_n - E_n = 0, n = -2, \ldots, -N, \tag{3.8}$$
$$Q_{-n} - E_{-n} = 0, n \geqslant N+1. \tag{3.9}$$

Since the P_n are given and E_n are fixed, we can use (3.5)–(3.9) to find the Q_n, $n = \ldots, -2, -1, 0, 1$. In addition, equating coefficients of ξ^n, $n = 2, 3, \ldots, N, \ldots$, we obtain

$$E_n = -[\overline{\omega} + (1-\delta)\bar{B}]\bar{P}_{-n}, \quad n = 2, 3, \ldots, N, \tag{3.10}$$
$$E_n = 0, \quad n > N. \tag{3.11}$$

Next, from England (1971), on $\partial\sigma$,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}n(w(\xi))} &= \xi \frac{w'(\xi)}{|w'(\xi)|} \\ &= \frac{\sum_{n=-N}^{1} nP_n\xi^n}{|w'(\xi)|}, \\ \mathrm{e}^{-\mathrm{i}n(w(\xi))} &= \frac{\sum_{n=-N}^{1} n\bar{P}_n\xi^{-n}}{|w'(\xi)|}. \end{aligned}$$

Thus, from (3.4), the interface function is given by

$$\begin{aligned} \alpha(w(\xi)) &= [B\mathrm{e}^{\mathrm{i}n(w(\xi))} + \overline{B}\mathrm{e}^{-\mathrm{i}n(w(\xi))}]^{-1}\left(\sum_{n=-\infty}^{1} E_n\xi^n - [\overline{\omega} + (1-\delta)\bar{B}]\sum_{n=2}^{N}\bar{P}_{-n}\,\xi^n\right) \\ &= \frac{|w'(\xi)|(\sum_{n=-\infty}^{1} E_n\xi^n - [\overline{\omega} + (1-\delta)\bar{B}]\sum_{n=2}^{N}\bar{P}_{-n}\,\xi^n)}{B\sum_{n=-N}^{1} nP_n\xi^n + \bar{B}\sum_{n=-N}^{1} n\bar{P}_n\xi^{-n}}. \end{aligned} \tag{3.12}$$

Finally, we must impose the conditions

$$\operatorname{Im}\alpha(w(\xi)) = 0, \quad \operatorname{Re}\alpha(w(\xi)) > 0, \tag{3.13}$$

in order to maintain the physical meaning of the interface function β.

REMARK 1 From (3.2) and (3.5)–(3.6),

$$AP_1 = \delta BP_1 + (1-\delta)\bar{B}\bar{P}_{-1} + E_1 + \omega P_1 + \overline{\omega}\bar{P}_{-1}, \tag{3.14}$$

$$AP_0 + D = \delta BP_0 + \delta C + (1-\delta)\bar{C} + (1-\delta)\bar{B}\bar{P}_0 + E_0 + \omega P_0 + \overline{\omega}\bar{P}_0. \tag{3.15}$$

These two equations relate the constants B and C (describing the uniform stress inside the inhomogeneity) to the constants A and D (describing the uniform remote stress).

Let $\alpha_0 = \dfrac{\mu_2}{2\beta} > 0$ corresponding to the case where $\beta > 0$ is uniform. The conditions (3.13) are satisfied if the coefficients in (3.12) take the form

$$E_1 = \tfrac{\alpha_0}{|P_1|}(BP_1 - \bar{B}\bar{P}_{-1}), \tag{3.16}$$

$$E_0 = 0, \tag{3.17}$$

$$E_{-1} = \tfrac{\alpha_0}{|P_1|}(\bar{B}P_1 - BP_{-1}), \tag{3.18}$$

$$E_{-n} = -nP_{-n}B\tfrac{\alpha_0}{|P_1|}, \quad n = 2, \ldots, N, \tag{3.19}$$

$$E_{-n} = 0, \quad n \geqslant N+1, \tag{3.20}$$

$$\left[\overline{\omega} + (1-\delta)\bar{B} - \bar{B}n\tfrac{\alpha_0}{|P_1|}\right]\bar{P}_{-n} = 0, \quad n = 2, \ldots, N. \tag{3.21}$$

From (3.21), it follows that either $\bar{P}_{-n} = 0, \quad n = 2, \ldots, N$ or $\exists\ \bar{P}_{-m} \neq 0$ ($m \in \{2, 3, \ldots, N\}$) and then

$$\overline{\omega} + (1-\delta)\bar{B} = \bar{B}m\frac{\alpha_0}{|P_1|}. \tag{3.22}$$

In either case, from (3.12), we obtain

$$\alpha(w(\xi)) = \frac{\alpha_0}{|P_1|}|w'(\xi)| > 0. \tag{3.23}$$

From (3.1), it follows that the most general shape of the inhomogeneity (with uniform interior stress field) is given by

$$z = w(\xi) = P_1\xi + P_0 + P_{-1}\xi^{-1} + P_{-m}\xi^{-m} \tag{3.24}$$

with corresponding interface function (3.23).

