ELECTROMAGNETIC SCATTERING FROM AN ANISOTROPIC IMPEDANCE HALF-PLANE AT OBLIQUE INCIDENCE: THE EXACT SOLUTION

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Summary

Scattering of a plane electromagnetic wave from an anisotropic impedance half-plane at skew incidence is considered. The two matrix surface impedances involved are assumed to be complex and different. The problem is solved in closed form. The boundary-value problem reduces to a system of two first-order difference equations with periodic coefficients subject to a symmetry condition. The main idea of the method developed is to convert the system of difference equations into a scalar Riemann–Hilbert problem on a finite contour of a hyperelliptic surface of genus 3. A constructive procedure for its solution and the solution of the associated Jacobi inversion problem is proposed and described in detail. Numerical results for the edge diffraction coefficients are reported.

1. Introduction

During the last 50 years significant progress has been made in the mathematical theory of diffraction which studies the influence of material properties on edge diffraction phenomena. The achievements have been made in large part due to the use of the Leontovich impedance boundary conditions in modeling and the Wiener–Hopf–Jones (**1**, **2**) and Maliuzhinets (**3**) methods for the solution. The former method works successfully for half-planes and right-angled wedges. Wedges of arbitrary angles are normally treated by the Maliuzhinets technique. Recently, Daniele (**4**) showed that the Wiener–Hopf–Jones method is applicable for scattering problems for wedges as well. In fact, for certain canonical problems of acoustic and electromagnetic diffraction (**5** to **8**), both techniques are equivalent to the solution of a scalar Riemann–Hilbert problem on a Riemann surface (in some particular cases on a sphere, that is, on a plane).

The scalar problem of electromagnetic diffraction of a plane wave from a semi-infinite impedance plane at oblique incidence was solved by Senior (**9**) by the Wiener–Hopf approach. By using the Sommerfeld integral, Maliuzhinets (**3**) found a closed-form solution to the problem of acoustic diffraction by an impedance wedge with two different impedance parameters. The solution was

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derived in terms of special functions later called the Maliuzhinets functions. The homogeneous isotropic two-face impedance half-plane problem of electromagnetic diffraction in the oblique incidence case was analysed by Lüneburg and Serbest (**10**) by using the 'range restrictions' concept.

For the more complicated anisotropic case, two different matrix surface impedances

$$\mathbf{Z}^{\pm} = Z_0 \begin{pmatrix} 0 & -\eta_2^{\pm} \\ \eta_1^{\pm} & 0 \end{pmatrix} \tag{1.1}$$

are involved. Here the superscripts identify the upper and the lower faces of the plane, $\eta_1^{\pm}$ and $\eta_2^{\pm}$ are dimensionless parameters, $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance, and μ_0 and ϵ_0 are the permeability and permittivity of free space. On the screen faces, the impedance boundary conditions are $\mathbf{E} = Z^{\pm}\mathbf{H}$, where $\mathbf{E} = (E_\rho, E_z)^\top$ and $\mathbf{H} = (H_\rho, H_z)^\top$ are the total electric and magnetic fields. In the normal incidence case, the governing system of Wiener–Hopf and first-order difference equations is uncoupled. The solution is obtained by quadratures.

When the incident wave is not orthogonal to the edge of the structure the equations are in general coupled. For special cases of skew incidence the equations reduce to scalar Maliuzhinets equations. Examples include those considered by Bernard (**11**) and Lyalinov and Zhu (**12**). For the case when the matrices $\mathbf{Z}^+$ and $\mathbf{Z}^-$ are different, Senior (**13**) used the Wiener–Hopf and Maliuzhinets ideas to reduce the problem to functional equations. The former method gives rise to a vector Wiener–Hopf problem for four pairs of unknown functions. Hurd and Lüneburg (**14**) found a closed-form solution in the case $\mathbf{Z}^+ = \mathbf{Z}^-$ when the problem is reducible to the problem of factorization of a 2×2 matrix. The matrix was factorized by employing the Daniele (**15**) method. In the more general case, when $\mathbf{Z}^+ \neq \mathbf{Z}^-$, the Daniele method leads to a system of highly nonlinear equations. These equations have to be solved in order to eliminate an essential singularity of the Wiener–Hopf factors. Apart from the elliptic case (the number of nonlinear equations equals one) (**15**, **14**) so far there is no constructive (exact or approximate) technique for its solution.

Senior and Topsakal (**16**) employed the Sommerfeld–Maliuzhinets formulation and reduced the problem to a second-order difference equation and then converted it into an integral equation. An approximate solution was given for the case $\mathbf{Z}^+ = \mathbf{Z}^-$. A method of the Riemann bilinear relations for abelian integrals for a partial solution of a second-order difference was proposed by Legault and Senior (**17**). The principal complexity of their method is to eliminate the polar and cyclic periods of the solution in order to make it single-valued. The procedure requires solving a system of nonlinear equations with respect to the unknown singularities of the abelian integrals of the third kind. Its constructive solution is found only for those cases which are equivalent to the elliptic case.

The main motivation behind the present work was to solve in closed form the problem on electromagnetic scattering of a plane wave from an anisotropic impedance half-plane at oblique incidence. The entries of the impedance matrices $\mathbf{Z}^+$ and $\mathbf{Z}^-$ could be complex and different. Antipov and Silvestrov (**6**, **7**) proposed a unified approach for a general class of problems in electromagnetic scattering. It generalizes the Wiener–Hopf and Maliuzhinets technique for systems of Wiener–Hopf and Maliuzhinets equations. The proposed method rephrases the systems of governing equations as a scalar Riemann–Hilbert problem on a Riemann surface. The Wiener–Hopf factors expressed through the solution of the boundary-value problem on a Riemann surface are single-valued. The essential singularity of the solution at infinity is removed by fixing certain free parameters which solve the associated Jacobi inversion problem. Its solution always exists and can be constructed in terms of the zeros of the Riemann θ-function. By this method, a problem on E-polarization of a right-angled magnetically conductive wedge was solved by quadratures (**8**). It was pointed out (**6**) that the problem of electromagnetic scattering at skew incidence from an anisotropic half-plane is solvable by quadratures.

The main aims of this work are (i) to construct the actual solution to the problem, (ii) to develop an efficient procedure for the associated Jacobi inversion problem, and (iii) to derive formulae for electric and magnetic fields and to compute the diffraction coefficients.

In section 2, the governing equations are converted into two vector first-order difference equations (2.18) subject to a symmetry condition (2.19) and additional conditions (2.5) and (2.17). Section 3 derives a vector Riemann–Hilbert problem (3.14) on a system of segments. In section 4, this problem reduces to a scalar Riemann–Hilbert problem (4.3) on two segments of a hyperelliptic surface of genus 3 and to a Jacobi inversion problem (4.23). The solution of the inversion problem requires calculating the Riemann constants and finding the zeros of the associated Riemann θ-function of genus 3. Formulae (5.19) and (5.33) for the Riemann constants are derived and a numerical algorithm for the Jacobi problem is proposed in section 5. Section 6 presents the exact solution (6.10), (2.10) to the physical problem and fixes arbitrary constants C_j $(j = 1, 2, \dots, 15)$ in (6.5). The case of normal incidence is considered in section 7. It is shown that the problem is equivalent to two scalar Riemann–Hilbert problems on a segment of a complex plane. These problems are solved in terms of one quadrature. The reflected, surface and diffracted waves are recovered in section 8. Numerical results for the diffraction coefficients are reported and discussed. It is proved that the electromagnetic field and the diffraction coefficients are invariant with respect to the transformation $(\theta, \theta_0, \beta, \eta_1^{\pm}, \eta_2^{\pm}) \to (-\theta, -\theta_0, \pi - \beta, \eta_1^{\mp}, \eta_2^{\mp})$, where θ is the angle of observation, β is the angle of incidence, and $\pi - \theta_0$ is the angle between the screen and the incident plane wave.

2. Formulation

PROBLEM 2.1 (Main problem) *Let* **S** *be a thin semi-infinite anisotropic impedance sheet* $\{0 < \rho < \infty, \theta = \pm\pi \mp 0, -\infty < z < +\infty\}$ *with surface impedances* η_1^+, η_2^+ *on the upper side* $\mathbf{S}^+$ $(\theta = \pi - 0)$ *and* η_1^-, η_2^- *on the lower surface* $\mathbf{S}^-$ $(\theta = -\pi + 0)$ *(Fig. 1). The impedances may be*

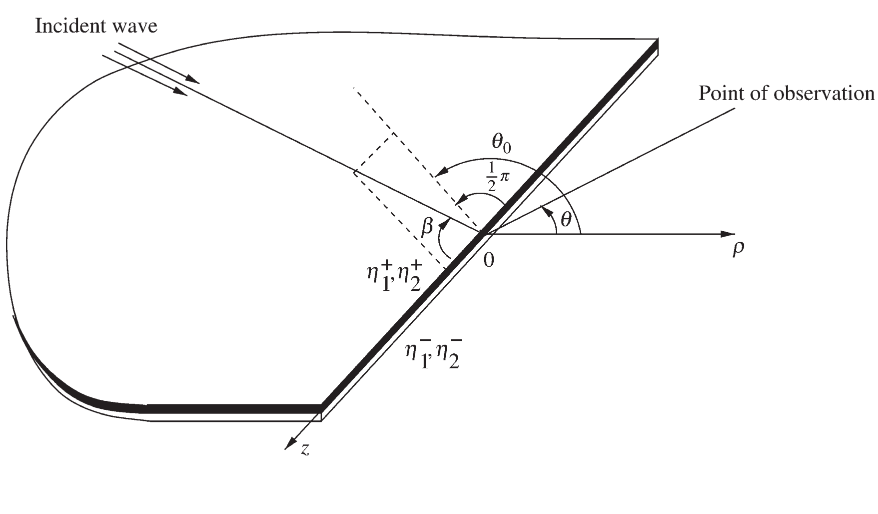

Fig. 1 Geometry of the problem

complex. The screen **S** *is illuminated by a plane wave incident obliquely whose z-components are*

$$E_z^i = e_1 e^{ik\rho\sin\beta\cos(\theta-\theta_0)-ikz\cos\beta}, \quad Z_0 H_z^i = e_2 e^{ik\rho\sin\beta\cos(\theta-\theta_0)-ikz\cos\beta}, \tag{2.1}$$

where k *is the wave number* ($\operatorname{Im} k \leqslant 0$), $\beta \in (0,\pi)$ *is the angle of incidence, the angle* $\theta_0 \in (-\pi,\pi)\backslash\{0\}$ *defines the direction of incidence,* e_1 *and* e_2 *are prescribed parameters, and a time factor* $e^{i\omega t}$ *is suppressed. The two components of the electric and magnetic field* $V_1 = E_z$ *and* $V_2 = Z_0 H_z$ *solve the Helmholtz equation*

$$(\nabla^2 + k^2)V_j = 0, \quad (\rho,\theta,z) \in \mathbb{R}^3\backslash\mathbf{S}, \tag{2.2}$$

and satisfy the following boundary conditions (**13**)*:*

$$\frac{1}{\rho}\frac{\partial V_j}{\partial\theta} + (-1)^j\cos\beta\frac{\partial V_{3-j}}{\partial\rho} \pm ik\hat{\eta}_j^{\pm}\sin^2\beta V_j = 0, \quad \theta = \pm\pi \mp 0, \quad j = 1,2, \tag{2.3}$$

where $\hat{\eta}_1^{\pm} = 1/\eta_1^{\pm}$, $\hat{\eta}_2^{\pm} = \eta_2^{\pm}$.