4. Examples

4.1 *Circular inhomogeneity*

In the case of a circular inhomogeneity of radius R, we have $P_{-n} = 0,\ n = 0, 1, \ldots, N$, $P_1 = R$, $w(\xi) = P_1\xi$, $w'(\xi) = P_1$ (see, for example, Muskhelishvili, 1953). From (3.23), the interface function is uniform and given by

$$\alpha = \alpha_0 = \frac{\mu_2}{2\beta} > 0. \tag{4.1}$$

Further, from (3.14)–(3.17), we have

$$B = \frac{A-\omega}{\delta + \dfrac{\alpha}{R}}, \quad C = \operatorname{Re} D + \mathrm{i}\frac{\operatorname{Im} D}{2\delta - 1}. \tag{4.2}$$

Consequently, the (uniform) stress field inside the inhomogeneity is given by

$$\phi_2(z) = \frac{(A-\omega)z}{\delta + \dfrac{\alpha}{R}} + \frac{D\delta + \bar{D}(\delta - 1)}{2\delta - 1}, \quad z \in S_2.$$

Also, from (3.2) and (3.7),

$$\Phi_1(\xi) = AR\xi + D + \frac{(1-\delta)\bar{B}R + \bar{\omega}R + \bar{B}\alpha}{\xi},$$

so that the stress field inside the matrix is given by

$$\phi_1(z) = Az + D + \frac{R^2}{z\left(\delta + \dfrac{\alpha}{R}\right)}\left[\bar{A}\left(1 - \delta + \frac{\alpha}{R}\right) + \bar{\omega}(2\delta - 1)\right], \quad z \in S_1.$$

These results agree with those presented in Ru & Schiavone (1997) and Schiavone (2002). That is, under the given conditions, the circular inhomogeneity has uniform interior stress when the interface function is uniform (homogeneously imperfect interface) and is given by (4.1).

4.2 *Elliptic inhomogeneity*

In the case of an elliptic inhomogeneity , we have $z = w(\xi) = R\left(\xi + \dfrac{k^2}{\xi}\right),\ k \in (0,1)$, $R > 0$ (see, for example, Muskhelishvili, 1953). Then $w'(\xi) = R\left(1 - \dfrac{k^2}{\xi^2}\right)$ and $\overline{w'(\xi)} = R(1 - k^2\xi^2)$. Thus, from (3.24), $P_1 = R,\ P_{-1} = Rk^2,\ P_0 = 0,\ P_{-n} = 0, n = 2, \dots, N$. From (3.23), the interior stress field is uniform when the interface function takes the form

$$\alpha(w(\xi)) = \alpha_0 \left|1 - \frac{k^2}{\xi^2}\right| > 0. \tag{4.3}$$

Note that this excludes the case when the interface parameter α is constant (homogeneously imperfect interface). This result is corroborated by the numerical evidence presented in Shen *et al.* (2000).

From (3.8), (3.9), (3.19), (3.20),

$$Q_n = 0, n \leqslant -2. \tag{4.4}$$

Also, from (3.5)–(3.7), (3.16)–(3.18),

$$Q_{-1} = BRk^2\left(\delta - \frac{\alpha_0}{R}\right) + \bar{B}R\left(1 - \delta + \frac{\alpha_0}{R}\right) + R(\omega k^2 + \bar{\omega}), \tag{4.5}$$

$$Q_0 = \delta C + (1-\delta)\bar{C} \ = D \quad \text{(see (3.2))}, \tag{4.6}$$

$$Q_1 = \delta BR + (1-\delta)\bar{B}Rk^2 \ + \alpha_0(B - \bar{B}k^2) + \omega R - \overline{\omega}Rk^2 \ . \tag{4.7}$$