In this section the boundary-value problem for the Helmholtz equation will be reduced to two vector difference equations with periodic coefficients. The total field $\mathbf{V} = (V_1, V_2)$ is sought in the form of the Sommerfeld integral (**3**)

$$\mathbf{V}(\rho,\theta,z) = \frac{e^{-ikz\cos\beta}}{2\pi i}\int_\gamma e^{ik\rho\sin\beta\cos s}\boldsymbol{\mathcal{S}}(s+\theta)ds. \tag{2.4}$$

Here γ is the Sommerfeld contour. It consists of two loops symmetric with respect to the origin. The asymptotes for the branches are the lines $s = \frac{3}{2}\pi$ and $s = -\frac{1}{2}\pi$ for the upper loop and the lines $s = \frac{1}{2}\pi$ and $s = -\frac{3}{2}\pi$ for the lower loop. The spectral vector function $\boldsymbol{\mathcal{S}}(s) = (\mathcal{S}_1(s), \mathcal{S}_2(s))$ is analytic everywhere in the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$ apart from the point $s = \theta_0$, where its components have a simple pole with the residues defined by the incident field (2.1)

$$\operatorname*{res}_{s=\theta_0} \mathcal{S}_1(s) = e_1, \quad \operatorname*{res}_{s=\theta_0} \mathcal{S}_2(s) = e_2. \tag{2.5}$$

At the infinite points $s = x \pm i\infty$ ($|x| < \infty$), the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ are bounded.

Because of the symmetry of the Sommerfeld contour in (2.4) the boundary conditions can be rephrased in terms of the spectral functions as the system of the Maliuzhinets type difference equations

$$\begin{aligned}&\left(\sin s \pm \hat{\eta}_j^{\pm}\sin\beta\right)\mathcal{S}_j(s\pm\pi) + (-1)^j\cos s\cos\beta\mathcal{S}_{3-j}(s\pm\pi)\\ &\quad= \left(-\sin s \pm \hat{\eta}_j^{\pm}\sin\beta\right)\mathcal{S}_j(-s\pm\pi) + (-1)^j\cos s\cos\beta\mathcal{S}_{3-j}(-s\pm\pi), \quad j = 1,2.\end{aligned} \tag{2.6}$$

To satisfy the boundary conditions, we introduce the following four new functions:

$$\Phi_j^{\pm}(s) = \left(-\sin s \pm \hat{\eta}_j^{\pm}\sin\beta\right)\mathcal{S}_j(s) - (-1)^j\cos s\cos\beta\mathcal{S}_{3-j}(s), \quad j = 1,2. \tag{2.7}$$

Clearly, the transformation

$$\mathcal{T}\colon (\theta,\theta_0,\beta,\eta_1^{\pm},\eta_2^{\pm}) \to (-\theta,-\theta_0,\pi-\beta,\eta_1^{\mp},\eta_2^{\mp}) \tag{2.8}$$

does not change the electromagnetic field. The general solution to the main problem invariant to the transformation $\mathcal{T}$ is given by the Sommerfeld integral (2.4) with the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ being expressed through the functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$ as follows:

$$\mathcal{S}_1(s) = \tfrac{1}{2}\left[\mathcal{S}_1^{+}(s) + \mathcal{S}_1^{-}(s)\right], \quad \mathcal{S}_2(s) = \tfrac{1}{2}\left[\mathcal{S}_2^{+}(s) + \mathcal{S}_2^{-}(s)\right], \tag{2.9}$$

where

$$\mathcal{S}_j^{\pm}(s) = \frac{1}{\Gamma_s(\mp\hat{\eta}_1^{\pm}, \mp\hat{\eta}_2^{\pm})}\left[\left(-\sin s \pm \hat{\eta}_{3-j}^{\pm}\sin\beta\right)\Phi_j^{\pm}(s) + (-1)^j \cos s\cos\beta\,\Phi_{3-j}^{\pm}(s)\right], \quad j = 1, 2, \tag{2.10}$$

and

$$\Gamma_s(a, b) = (\sin s + a\sin\beta)(\sin s + b\sin\beta) + \cos^2 s\cos^2\beta. \tag{2.11}$$

The boundary conditions (2.6) for the functions $\mathcal{S}_1^{\pm}(s)$ and $\mathcal{S}_2^{\pm}(s)$ are equivalent to the following conditions:

$$\frac{1}{\Gamma_s(\mp 1/\eta_1^{\pm}, \mp\eta_2^{\pm})}\left[\Gamma_s\left(\pm\frac{1}{\eta_1^{\mp}}, \mp\eta_2^{\pm}\right)\Phi_1^{\pm}(-s \mp \pi) \mp \frac{1}{\eta_1}\cos s\sin 2\beta\,\Phi_2^{\pm}(-s \mp \pi)\right]$$
$$= \frac{1}{\Gamma_s(\pm 1/\eta_1^{\pm}, \pm\eta_2^{\pm})}\left[\Gamma_s\left(\mp\frac{1}{\eta_1^{\mp}}, \pm\eta_2^{\pm}\right)\Phi_1^{\pm}(s \mp \pi) \mp \frac{1}{\eta_1}\cos s\sin 2\beta\,\Phi_2^{\pm}(s \mp \pi)\right],$$
$$\frac{1}{\Gamma_s(\mp 1/\eta_1^{\pm}, \mp\eta_2^{\pm})}\left[\Gamma_s\left(\mp\frac{1}{\eta_1^{\pm}}, \pm\eta_2^{\mp}\right)\Phi_2^{\pm}(-s \mp \pi) \pm \eta_2\cos s\sin 2\beta\,\Phi_1^{\pm}(-s \mp \pi)\right]$$
$$= \frac{1}{\Gamma_s(\pm 1/\eta_1^{\pm}, \pm\eta_2^{\pm})}\left[\Gamma_s\left(\pm\frac{1}{\eta_1^{\pm}}, \mp\eta_2^{\mp}\right)\Phi_2^{\pm}(s \mp \pi) \pm \eta_2\cos s\sin 2\beta\,\Phi_1^{\pm}(s \mp \pi)\right],$$

$$\Phi_j^{\pm}(s \pm \pi) = \Phi_j^{\pm}(-s \pm \pi), \quad j = 1, 2, \tag{2.12}$$

Here

$$\eta_1 = 2\left(\frac{1}{\eta_1^{+}} + \frac{1}{\eta_1^{-}}\right)^{-1}, \quad \eta_2 = \frac{\eta_2^{+} + \eta_2^{-}}{2}. \tag{2.13}$$

From formulae (2.10) it follows that $\Phi_j^{\pm}(s) = O(\mathrm{e}^{|s|})$ ($j = 1, 2$) as $s \to \infty$ ($\operatorname{Re} s$ is finite). Because of the poles of the functions $\mathcal{S}_1^{\pm}(s)$ and $\mathcal{S}_2^{\pm}(s)$, the new functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$ admit simple poles at the point $s = \theta_0$. Analysis of the relations (2.10) implies that the functions $\mathcal{S}_1^{\pm}(s)$, $\mathcal{S}_2^{\pm}(s)$ have inadmissible poles at the zeros of the functions $\Gamma_s(\mp 1/\eta_1^{\pm}, \mp\eta_2^{\pm})$ which lie in the strip $-\pi < \operatorname{Re} s < \pi$. Let these zeros be denoted by $\xi_j^{\pm}$, $j = 1, 2, 3, 4$. The singular points $\xi_j^{\pm}$ are the numbers

$$2\pi j - i\log\left(iM_\nu^{\pm} \pm \sqrt{1 - (M_\nu^{\pm})^2}\right), \quad j = 0, \pm 1, \ldots, \quad \nu = 1, 2, \tag{2.14}$$

which meet the requirement $\operatorname{Re}\xi_j^{\pm} \in (-\pi, \pi)$. Here

$$M_\nu^{\pm} = \pm\frac{1}{2\eta_1^{\pm}\sin\beta}\left[1 + \eta_1^{\pm}\eta_2^{\pm} + (-1)^\nu\sqrt{d_1^{\pm}}\right],$$
$$d_1^{\pm} = (1 - \eta_1^{\pm}\eta_2^{\pm})^2 + 4\eta_1^{\pm}(\eta_2^{\pm} - \eta_1^{\pm})\cos^2\beta, \tag{2.15}$$

and $\sqrt{d_1^{\pm}}$ and $\log(iM_\nu^{\pm} \pm \sqrt{1 - (M_\nu^{\pm})^2})$ are fixed branches of the square root and the logarithmic functions, respectively. Notice that the transformation $\mathcal{T}$ maps the points ξ_j^{+} into $-\xi_j^{-}$:

$\xi_j^+ = -\mathcal{T}\xi_j^-$ and $\xi_j^- = -\mathcal{T}\xi_j^+$. The points $\xi_j^\pm$ become removable points of the functions $\mathcal{S}_1^\pm(s)$ and $\mathcal{S}_2^\pm(s)$ if the following conditions hold:

$$(-\sin s \pm \eta_2^\pm \sin\beta)\Phi_1^\pm(s) - \cos s\cos\beta\Phi_2^\pm(s) = 0,$$

$$\left(-\sin s \pm \frac{1}{\eta_1^\pm}\sin\beta\right)\Phi_2^\pm(s) + \cos s\cos\beta\Phi_1^\pm(s) = 0, \quad s = \xi_j^\pm, \quad j = 1,2,3,4. \quad (2.16)$$

Since the determinant of this system $\Gamma_{\xi_j^\pm}(\mp 1/\eta_1^\pm, \mp\eta_2^\pm)$ is equal to 0, the above conditions are equivalent to the following four equations:

$$(-\sin\xi_j^\pm \pm \eta_2^\pm \sin\beta)\Phi_1^\pm(\xi_j^\pm) - \cos\xi_j^\pm \cos\beta\Phi_2^\pm(\xi_j^\pm) = 0, \quad j = 1,2,3,4. \quad (2.17)$$

Let Π^+ be the infinite strip $\Pi^+ = \{s \in \mathbb{C} | -\pi < \operatorname{Re} s < 3\pi\}$, and $\Pi^- = \{s \in \mathbb{C} | -3\pi < \operatorname{Re} s < \pi\}$. Denote the boundaries of the strips by $\mathbf{\Omega}^\pm = \{\operatorname{Re}\sigma = 2\pi \pm \pi\}$ and $\mathbf{\Omega}_{-1}^\pm = \{\operatorname{Re}\sigma = -2\pi \pm \pi\}$. From the integral representation (2.4) and the boundary conditions (2.3) it may be deduced that the vectors $\mathbf{\Phi}^\pm(s) = (\Phi_1^\pm(s), \Phi_2^\pm(s))^\top$ solve the following problem.

PROBLEM 2.2 *Find the vectors $\mathbf{\Phi}^\pm(s)$ analytic in the strips $\Pi^\pm$ apart from simple poles at the points $s = \theta_0$ and $s = \pm 2\pi - \theta_0$, continuous up to the boundaries $\mathbf{\Omega}^\pm \cup \mathbf{\Omega}_{-1}^\pm$ and satisfying the boundary conditions*

$$\mathbf{\Phi}^\pm(\sigma) = \mathbf{G}^\pm(\sigma)\mathbf{\Phi}^\pm(\sigma - 4\pi), \quad \sigma \in \mathbf{\Omega}^\pm, \quad (2.18)$$

and the conditions of symmetry

$$\mathbf{\Phi}^\pm(s) = \mathbf{\Phi}^\pm(\pm 2\pi - s), \quad s \in \Pi^\pm, \quad (2.19)$$

where the components $G_{mn}^\pm(\sigma)$ of the matrices $\mathbf{G}^\pm(\sigma)$ are given by

$$G_{11}^+(\sigma) = G_{11}^-(\sigma) = \frac{\Gamma_\sigma(1/\eta_1^-, -\eta_2^+)\Gamma_\sigma(1/\eta_1^+, -\eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta}{D(\sigma)},$$

$$G_{22}^+(\sigma) = G_{22}^-(\sigma) = \frac{\Gamma_\sigma(-1/\eta_1^-, \eta_2^+)\Gamma_\sigma(-1/\eta_1^+, \eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta}{D(\sigma)},$$

$$G_{12}^\pm(\sigma) = \mp\frac{\eta_0^\mp \sin\beta\sin 2\beta\sin 2\sigma}{\eta_1 D(\sigma)}, \quad G_{21}^\pm(\sigma) = \frac{\eta_0^\pm}{\eta_0^\mp}\eta_1\eta_2 G_{12}^\pm(\sigma),$$

$$\begin{aligned} D(\sigma) &= \frac{\Gamma_\sigma(-1/\eta_1^+, -\eta_2^+)}{\Gamma_\sigma(1/\eta_1^+, \eta_2^+)}[\Gamma_\sigma(-1/\eta_1^-, \eta_2^+)\Gamma_\sigma(1/\eta_1^+, -\eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta] \\ &= \frac{\Gamma_\sigma(-1/\eta_1^-, -\eta_2^-)}{\Gamma_\sigma(1/\eta_1^-, \eta_2^-)}[\Gamma_\sigma(-1/\eta_1^+, \eta_2^-)\Gamma_\sigma(1/\eta_1^-, -\eta_2^+) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta], \end{aligned}$$

$$\eta_0^+ = \eta_2^+ - \frac{1}{\eta_1^-}, \quad \eta_0^- = \eta_2^- - \frac{1}{\eta_1^+}. \quad (2.20)$$

At the ends of the strip, that is, as $s \to \infty$ (Re s *is finite*), $\Phi_j^\pm(s) = O(\mathrm{e}^{|s|})$ $(j = 1,2)$. *The components $\Phi_j^\pm(s)$ of the vectors $\mathbf{\Phi}^\pm(s)$ satisfy the additional conditions* (2.17).

REMARK 2.1 Because the entries of the matrices $\mathbf{G}^{\pm}(s)$ are meromorphic functions, by analytical continuation the boundary conditions (2.18) are valid not only on the contours $\mathbf{\Omega}^{\pm}$ but everywhere in the complex plane apart from the poles of the solution and the functions $G_{ij}^{\pm}(s)$.

REMARK 2.2 In the case of normal incidence ($\beta = \frac{1}{2}\pi$), $\mathbf{G}^{\pm}(\sigma) = \mathrm{diag}\{G_{11}^{\pm}(\sigma), G_{22}^{\pm}(\sigma)\}$. This case is considered in section 7.

3. Vector Riemann–Hilbert problem on a plane

3.1 *Riemann–Hilbert problem for two even functions on a system of contours*

Let $\beta \neq \frac{1}{2}\pi$. In this section Problem 2.2 will be converted into two vector Riemann–Hilbert problems on the segment $(-1, 1)$ and a system of branch cuts. In what follows the general method (**6**) for vector difference equations with periodic meromorphic coefficients will be adjusted to Problem 2.2. The first step of the procedure is decoupling of the vector equations (2.18). It may be achieved by splitting the matrices $\mathbf{G}^{\pm}(\sigma)$:

$$\mathbf{G}^{\pm}(\sigma) = \mathbf{T}^{\pm}(\sigma)\Lambda^{\pm}(\sigma)[\mathbf{T}^{\pm}(\sigma - 4\pi)]^{-1}, \quad \sigma \in \mathbf{\Omega}^{\pm}, \tag{3.1}$$

where $\Lambda^{\pm}(\sigma) = \mathrm{diag}\{\lambda_1^{\pm}(\sigma), \lambda_2^{\pm}(\sigma)\}$, and $\lambda_1^{\pm}(\sigma)$ and $\lambda_2^{\pm}(\sigma)$ are the eigenvalues of the matrices $\mathbf{G}^{\pm}(\sigma)$,

$$\lambda_1^{\pm}(\sigma) = a_1(\sigma) + a_2^{\pm}(\sigma) f_{\pm}^{\frac{1}{2}}(\sigma), \quad \lambda_2^{\pm}(\sigma) = a_1(\sigma) - a_2^{\pm}(\sigma) f_{\pm}^{\frac{1}{2}}(\sigma). \tag{3.2}$$

The matrices $\mathbf{T}^{\pm}(s)$ meet the condition $\mathbf{T}^{\pm}(s) = \mathbf{T}^{\pm}(s - 4\pi)$ (their elements are 2π-periodic functions), and they are given by

$$\mathbf{T}^{\pm}(s) = \begin{pmatrix} 1 & 1 \\ -f_1^{\pm}(s) + f_{\pm}^{\frac{1}{2}}(s) & -f_1^{\pm}(s) - f_{\pm}^{\frac{1}{2}}(s) \end{pmatrix}. \tag{3.3}$$

Here

$$\begin{gathered}
a_1(\sigma) = \tfrac{1}{2}[G_{11}(\sigma) + G_{22}(\sigma)], \quad a_2^{\pm}(\sigma) = G_{12}^{\pm}(\sigma), \\
f_1^{\pm}(s) = \frac{G_{11}(s) - G_{22}(s)}{2G_{12}^{\pm}(s)} = \pm\frac{\eta_1(\eta_0^{+} + \eta_0^{-})}{2\eta_0^{\mp}\sin 2\beta\cos s}(\cos^2 s\cos^2\beta + \sin^2 s - g_0\sin^2\beta), \\
f_2^{\pm} = \frac{G_{21}^{\pm}(s)}{G_{12}^{\pm}(s)} = \frac{\eta_0^{\pm}}{\eta_0^{\mp}}\eta_1\eta_2, \quad f_{\pm}(s) = [f_1^{\pm}(s)]^2 + f_2^{\pm} = \left[\frac{\eta_1(\eta_0^{+} + \eta_0^{-})\tan\beta}{4\eta_0^{\mp}\cos s}\right]^2 f^{*}(s), \\
f^{*}(s) = \left(\cos^2 s + g_0 - \frac{1}{\sin^2\beta}\right)^2 + 16g_1\cos^2 s\cot^2\beta, \\
g_0 = \frac{1}{\eta_0^{+} + \eta_0^{-}}\left(\eta_0^{+}\frac{\eta_2^{-}}{\eta_1^{+}} + \eta_0^{-}\frac{\eta_2^{+}}{\eta_1^{-}}\right), \quad g_1 = \frac{\eta_2\eta_0^{+}\eta_0^{-}}{\eta_1(\eta_0^{+} + \eta_0^{-})^2}.
\end{gathered} \tag{3.4}$$

From the above formulae it follows that the matrices $\mathbf{G}^{+}(\sigma)$ and $\mathbf{G}^{-}(\sigma)$ have the same eigenvalues: $\lambda_j^{+}(\sigma) = \lambda_j^{-}(\sigma) = \lambda_j(\sigma)$ and therefore $\Lambda^{+}(\sigma) = \Lambda^{-}(\sigma) = \Lambda(\sigma)$. Clearly, the elements of the matrices $\mathbf{T}^{\pm}(s)$ are two-valued functions. They have branch points at the zeros of odd order of the function $f^*(s)$ which are defined by the roots of the equations $\cos 2s = N_\nu$, $s \in \Pi^\pm$ $(\nu = 1, 2)$, where

$$N_\nu = -1 + \frac{2}{\sin^2\beta}\left[1 - g_0 \sin^2\beta - 8g_1 \cos^2\beta + 4i(-1)^\nu \cos\beta\sqrt{d_0}\right],$$
$$d_0 = g_1(1 - g_0 \sin^2\beta - 4g_1 \cos^2\beta). \tag{3.5}$$

Here $\sqrt{d_0}$ is one of the branches of the square root. In general, d_0 is complex. If $d_0 = 0$ then the functions $f_\pm^{1/2}(s)$ are single-valued. This case for the strip Π^+ is analysed in (**6**). Attention will now be directed towards describing the details of the solution of Problem 2.2 in the case of the complex impedance parameters (in particular, they might be real) when $d_0 \neq 0$.