Using (3.2), equation (4.7), leads to the following expression for the constant B in terms of the (prescribed) constant A:

$$B = \operatorname{Re} B + \mathrm{i} \operatorname{Im} B, \tag{4.8}$$

$$\operatorname{Re} B = \frac{\operatorname{Re} A}{\left(\delta + \dfrac{\alpha_0}{R}\right)(1-k^2) + k^2 + \operatorname{Re}\omega(1+k^2)},$$

$$\operatorname{Im} B = \frac{\operatorname{Im} A}{\left(\delta + \dfrac{\alpha_0}{R}\right)(1+k^2) - k^2 + \operatorname{Im}\omega(1-k^2)}.$$

The constant C is related to the constant D by (4.6):

$$C = \operatorname{Re} D + \mathrm{i}\frac{1}{2\delta - 1}\operatorname{Im} D. \tag{4.9}$$

Consequently, the (uniform) stress field inside the inhomogeneity is given by

$$\phi_2(z) = Bz + C \quad z \in S_2,$$

where the constants B and C are given by (4.8)–(4.9). Also, from (4.4)–(4.7), the stress field inside the matrix is given by

$$\Phi_1(\xi) = \frac{Q_{-1}}{\xi} + Q_0 + Q_1\xi,\ \ \xi \in \mathcal{D}_1 = \{|\xi| > 1\}. \tag{4.10}$$

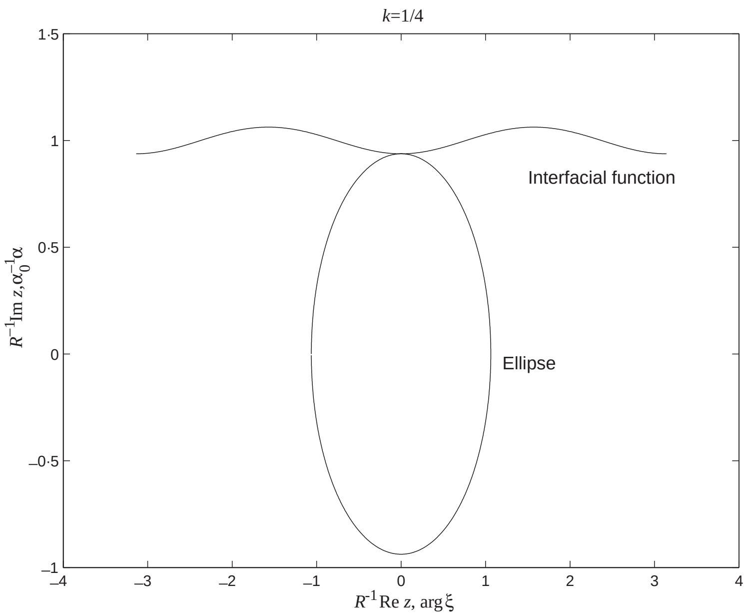

FIG. 1. Ellipse with corresponding interface function.

Finally, the function $\phi_1(z)$, $z \in S_1$ and the interface parameter $\alpha(z)$, $z \in \Gamma$, can be calculated using (4.10) and (4.3), respectively, and the fact that z and ξ are related through the relation $z = R\left(\xi + \dfrac{k^2}{\xi}\right)$, $k \in (0, 1)$, $R > 0$. Figure 1 plots the ellipse with corresponding interface function in the case $k = \frac{1}{4}$.

4.3 *Inhomogeneity in the shape of a hypotrochoid*

In the case of an inhomogeneity in the shape of a hypotrochoid, we have $z = w(\xi) = R\left(\xi + \dfrac{\kappa}{\xi^m}\right)$, $R > 0$, $\kappa \in \left(0, \dfrac{1}{m}\right)$, $m = 2, 3, \ldots$ (see, for example, Muskhelishvili, 1953). Here, $m + 1$ represents the number of 'corners' of the hypotrochoid and κ reflects its smoothness. Then $w'(\xi) = R\left(1 - \dfrac{m\kappa}{\xi^{m+1}}\right)$. Thus, from (3.24), $P_1 = R$, $P_{-1} = 0$, $P_0 = 0$, $P_{-m} = R\kappa > 0$ and $P_{-n} = 0$, $n \in [2, N]$, if $n \neq m \geqslant 2$. That is,

$$P_{-1} = P_0 = 0, \quad P_1 = R, \quad P_{-n} = R\kappa\delta_{mn}, \quad n \in [0, N], \tag{4.11}$$

where δ_{mn} is the Kronecker delta. From (3.23), the interior stress field is uniform when the interface function takes the form

$$\alpha(w(\xi)) = \alpha_0 \left|1 - \frac{m\kappa}{\xi^{m+1}}\right| > 0. \tag{4.12}$$