Among the roots

$$\pi j + \tfrac{1}{2} i \log(N_\nu \pm \sqrt{N_\nu^2 - 1}) \quad (\nu = 1, 2; \quad j = 0, \pm 1, \pm 2, \ldots), \tag{3.6}$$

only those roots $s_j^\pm$ should be taken which lie in the strip $\Pi^\pm = \{-2\pi \pm \pi < \operatorname{Re} s < 2\pi \pm \pi\}$. The branch of the logarithmic functions is fixed by the condition $-\pi < \arg(N_\nu \pm \sqrt{N_\nu^2 - 1}) \leqslant \pi$, and $\sqrt{N_\nu^2 - 1}$ $(\nu = 1, 2)$ is a fixed branch. There are 16 roots in the strip $\Pi^\pm$, $s_0^\pm, s_1^\pm, \ldots, s_{15}^\pm$. Single branches of the functions $f_\pm^{1/2}$ are fixed by

$$f_\pm^{\frac{1}{2}}(s) = \mp \frac{\eta_1(\eta_0^+ + \eta_0^-)\tan\beta}{4\eta_0^\mp \cos s}\sqrt{f^*(s)}, \quad \sqrt{f^*(s)} \sim \cos^2 s, \quad s \to x \pm i\infty,$$
$$-2\pi \pm \pi \leqslant x \leqslant 2\pi \pm \pi. \tag{3.7}$$

Then, clearly, $\mathcal{T} f_+^{1/2}(s) = f_-^{1/2}(s)$. The branches are single-valued functions in the strips $\Pi^\pm$ cut along a system of cuts $\mathcal{C}_j^\pm$ $(j = 0, 1, \ldots, 7)$ which join pairs of the branch points. The cuts are smooth curves and they do not intersect each other.

According to the splitting (3.1) of the matrices $\mathbf{G}^\pm(\sigma)$ the new vector-functions

$$\phi^\pm(s) = [\mathbf{T}^\pm(s)]^{-1}\Phi^\pm(s), \quad s \in \Pi^\pm, \tag{3.8}$$

are introduced. Their components are given by

$$\phi_1^\pm(s) = \left(\frac{f_1^\pm(s)}{2 f_\pm^{1/2}(s)} + \frac{1}{2}\right)\Phi_1^\pm(s) + \frac{\Phi_2^\pm(s)}{2 f_\pm^{1/2}(s)},$$
$$\phi_2^\pm(s) = \left(-\frac{f_1^\pm(s)}{2 f_\pm^{1/2}(s)} + \frac{1}{2}\right)\Phi_1^\pm(s) - \frac{\Phi_2^\pm(s)}{2 f_\pm^{1/2}(s)}, \quad s \in \Pi^\pm. \tag{3.9}$$

Analysis of these formulae shows that the functions $f_1^\pm(s) f_\pm^{-1/2}(s)$ and $f_\pm^{-1/2}(s)$ are free of poles in the strips $\Pi^\pm$, respectively, and they have 16 branch points $s_0^\pm, s_1^\pm, \ldots, s_{15}^\pm$. Because of the poles of the functions $\Phi_1^\pm(s)$ and $\Phi_2^\pm(s)$ at the points $s = \theta_0$ and $s = \pm 2\pi - \theta_0$, the functions $\phi_1^\pm(s)$ and $\phi_2^\pm(s)$ may have two simple poles at these points.

Clearly, the branch cuts $\mathcal{C}_j^\pm$ $(j = 0, 1, \ldots, 7)$ are lines of discontinuity for the functions $\phi_1^\pm(s)$ and $\phi_2^\pm(s)$. However, the vectors $\Phi^\pm(s) = \mathbf{T}^\pm(s)\phi^\pm(s)$ have to be continuous through these lines. Therefore the functions $\phi_1^\pm(s)$ and $\phi_2^\pm(s)$ solve the following system of two scalar difference equations:

$$\phi_\mu^\pm(\sigma) = \lambda_\mu(\sigma)\phi_\mu^\pm(\sigma - 4\pi), \quad \mu = 1, 2, \quad \sigma \in \Omega^\pm, \tag{3.10}$$

subject to the Riemann–Hilbert boundary conditions on the cuts $\mathcal{C}_j^{\pm}$

$$\phi_1^{\pm}(\sigma)|_L = \phi_2^{\pm}(\sigma)|_R, \quad \phi_1^{\pm}(\sigma)|_R = \phi_2^{\pm}(\sigma)|_L, \quad \sigma \in \mathcal{C}_j^{\pm} \quad (j = 0, 1, \ldots, 7), \tag{3.11}$$

where $\phi_\mu^{\pm}(\sigma)|_L$ and $\phi_\mu^{\pm}(\sigma)|_R$ are the limiting values of the functions $\phi_\mu^{\pm}(s)$ on the contours $\mathcal{C}_j^{\pm}$ from the left- and from the right-hand sides with respect to the positive directions of the contours $\mathcal{C}_j^{\pm}$ ($\mu = 1, 2$; $j = 1, 2, \ldots, 7$). It becomes evident from (3.9) that at the ends of the strip the functions $\phi_1^{\pm}(s)$ grow, and the functions $\phi_2^{\pm}(s)$ are bounded:

$$\phi_1^{\pm}(s) = O(\mathrm{e}^{|s|}), \quad \phi_2^{\pm}(s) = O(1), \quad s \to \infty, \quad s \in \Pi^{\pm}. \tag{3.12}$$

By the mappings

$$w = -i \tan \frac{s - 2\pi \mp \pi}{4}, \quad s = 2\pi \pm \pi + 2i \log \frac{1 + w}{1 - w}, \tag{3.13}$$

the contours $\Omega^{\pm}$ are mapped onto the upper side of the cut $[-1, 1]$ (the left bank with respect to the positive direction), and the left boundaries of the strips, $\Omega_{-1}^{\pm}$, are mapped onto the lower side of the cut. The images of the upper and the lower infinite points of the strips $\Pi^{\pm}$, $x - i\infty$ and $x + i\infty$ $(-2\pi \pm \pi \leqslant x \leqslant 2\pi \pm \pi)$, are the points $w = -1$ and $w = 1$, respectively. The function $\log[(1 + w)(1 - w)^{-1}]$ is real on the upper side of the cut. The systems of equations (3.10) and (3.11) reduce to the following vector Riemann–Hilbert problem.

PROBLEM 3.1 *Find the functions $F_{\mu\pm}(s)$ analytic in the w-planes defined by* (3.13), *apart from the segment* $[-1, 1]$ *and the poles $w = \hat{w}_{\pm}$ and $w = -\hat{w}_{\pm}$, where $\hat{w}_{\pm} = -i \tan \frac{1}{4}(\theta_0 \pm \pi)$, continuous up to the segment* $(-1, 1)$ *and the contours $\mathcal{C}_{j\pm}^{*}$ ($j = 0, 1, \ldots, 7$). The functions $F_{\mu\pm}(s)$ are even: $F_{\mu\pm}(w) = F_{\mu\pm}(-w)$, $w \in \mathbb{C} \setminus [-1, 1]$, and satisfy the boundary conditions*

$$F_{\mu\pm}^{+}(t) = l_\mu(t) F_{\mu\pm}^{-}(t), \quad \mu = 1, 2, \quad t \in (-1, 1),$$
$$F_{1\pm}^{+}(t) = F_{2\pm}^{-}(t), \quad F_{2\pm}^{+}(t) = F_{1\pm}^{-}(t), \quad t \in \mathcal{C}_{j\pm}^{*}, \quad j = 0, 1, \ldots, 7. \tag{3.14}$$

Here the curves $\mathcal{C}_{j\pm}^{*}$ ($j = 0, 1, \ldots, 7$) are the images of the branch cuts $\mathcal{C}_j^{\pm}$, and

$$F_{\mu\pm}(w) = \phi_\mu^{\pm}(s), \quad w \in \mathbb{C}, \quad s \in \Pi^{\pm},$$
$$F_{\mu\pm}^{+}(t) = F_{\mu\pm}(w)|_L, \quad F_{\mu\pm}^{-}(t) = F_{\mu\pm}(w)|_R, \quad t \in [-1, 1],$$
$$l_\mu(t) = \lambda_\mu(\sigma), \quad t \in [-1, 1], \quad \sigma \in \Omega^{\pm}, \tag{3.15}$$

where w and s are linked by (3.13) *and $t = -i \tan \frac{1}{4}(\sigma - 2\pi \mp \pi)$. At the ends,*

$$F_{1\pm}(w) \sim D_{1\pm}|w \pm 1|^{-2}, \quad F_{2\pm}(w) \sim D_{2\pm}, \quad w \to \mp 1, \tag{3.16}$$

where $D_{1\pm}$ and $D_{2\pm}$ are constants.

The functions $F_{1\pm}(w)$ and $F_{2\pm}(w)$ are even, because of the relations

$$\Phi_\mu^{\pm}(s) = \Phi_\mu^{\pm}(\pm 2\pi - s), \quad \mu = 1, 2,$$
$$f_1^{\pm}(s) = f_1^{\pm}(\pm 2\pi - s), \quad f_{\pm}^{1/2}(s) = f_{\pm}^{1/2}(\pm 2\pi - s), \quad s \in \Pi^{\pm}. \tag{3.17}$$

The class of solutions (3.16) for Problem 3.1 follows from the conditions (3.12).

It was proved in (**6**) that the even Riemann–Hilbert problem (3.14) has a solution if the coefficients $l_\mu(t)$ ($\mu = 1, 2$) satisfy the condition $l_\mu(t)l_\mu(-t) = 1$. The functions $l_\mu(t)$ meet the following conditions:

$$l_\mu(-1) = l_\mu(0) = l_\mu(1) = 1, \quad \Delta_\mu = -2\pi, \tag{3.18}$$

where Δ_μ are the increments of the argument of the functions $l_\mu(t)$ as the point t traverses the segment [0, 1] in the positive direction. The quantities Δ_μ are evaluated numerically. The argument of the functions $l_\mu(t)$ ($\mu = 1, 2$) is fixed by the condition $\arg l_\mu(0) = 0$.

3.2 *Branch points and the function* $p^{\frac{1}{2}}(\zeta)$

Analysis of the branch points

$$w_j^\pm = -i \tan \frac{s_j^\pm \pm \pi}{4}, \quad j = 0, 1, \ldots, 15, \tag{3.19}$$

shows that half of them are located in the upper half-plane. Moreover, for each branch point $w_j^\pm$ ($j = 0, 1, \ldots, 7$) there is one branch point, say $w_{m+8}^\pm$ ($m = 0, 1, \ldots, 7$), such that $w_j^\pm = -w_{m+8}^\pm$. Denote $\zeta = w^2 = -\tan^2 \frac{1}{4}(s \pm \pi)$ and $\zeta_j^\pm = (w_j^\pm)^2$ ($j = 0, 1, \ldots, 15$). Among the 16 numbers $\zeta_j^\pm$ there are eight distinct ones. Let they be reordered in the following manner:

$$\operatorname{Im} \zeta_0^\pm \leqslant \operatorname{Im} \zeta_1^\pm \leqslant \cdots \leqslant \operatorname{Im} \zeta_7^\pm, \tag{3.20}$$

and

$$\operatorname{Re} \zeta_j^\pm < \operatorname{Re} \zeta_{j+1}^\pm \quad \text{if } \operatorname{Im} \zeta_j^\pm = \operatorname{Im} \zeta_{j+1}^\pm. \tag{3.21}$$

Then, clearly, $\zeta_j^+ = \zeta_j^- = \zeta_j$ ($j = 0, 1, \ldots, 7$). For example, if

$$\beta = \tfrac{1}{3}\pi, \quad \eta_1^+ = 1 - i, \quad \eta_2^+ = 0{\cdot}1 - i, \quad \eta_1^- = 2 - i, \quad \eta_2^- = 1 + i, \tag{3.22}$$

then the branch points ζ_j are

$$\zeta_0 = -2{\cdot}3406 - 1{\cdot}9658i, \quad \zeta_1 = -0{\cdot}16520 - 1{\cdot}5996i,$$
$$\zeta_2 = 0{\cdot}12473 - 0{\cdot}10242i, \quad \zeta_3 = -0{\cdot}051898 - 0{\cdot}091307i,$$
$$\zeta_4 = -0{\cdot}25053 + 0{\cdot}21041i, \quad \zeta_5 = -0{\cdot}063887 + 0{\cdot}61857i,$$
$$\zeta_6 = 4{\cdot}7887 + 3{\cdot}9321i, \quad \zeta_7 = -4{\cdot}7050 + 8{\cdot}2778i.$$

Notice that the branch points are invariant with respect to the transformation $\mathcal{T}$: $\mathcal{T}\zeta_j = \zeta_j$, $j = 0, 1, \ldots, 7$. Since the shape of the branch cuts $\mathcal{C}_j^\pm$ and the starting and the ending points of the cuts were not chosen it may be assumed that the branch cuts are the straight lines in the ζ-plane which join the branch points ζ_{2j} with ζ_{2j+1}, $j = 0, 1, 2, 3$. These branch cuts are denoted by γ_j. The convention (3.20), (3.21) guarantees that the branch cuts will not cross each other (Fig. 2). Notice that regardless of the values of the impedance parameters the first four branch points ζ_j ($j = 0, 1, 2, 3$) are located below the real axis, and the other four branch points are in the upper half-plane. If all the impedance parameters $\eta_j^\pm$ are real, then $\zeta_j = \overline{\zeta_{7-j}}$, $j = 0, 1, 2, 3$.

Let $p^{1/2}(\zeta)$ be the branch of the function

$$u^2 = p(\zeta), \quad p(\zeta) = (\zeta - \zeta_0)(\zeta - \zeta_1) \ldots (\zeta - \zeta_7), \tag{3.23}$$

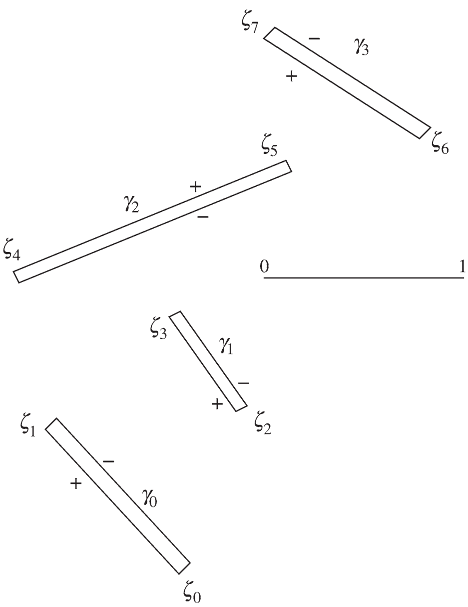

Fig. 2 Branch cuts $\gamma_0, \gamma_1, \gamma_2$ and γ_3

chosen such that $p^{1/2}(\zeta) \sim \zeta^4, \zeta \to \infty$. In what follows a relation between the single branches of the functions $f_{\pm}^{1/2}(s)$ and the branch of the function $p^{1/2}(\zeta)$ will be established. Since

$$\zeta = w^2 = -\cot^2\frac{s \mp \pi}{4} = \left(\frac{\sin\frac{1}{2}s + 1}{\sin\frac{1}{2}s - 1}\right)^{\pm 1},$$

$$\cos s = 1 - 2\left(\frac{\zeta+1}{\zeta-1}\right)^2 = -\frac{\zeta^2 + 6\zeta + 1}{(\zeta-1)^2}, \tag{3.24}$$

formula (3.7) implies

$$f_{\pm}^{\frac{1}{2}}(s) = \pm\frac{\eta_1(\eta_0^+ + \eta_0^-)\tan\beta(\zeta-1)^2}{4\eta_0^{\mp}(\zeta^2 + 6\zeta + 1)}\sqrt{f^*(s)}. \tag{3.25}$$

Use of the directly verified relations

$$\cos 2s = 1 + \frac{32\zeta(\zeta+1)^2}{(\zeta-1)^4} \tag{3.26}$$

and

$$f^*(s) = \tfrac{1}{4}(\cos 2s - N_1)(\cos 2s - N_2) = \frac{(1-N_1)(1-N_2)}{4(\zeta-1)^8}p(\zeta) = \frac{2^{12}p(\zeta)}{p(1)(\zeta-1)^8} \tag{3.27}$$

makes it possible to show that

$$\sqrt{f^*(s)} = \frac{64}{p^{1/2}(1)(\zeta-1)^4}p^{1/2}(\zeta). \tag{3.28}$$

Indeed, since $\cos^2 s \sim 64(\zeta-1)^{-4}$ as $\zeta \to 1$ ($s \to x \pm \infty$), the function $\sqrt{f^*(s)}$ meets the condition (3.7). Therefore,

$$f_\pm^{1/2}(s) = \pm \frac{16\eta_1(\eta_0^+ + \eta_0^-)\tan\beta p^{1/2}(\zeta)}{\eta_0^\mp p^{1/2}(1)(\zeta^2+6\zeta+1)(\zeta-1)^2}. \tag{3.29}$$

Analysis of the branches $f_\pm^{1/2}(s)$ and $p^{1/2}(\zeta)$ shows that the functions $f_\pm^{1/2}(s)p^{1/2}(\zeta)$ do not have branch points and also that at infinity

$$f_\pm^{1/2}(s) \sim \pm \frac{2\eta_1(\eta_0^+ + \eta_0^-)\tan\beta}{\eta_0^\mp(\zeta-1)^2}, \quad s \to x \pm i\infty \quad (\zeta \to 1). \tag{3.30}$$

4. Scalar Riemann–Hilbert problem on a surface of genus 3

In this section the vector problem (3.14) is formulated as a scalar Riemann–Hilbert problem on a hyperelliptic surface, say $\mathcal{R}$, of the algebraic function (3.23).

4.1 *Formulation of the problem*

The surface $\mathcal{R}$ is formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex ζ-plane $\mathbb{C} \cup \infty$ cut along the system of the cuts γ_j ($j = 0, 1, 2, 3$). The positive (left) sides of the cuts γ_j on $\mathbb{C}_1$ are glued with the negative (right) sides of the curves γ_j on $\mathbb{C}_2$, and vice versa. The genus of the surface is 3. The function u, defined by (3.23), becomes single-valued on the surface $\mathcal{R}$:

$$u = \begin{cases} p^{1/2}(\zeta), & \zeta \in \mathbb{C}_1, \\ -p^{1/2}(\zeta), & \zeta \in \mathbb{C}_2, \end{cases} \tag{4.1}$$

where a point of the surface $\mathcal{R}$ with affix ζ on $\mathbb{C}_\mu$ is denoted by the pair $(\zeta, (-1)^{\mu-1}p^{1/2}(\zeta))$, $\mu = 1, 2$. It follows from (3.14) that the functions

$$F_\pm(\zeta, u) = \begin{cases} F_{1\pm}(w), & (\zeta, u) \in \mathbb{C}_1, \\ F_{2\pm}(w), & (\zeta, u) \in \mathbb{C}_2, \end{cases} \quad \zeta = w^2 = -\tan^2\frac{s \pm \pi}{4}, \tag{4.2}$$

are single-valued on the Riemann surface $\mathcal{R}$. They are meromorphic everywhere on $\mathcal{R}$ apart from the contour $\mathcal{L} = L_1 \cup L_2$, where $L_1 = (0, 1) \subset \mathbb{C}_1$ and $L_2 = (0, 1) \subset \mathbb{C}_2$. Therefore, Problem 3.1 is equivalent to the following scalar Riemann–Hilbert problem on the surface $\mathcal{R}$.