From (3.16)–(3.20) and (4.11),

$$E_1 = \alpha_0 B, \tag{4.13}$$

$$E_0 = 0, \tag{4.14}$$

$$E_{-1} = \alpha_0 \bar{B}, \tag{4.15}$$

$$E_{-n} = -n\kappa\alpha_0\delta_{mn} B, \quad n = 2, \ldots, N, \tag{4.16}$$

$$E_{-n} = 0, \quad n \geqslant N + 1. \tag{4.17}$$

Also, from (3.22)

$$\overline{\omega} + (1-\delta)\bar{B} = \bar{B}m\frac{\alpha_0}{R},$$

so that

$$B = \frac{\omega}{\dfrac{m\alpha_0}{R} + \delta - 1}. \tag{4.18}$$

Next, from (3.8)–(3.9), (4.11) and (4.16)–(4.17),

$$Q_{-n} = R\kappa\delta_{mn}\left[\omega + B\left(\delta - \frac{n\alpha_0}{R}\right)\right], \tag{4.19}$$

$$Q_{-n} = 0, \; n \geqslant N + 1. \tag{4.20}$$

Also, from (3.5)–(3.7), (4.11), (4.13)–(4.15) and (3.2),

$$Q_1 = \delta BR + B\alpha_0 + \omega R = AR\ , \tag{4.21}$$

$$Q_0 = \delta C + (1-\delta)\bar{C}\ = D, \tag{4.22}$$

$$Q_{-1} = \bar{B}R\left(1 - \delta + \frac{\alpha_0}{R}\right) + \overline{\omega}R. \tag{4.23}$$

Using (4.21), the constant B can be written in terms of the (prescribed) constant A as

$$B = \frac{R(A-\omega)}{\delta R + \alpha_0}. \tag{4.24}$$

The constants C and D are again related by (4.22):

$$D = \delta C + (1-\delta)\bar{C}, \tag{4.25}$$

$$C = \operatorname{Re} D + \mathrm{i}\frac{1}{2\delta - 1}\operatorname{Im} D. \tag{4.26}$$

Consequently, the (uniform) stress field inside the inhomogeneity is given by

$$\phi_2(z) = Bz + C \quad z \in S_2,$$

where the constants B and C are given by (4.24) and (4.26). Also, from (4.19)–(4.23), the stress field inside the matrix is given by

$$\Phi_1(\xi) = \sum_{n=2}^{N} Q_{-n}\xi^{-n} + \frac{Q_{-1}}{\xi} + Q_0 + Q_1\xi, \;\; \xi \in \mathcal{D}_1 = \{|\xi| > 1\}. \tag{4.27}$$

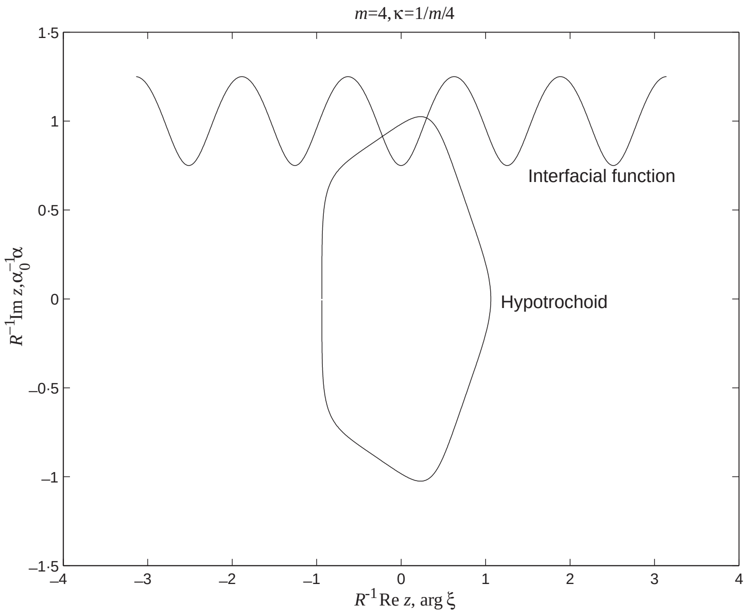

FIG. 2. Hypotrochoid with corresponding interface function.