PROBLEM 4.1 *Find the functions $F_\pm(\zeta, u)$ piecewise analytic on the surface $\mathcal{R}$ apart from the geometrical optics poles at the points $(\alpha_{1\pm}, p^{1/2}(\alpha_{1\pm}))$ and $(\alpha_{1\pm}, -p^{1/2}(\alpha_{1\pm}))$ and simple poles at the branch points $\zeta_0, \zeta_1, \ldots, \zeta_7$ of the surface $\mathcal{R}$, continuous up to the boundary $\mathcal{L}$ and satisfying the boundary condition*

$$F_\pm^+(\tau, \eta) = l(\tau, \eta)F_\pm^-(\tau, \eta), \quad (\tau, \eta) \in \mathcal{L}, \tag{4.3}$$

where

$$l(\tau, \eta) = \begin{cases} l_1(\sqrt{\tau}), & (\tau, \eta) \in L_1, \\ l_2(\sqrt{\tau}), & (\tau, \eta) \in L_2, \end{cases} \tag{4.4}$$

$\eta = u(\tau)$, and $\alpha_{1\pm} = -\tan^2\frac{1}{4}(\pi \pm \theta_0)$. The functions $F_\pm(\zeta, u)$ are bounded at the infinite points ∞_μ ($\mu = 1, 2$) of the surface. As $\zeta \to 1$, the functions $F_\pm(\zeta, u)$ are singular on the first sheet: $F_\pm(\zeta, u) \sim B_\pm(\zeta-1)^{-2}$, and they are bounded on the second sheet. As $\zeta \to 0$ and $(\zeta, u) \in \mathcal{R}$, the functions $F_\pm(\zeta, u)$ are bounded.

Notice that a branch point ζ_j of a two-sheeted Riemann surface is called (**18**) a pole of order μ_j for a function $F(\zeta, u)$ if $F(\zeta, u) \sim Az^{-\mu_j}$, $z \to 0$, $A = \text{const}$, and $z = (\zeta - \zeta_j)^{1/2}$ is a local uniformizer of the point ζ_j.

4.2 *Factorization of the coefficient* $l(t, \eta)$

The first step of the procedure for the problem (4.3) is to factorize the function $l(\tau, \eta)$. This means finding a meromorphic solution to the following problem:

$$l(\tau, \eta) = \frac{X^+(\tau, \eta)}{X^-(\tau, \eta)}, \quad (\tau, \eta) \in \mathcal{L}. \tag{4.5}$$

Its solution is given in terms of the singular integrals with the Weierstrass kernel on the Riemann surface

$$dU = \frac{u + \eta}{2\eta} \frac{d\tau}{\tau - \zeta}, \quad (\tau, \eta) \in \mathcal{L}, \quad (\zeta, u) \in \mathcal{R}, \tag{4.6}$$

as follows (**6**):

$$\begin{gathered} X(\zeta, u) = \mathrm{e}^{\chi(\zeta, u)}, \\ \chi(\zeta, u) = \mathcal{I}_1(\zeta, u) + \mathcal{I}_2(\zeta, u) + \mathcal{I}_3(\zeta, u), \\ \mathcal{I}_1(\zeta, u) = \frac{1}{2\pi i} \int_{\mathcal{L}} \log l(\tau, \eta) dU, \quad \mathcal{I}_2(\zeta, u) = \int_{p_{10}}^{p_{11}} dU - \int_{p_{20}}^{p_{21}} dU, \\ \mathcal{I}_3(\zeta, u) = \sum_{j=1}^{3} \left(\int_{\Xi_j} dU + m_j \oint_{\mathbf{a}_j} dU + n_j \oint_{\mathbf{b}_j} dU \right). \end{gathered} \tag{4.7}$$

The function $\exp\{\mathcal{I}_1(\zeta, u)\}$ satisfies the boundary condition (4.3). However, its asymptotics at the ends ± 1 of the contours L_1 and L_2 differs from the one required in Problem 4.1. Also, in general, it has an essential singularity at the infinite points ∞_μ ($\mu = 1, 2$) of the Riemann surface that is not acceptable. The functions $\exp\{\mathcal{I}_2(\zeta, u)\}$ and $\exp\{\mathcal{I}_3(\zeta, u)\}$ are continuous through the contour $\mathcal{L}$ and the contours of integration for the integrals $\mathcal{I}_2$ and $\mathcal{I}_3$. The function $\mathcal{I}_2$ is added to correct the asymptotics of the solution at the ends of the contour $\mathcal{L}$. The presence of the functions $\mathcal{I}_3$ is explained by the necessity to illuminate the essential singularity of the functions $\exp\{\mathcal{I}_1(\zeta, u) + \mathcal{I}_2(\zeta, u)\}$ at the infinite points.

The contours $\mathbf{a}_j$ and $\mathbf{b}_j$ ($j = 1, 2, 3$) are the canonical cross-sections of the surface $\mathcal{R}$ shown in Fig. 3. The loops $\mathbf{a}_j$ are closed contours formed by the branch cuts γ_1, γ_2 and γ_3:

$$\begin{aligned} \mathbf{a}_1 &= [\zeta_2, \zeta_3]^+ \cup [\zeta_3, \zeta_2]^-, \\ \mathbf{a}_2 &= [\zeta_4, \zeta_5]^+ \cup [\zeta_5, \zeta_4]^-, \\ \mathbf{a}_3 &= [\zeta_6, \zeta_7]^+ \cup [\zeta_7, \zeta_6]^-. \end{aligned} \tag{4.8}$$

The positive direction is chosen such that the first sheet $\mathbb{C}_1$ is on the left. The cross-sections $\mathbf{b}_j$ are closed contours consisting of two parts:

$$\begin{aligned} \mathbf{b}_1 &= [\zeta_1, \zeta_2]_{\mathbb{C}_2} \cup [\zeta_2, \zeta_1]_{\mathbb{C}_1}, \\ \mathbf{b}_2 &= \mathbf{b}_1 \cup [\zeta_2, \zeta_3]^+_{\mathbb{C}_2} \cup [\zeta_3, \zeta_4]_{\mathbb{C}_2} \cup [\zeta_4, \zeta_3]_{\mathbb{C}_1} \cup [\zeta_3, \zeta_2]^-_{\mathbb{C}_1}, \\ \mathbf{b}_3 &= \mathbf{b}_2 \cup [\zeta_4, \zeta_5]^+_{\mathbb{C}_2} \cup [\zeta_5, \zeta_6]_{\mathbb{C}_2} \cup [\zeta_6, \zeta_5]_{\mathbb{C}_1} \cup [\zeta_5, \zeta_4]^-_{\mathbb{C}_1}. \end{aligned} \tag{4.9}$$

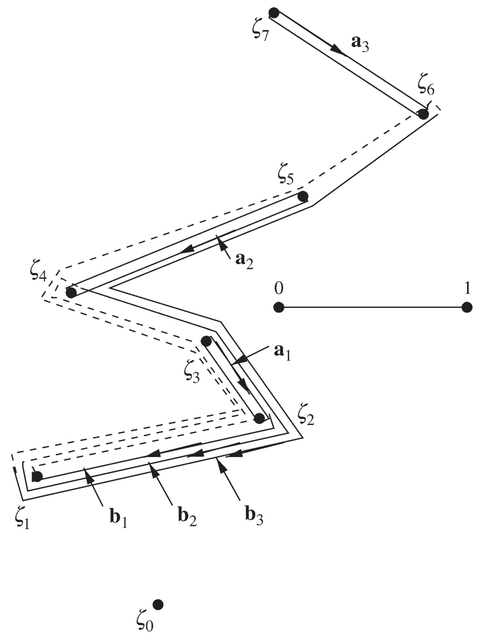

Fig. 3 Canonical cross-sections $\mathbf{a}_j$ and $\mathbf{b}_j$ $(j = 1, 2, 3)$

The first part (the solid line in Fig. 3) lies on the sheet $\mathbb{C}_1$. The starting point of the cross-section $\mathbf{b}_j$ is ζ_{2j} $(j = 1, 2, 3)$, and the terminal point is ζ_1. The second part (the dashed line in Fig. 3) lies on the second sheet $\mathbb{C}_2$. The contour $\mathbf{b}_j$ crosses the loop $\mathbf{a}_j$ from right to left and does not cross the other loops $\mathbf{a}_k$ and $\mathbf{b}_k$ $(k \neq j)$ or the contour $\mathcal{L}$. Notice that for all tested values of the impedance parameters and the angle β, the segment $\zeta_3\zeta_4$ does not cross the segment (0, 1), and the point $\zeta = 0$ always lies to right of the segment $\zeta_3\zeta_4$. Because of the conditions (3.18) and the choice of the argument of the functions $l_\mu(t)$, the single branch of the logarithmic function in (4.7) has the following properties:

$$\log l(\tau, \eta) = \log l_\mu(\sqrt{\tau}) = \log |l_\mu(\sqrt{\tau})| + i\epsilon_\mu(\tau), \quad (\tau, \eta) \in L_\mu \subset \mathbb{C}_\mu, \quad \mu = 1, 2,$$
$$\epsilon_\mu(\tau) = \arg l_\mu(\sqrt{\tau}), \quad \epsilon_\mu(\tau) \in [-2\pi, 0], \quad \epsilon_\mu(0) = 0, \quad \epsilon_\mu(1) = -2\pi. \tag{4.10}$$

The contours $\Xi_j = r_j q_j$ $(j = 1, 2, 3)$ are continuous curves with starting points $r_j = (\delta_j, \upsilon_j) \in \mathbb{C}_1, \upsilon_j = p^{1/2}(\delta_j)$. These points are *distinct and fixed arbitrarily*. The ending points $q_j = (\sigma_j, u_j) \in \mathcal{R}, u_j = u(\sigma_j)$, are to be determined (they may lie on either sheet). The contours Ξ_j cannot cross the **a**- and **b**-loops or the contour $\mathcal{L}$. It is convenient to take the contour Ξ_j as a line that passes through the branch point ζ_0 (Fig. 4). If the terminal point $q_j \in \mathbb{C}_1$, then the whole contour $\Xi_j \subset \mathbb{C}_1$. If this point lies on the second sheet $\mathbb{C}_2$, then the contour Ξ_j consists of two parts: the segment $\delta_j\zeta_0 \subset \mathbb{C}_1$, and the line $\zeta_0\sigma_j \subset \mathbb{C}_2$. Depending on whether the point (σ_j, u_j) is to the left or to the right of the broken line $\zeta_0\zeta_1 \ldots \zeta_7$, the part $\zeta_0\sigma_j$ of the contour Ξ_j lies in the domain that is either to the left, or to the right of the contour $\zeta_0\zeta_1 \ldots \zeta_7$. In Fig. 4, as an example, the following case is illustrated. The point $q_1 \in \mathbb{C}_1$, and the points q_2 and q_3 lie on the sheet $\mathbb{C}_2$.

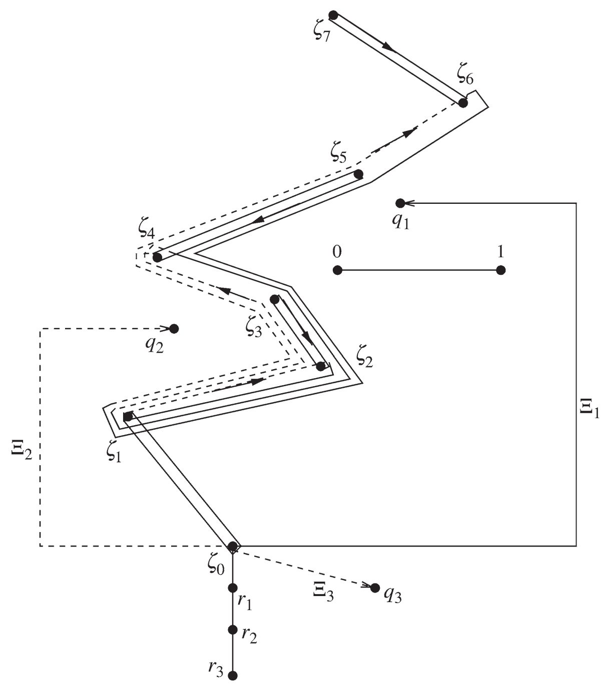

Fig. 4 Contours Ξ_1, Ξ_2 and Ξ_3

The points $p_{\mu 1}$ $(\mu = 1, 2)$ are *arbitrary, distinct and fixed*

$$p_{\mu 1} = (\rho_\mu, (-1)^{\mu-1} z_\mu) \in \mathbb{C}_\mu, \quad \rho_1 \neq \rho_2, \quad \rho_\mu \neq 1, \quad z_\mu = p^{1/2}(\rho_\mu), \quad \mu = 1, 2. \tag{4.11}$$

The point $p_{\mu 0}$ coincides with the right end $\zeta = 1$ of the contour $L_\mu \subset \mathbb{C}_\mu$: $p_{\mu 0} = (1, (-1)^{\mu-1} p^{1/2}(1))$. The integers m_j and n_j $(j = 1, 2, 3)$ and the points q_j are *to be determined*. Formulae (4.7) can be represented in another form more convenient for analysis

$$\chi(\zeta, u) = \chi_1(\zeta) + u(\zeta)\chi_2(\zeta),$$

$$\chi_1(\zeta) = \frac{1}{4\pi i}\int_0^1 [\log l_1(\sqrt{\tau}) + \log l_2(\sqrt{\tau})]\frac{d\tau}{\tau - \zeta} + \frac{1}{2}\int_1^{\rho_1} \frac{d\tau}{\tau - \zeta} - \frac{1}{2}\int_1^{\rho_2} \frac{d\tau}{\tau - \zeta} + \frac{1}{2}\sum_{j=1}^{3}\int_{\Xi_j}\frac{d\tau}{\tau - \zeta},$$

$$\chi_2(\zeta) = \frac{1}{4\pi i}\int_0^1 [\log l_1(\sqrt{\tau}) - \log l_2(\sqrt{\tau})]\frac{d\tau}{p^{1/2}(\tau)(\tau - \zeta)} + \frac{1}{2}\int_1^{\rho_1} \frac{d\tau}{p^{1/2}(\tau)(\tau - \zeta)}$$
$$+ \frac{1}{2}\int_1^{\rho_2} \frac{d\tau}{p^{1/2}(\tau)(\tau - \zeta)} + \frac{1}{2}\sum_{j=1}^{3}\left(\int_{\Xi_j} + m_j \oint_{\mathbf{a}_j} + n_j \oint_{\mathbf{b}_j}\right)\frac{d\tau}{\eta(\tau)(\tau - \zeta)}. \tag{4.12}$$

By the Sokhotski–Plemelj formulae for integrals with the Weierstrass kernel the first integrals in the above formulae for the functions $\chi_1(\zeta)$ and $\chi_2(\zeta)$ are discontinuous through the contour [0, 1], and the limiting values of the function $X(\zeta, u)$ meet the condition (4.5). The other integrals are continuous through the contour $\mathcal{L}$. They are discontinuous through their lines of integration with the jump multiple to $2\pi i$. The continuity of the function $X(\zeta, u) = \exp \chi(\zeta, u)$, however, is not affected by these jumps. Analysis of the first, second, and third integrals in the representations (4.12) of the functions $\chi_1(\zeta)$ and $\chi_2(\zeta)$ at the points $\zeta = 0$ and $\zeta = 1$ yields

$$\begin{aligned} &\chi(\zeta, u) \sim \text{const}, \quad (\zeta, u) \to (0, \pm p^{1/2}(0)) \in \mathcal{R}, \\ &\chi(\zeta, u) \sim -2\log(\zeta - 1), \quad (\zeta, u) \to (1, p^{1/2}(1)) \in \mathbb{C}_1, \\ &\chi(\zeta, u) \sim \text{const}, \quad (\zeta, u) \to (1, -p^{1/2}(1)) \in \mathbb{C}_2. \end{aligned} \tag{4.13}$$

Thus, the function $X(\zeta, u)$ is bounded at the points $(0, \pm p^{1/2}(0)) \in \mathcal{R}$, and its asymptotics at the ends $(1, \pm p^{1/2}(1))$ is the same as for the functions $F_\pm(\zeta, u)$ given by (3.16), (4.2).

4.3 *Elimination of the essential singularity*

The Weierstrass kernel (4.6) has a pole of order 3 at infinity. Therefore, in general, the function $X(\zeta, u)$ has an essential singularity at infinity that is not acceptable. Use of the identity

$$\frac{1}{\tau - \zeta} = -\frac{1}{\zeta} - \frac{\tau}{\zeta^2} - \frac{\tau^2}{\zeta^3} + \frac{\tau^3}{\zeta^3(\tau - \zeta)} \tag{4.14}$$

gives the following asymptotic expansion of the function $\chi(\zeta, w)$ at infinity:

$$\chi(\zeta, w) = -\frac{1}{2}\sum_{\nu=1}^{3}\left\{\frac{1}{2\pi i}\int_0^1 [\log l_1(\sqrt{\tau}) - \log l_2(\sqrt{\tau})]\frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)} + \sum_{\mu=1}^{2}\int_1^{\rho_\mu}\frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)}\right.$$
$$\left. + \sum_{j=1}^{3}\left(\int_{\Xi_j} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{\tau^{\nu-1}d\tau}{\eta(\tau)}\right\}\frac{u(\zeta)}{\zeta^\nu} + O(1), \quad \zeta \to \infty, \quad \nu = 1, 2, 3. \tag{4.15}$$

Therefore the function $\chi(\zeta, w)$ is bounded at infinity if and only if the points $q_j = (\sigma_j, u_j)$ and the integers m_j and n_j solve the following system of nonlinear equations:

$$\sum_{j=1}^{3}[\omega_\nu(q_j) + m_j A_{\nu j} + n_j B_{\nu j}] = d_\nu, \quad \nu = 1, 2, 3, \tag{4.16}$$

where

$$d_\nu = \sum_{j=1}^{3}\omega_\nu(\delta_j, \upsilon_j) - \frac{1}{2\pi i}\int_0^1 [\log l_1(\sqrt{\tau}) - \log l_2(\sqrt{\tau})]\frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)} - \sum_{\mu=1}^{2}\int_1^{\rho_\mu}\frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)}. \tag{4.17}$$

The differentials

$$d\omega_\nu = \frac{\tau^{\nu-1}d\tau}{\eta(\tau)}, \quad \nu = 1, 2, 3, \tag{4.18}$$

form a basis of abelian differentials of the first kind on the surface $\mathcal{R}$. The integrals

$$A_{\nu j} = \oint_{\mathbf{a}_j}\frac{\tau^{\nu-1}d\tau}{\eta(\tau)}, \quad B_{\nu j} = \oint_{\mathbf{b}_j}\frac{\tau^{\nu-1}d\tau}{\eta(\tau)} \tag{4.19}$$

are the A- and B-periods of the abelian integrals

$$\omega_\nu = \omega_\nu(\zeta, u) = \int_{(\zeta_0,0)}^{(\zeta,u)} \frac{\tau^{\nu-1} d\tau}{\eta(\tau)}, \quad \nu = 1, 2, 3. \tag{4.20}$$

The contour of integration does not intersect the canonical cross-sections. The nonlinear system (4.16) with respect to the points $q_j = (\sigma_j, u_j) \in \mathcal{R}$ and the integers m_j and n_j $(j = 1, 2, 3)$ is the Jacobi inversion problem for the surface $\mathcal{R}$ of genus 3. Clearly, the solution to this problem is invariant with respect to the transformation $\mathcal{T}$. To reduce the system (4.16) to standard form, the basis of abelian integrals needs to be normalized. The A-periods of the normalized (canonical) basis form the unit matrix. The canonical basis $\hat{\omega} = \{\hat{\omega}_\nu\}$ $(\nu = 1, 2, 3)$ can be expressed through the basis $\omega = \{\omega_\nu\}$ $(\nu = 1, 2, 3)$ of abelian integrals (4.20) as follows:

$$\hat{\omega} = \mathbf{M}\omega, \tag{4.21}$$

where $\mathbf{M} = \mathbf{A}^{-1}$, $\mathbf{A} = \{A_{\nu j}\}$ $(\nu, j = 1, 2, 3)$, and $A_{\nu j}$ are given by (4.19). The A- and B-periods of the canonical basis $\hat{\omega}$ are

$$\mathcal{A}_{\nu j} = \oint_{\mathbf{a}_j} d\hat{\omega}_\nu = \delta_{\nu j},$$

$$\mathcal{B}_{\nu j} = \oint_{\mathbf{b}_j} d\hat{\omega}_\nu = \sum_{r=1}^{3} M_{\nu r} B_{rj}. \tag{4.22}$$