Again, the function $\phi_1(z), z \in S_1$ and the interface parameter $\alpha(z), z \in \Gamma$, can be calculated using (4.27) and (4.12), respectively, and the fact that z and ξ are related through the relation $z = w(\xi) = R\left(\xi + \frac{\kappa}{\xi^m}\right)$, $R > 0, \kappa \in \left(0, \frac{1}{m}\right)$, $m = 2, 3, \ldots$.

Figure 2 plots a hypotrochoid with corresponding interface function in the case $m = 4$, $\kappa = 1/4m$.

5. Uniqueness of the circular inhomogeneity

In Ru & Schiavone (1997), it was proved that for antiplane shear deformations, under the assumption of uniform remote loading, the stress field inside a circular inhomogeneity is uniform even when the interface is homogeneously imperfect (characterized by a *constant* interface function $\alpha = \alpha_0$). This surprising result is in sharp contrast to the results obtained by Hashin (1991) and Gao (1995) for the corresponding problems in three-dimensional and plane elasticity, respectively, where, in each case, it is shown that the stress field inside the inhomogeneity is intrinsically non-uniform. The same has been shown for the corresponding antiplane problem in the case of an *elliptic* inhomogeneity (see, for example, Shen *et al.*, 2000). Consequently, it is of interest to investigate whether the circular inhomogeneity is unique in delivering a uniform interior stress field under the stated conditions.

To this end, suppose that the interface function is constant and that the ‘arbitrary’ curve given by (3.24) encloses a uniform stress field. Then, from (3.23),

$$\alpha(w(\xi)) = \frac{\alpha_0}{|P_1|}|w'(\xi)| = c_1 = \text{ constant} \in \mathbb{R}^+.$$

That is,

$$|P_1 - P_{-1}\xi^{-2} - mP_{-m}\xi^{-(m+1)}| = \frac{c_1|P_1|}{\alpha_0}, \ \forall \xi \in \partial\sigma = \{|\xi| = 1\}, m \in \{2, 3, \dots, N\}. \tag{5.1}$$

It is clear that, in order to satisfy (5.1), we must have

$$P_{-1} = 0, \ P_{-m} = 0 \text{ and } c_1 = \alpha_0$$

This means that

$$z = w(\xi) = P_1\xi + P_0, \ \ \xi \in \partial\sigma$$

which, in the *z-plane* corresponds to a circle of centre (Re P_0, Im P_0), radius P_1.

This result indicates that, under the given conditions, the circular inhomogeneity is the only inhomogeneity within the class of curves corresponding to (3.1) which delivers a uniform interior stress field when the interface is characterized as homogeneously imperfect (i.e. when the interface parameter α is constant with value given by $\alpha = \alpha_0 = \frac{\mu_2}{2\beta} > 0$).

6. Conclusions

In this paper, we consider an inhomogeneity of 'arbitrary' shape embedded within an infinite isotropic elastic medium (matrix) subjected to antiplane shear deformations under the assumption of uniform remote loading. The inhomogeneity–matrix interface is assumed to be imperfect, characterized by a single interface function. Under these assumptions, we use complex variable methods to solve the problem concerned with identifying the shape of the inhomogeneity and the form of the corresponding interface function which leads to a uniform interior stress field. Specific examples are presented and verified, when possible, by comparison with existing solutions. As a consequence of our analysis, we present an interesting result on the uniqueness (within a certain class of smooth curves) of the circular inhomogeneity as the only inhomogeneity which, under the given conditions, leads to a uniform interior stress field when the interface function is constant (homogeneously imperfect interface).

These results are important in that they allow for the design of the interface (choice of interface function) to achieve a state of uniform stress inside an inhomogeneity, the practical significance of which lies in the fact that a uniform stress distribution is optimal in the sense that it eliminates stress peaks within the inhomogeneity. In many practical cases, it is the maximum stress (rather than the average stress) that dominates the mechanical failure of the inhomogeneity. Consequently, the results in this paper will provide valuable information for, say, the design of radically inhomogeneous functionally graded interfaces in composite materials, an area which has recently received considerable attention in the literature (see, for example, Suresh & Mortensen, 1997). Finally, it is clear that our method is sufficiently general to allow straightforward extension to the more practical case of plane elastic deformations and in the study of neutral elastic inhomogeneities (see, for example, Ru, 1998 and Milton & Serkov, 2001).

Acknowledgements

The work is supported by the Natural Science and Engineering Research Council of Canada through grant NSERC OGP 115112.

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