Here $\delta_{\nu j}$ is the Kronecker symbol. The matrix $\boldsymbol{\mathcal{B}} = \{\mathcal{B}_{\nu j}\}$ $(\nu, j = 1, 2, 3)$ is symmetric, and $\mathrm{Im}\,\boldsymbol{\mathcal{B}}$ is a positive definite matrix. The Jacobi problem (4.16) can now be formulated for the canonical basis

$$\sum_{j=1}^{3} [\hat{\omega}_\nu(q_j) + n_j \mathcal{B}_{\nu j}] + m_\nu = \hat{d}_\nu, \quad \nu = 1, 2, 3, \tag{4.23}$$

where

$$\hat{d}_\nu = \sum_{j=1}^{3} M_{\nu j} d_j, \quad \nu = 1, 2, 3, \tag{4.24}$$

and $M_{\nu j}$ are the elements of the matrix $\mathbf{M}$.

5. Solution of the Jacobi inversion problem

Consider the Jacobi inversion problem for a Riemann surface of genus h.

PROBLEM 5.1 *Given h constants $\hat{d}_\nu$ find h points $q_\nu \in \mathcal{R}$ and $2h$ integers n_ν and m_ν $(\nu = 1, 2, \ldots, h)$ such that*

$$\sum_{j=1}^{h} \hat{\omega}_\nu(q_j) = e_\nu - k_\nu - \sum_{j=1}^{h} n_j \mathcal{B}_{\nu j} - m_\nu \equiv e_\nu - k_\nu \quad \text{(modulo the periods)}, \quad \nu = 1, 2, \ldots, h, \tag{5.1}$$

where $e_\nu = \hat{d}_\nu + k_\nu$, and k_ν $(\nu = 1, 2, \ldots, h)$ are the Riemann constants.

The unknown points q_j $(j = 1, 2, \ldots, h)$ coincide with h zeros of the Riemann θ-function (see, for example, (**19**, **5**))

$$F(q) = \theta(\hat{\omega}_1(q) - e_1, \hat{\omega}_2(q) - e_2, \ldots, \hat{\omega}_h(q) - e_h) = \sum_{t_1, t_2, \ldots, t_h = -\infty}^{\infty} \mathcal{G}_{\mathbf{t}}(q). \tag{5.2}$$

where

$$\mathcal{G}_{\mathbf{t}}(q) = \exp\left\{\pi i \sum_{\mu=1}^{h}\sum_{\nu=1}^{h} \mathcal{B}_{\mu\nu} t_\mu t_\nu + 2\pi i \sum_{\nu=1}^{h} t_\nu[\hat{\omega}_\nu(q) - e_\nu]\right\}, \quad \mathbf{t} = (t_1, t_2, \dots, t_h). \tag{5.3}$$

Since the matrix $\operatorname{Im}\mathcal{B}$ is positive definite the series (5.2) converges exponentially. The integers n_ν solve the system of linear algebraic equations

$$\sum_{j=1}^{h} n_j \operatorname{Im}(\mathcal{B}_{\nu j}) = \operatorname{Im}(\varpi_\nu), \quad \nu = 1, 2, \dots, h, \tag{5.4}$$

and the integers m_ν are defined explicitly through the integers n_ν by

$$m_\nu = \operatorname{Re}(\varpi_\nu) - \sum_{j=1}^{h} n_j \operatorname{Re}(\mathcal{B}_{\nu j}). \tag{5.5}$$

Here

$$\varpi_\nu = e_\nu - k_\nu - \sum_{j=1}^{h} \hat{\omega}_\nu(q_j). \tag{5.6}$$

5.1 *Riemann's constants*

The Riemann constants $k_\nu = k_\nu^{(z)}$ were defined in (**19**, p. 144) as follows:

$$k_\nu^{(z)} = -\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}_{\nu\nu} - \sum_{j=1,\, j\neq\nu}^{h} \oint_{\mathbf{a}_j} \hat{\omega}_\nu^-(r) d\hat{\omega}_j(r). \tag{5.7}$$

Here $\hat{\omega}_\nu^-(r)$ is the limiting value of the function $\hat{\omega}_\nu(q)$ on the cross-section $\mathbf{a}_j$ from the side of the second sheet $\mathbb{C}_2$. It turns out that the use of these formulae gives non-integer values for the numbers m_ν and n_ν. In what follows a correct formula for the constants k_ν will be derived. It will also be shown that the zeros of the function (5.2) are the points q_j. Following (**19**) consider the integral

$$\mathcal{J}_\nu = \frac{1}{2\pi i}\int_{\partial\tilde{\mathcal{R}}} \hat{\omega}_\nu(r) d \log \mathfrak{F}(r), \tag{5.8}$$

where $\tilde{\mathcal{R}}$ is the surface $\mathcal{R}$ cut along all the loops $\mathbf{a}_\nu$ and $\mathbf{b}_\nu$ ($\nu = 1, 2, \dots, h$). Because the winding number of the Riemann θ-function (5.2) for each contour $\mathbf{a}_\nu$ equals 2π, by the argument principle, the θ-function has h zeros, say q'_ν ($\nu = 1, 2 \dots, h$). The residue theorem yields

$$\mathcal{J}_\nu = \sum_{j=1}^{h} \hat{\omega}_\nu(q'_j). \tag{5.9}$$

On the other hand, the same integral may be represented as

$$\mathcal{J}_\nu = \sum_{j=1}^{h} (\mathcal{J}'_{\nu j} + \mathcal{J}''_{\nu j}), \tag{5.10}$$

where

$$\mathcal{J}'_{\nu j} = \frac{1}{2\pi i}\oint_{\mathbf{a}_j} \hat{\omega}^+_\nu(r) d\log\mathfrak{F}^+(r) - \frac{1}{2\pi i}\oint_{\mathbf{a}_j} \hat{\omega}^-_\nu(r) d\log\mathfrak{F}^-(r),$$

$$\mathcal{J}''_{\nu j} = \frac{1}{2\pi i}\oint_{\mathbf{b}_j} \hat{\omega}^+_\nu(r) d\log\mathfrak{F}^+(r) - \frac{1}{2\pi i}\oint_{\mathbf{b}_j} \hat{\omega}^-_\nu(r) d\log\mathfrak{F}^-(r). \tag{5.11}$$

Because of the properties of the abelian integrals and the θ-function, on the loops $\mathbf{a}_j$

$$\hat{\omega}^+_\nu(r) = \hat{\omega}^-_\nu(r) - \mathcal{B}_{\nu j},$$

$$\mathfrak{F}^+(r) = \mathfrak{F}^-(r)\exp\{\pi i \mathcal{B}_{jj} + 2\pi i[\hat{\omega}^+_j(r) - e_j]\}, \tag{5.12}$$

and on the loops $\mathbf{b}_j$

$$\hat{\omega}^+_\nu(r) = \hat{\omega}^-_\nu(r) + \delta_{\nu j}, \quad \mathfrak{F}^+(r) = \mathfrak{F}^-(r), \tag{5.13}$$

the integrals (5.11) reduce to

$$\mathcal{J}'_{\nu j} = \frac{1}{2\pi i}\oint_{\mathbf{a}_j} \hat{\omega}^-_\nu(r) d\log\frac{\mathfrak{F}^+(r)}{\mathfrak{F}^-(r)} - \frac{\mathcal{B}_{\nu j}}{2\pi i}\oint_{\mathbf{a}_j} d\log\mathfrak{F}^+(r) = \int_{\mathbf{a}_j} \hat{\omega}^-_\nu(r) d\hat{\omega}_j(r) - n'_j\mathcal{B}_{\nu j},$$

$$\mathcal{J}''_{\nu j} = \frac{\delta_{\nu j}}{2\pi i}\oint_{\mathbf{b}_j} d\log\mathfrak{F}^+(r) = \frac{\delta_{\nu j}}{2\pi i}\left[\log\mathfrak{F}(r_j^{-+}) - \log\mathfrak{F}(r_j^{++})\right]$$
$$= \frac{\delta_{\nu j}}{2\pi i}\left[\log\frac{\mathfrak{F}(r_j^{-+})}{\mathfrak{F}(r_j^{++})} + 2\pi i m'_j\right], \tag{5.14}$$

where n'_j and m'_j are some integers. The points r_j^{++} and r_j^{-+} are shown in Fig. 5. Since the points r_j^{-+} and r_j^{++} are located on the different sides of the cross-section $\mathbf{a}_j$, from (5.12),

$$\log\frac{\mathfrak{F}(r_j^{-+})}{\mathfrak{F}(r_j^{++})} = -\pi i \mathcal{B}_{jj} - 2\pi i[\hat{\omega}_j(r_j^{++}) - e_j] + 2\pi i m''_j, \tag{5.15}$$

where m''_j are some integers. Therefore

$$\mathcal{J}_\nu = \sum_{j=1}^{h}\left(\oint_{\mathbf{a}_j} \hat{\omega}^-_\nu(r) d\hat{\omega}_j(r) - n'_j\mathcal{B}_{\nu j}\right) - \tfrac{1}{2}\mathcal{B}_{\nu\nu} - \hat{\omega}_\nu(r_\nu^{++}) + e_\nu + m'_\nu + m''_\nu. \tag{5.16}$$

Performing the integration and using $\hat{\omega}_\nu(r_\nu^{--}) = \hat{\omega}_\nu(r_\nu^{-+}) - 1$ yield

$$\oint_{\mathbf{a}_\nu} \hat{\omega}^-_\nu(r) d\hat{\omega}_\nu(r) = \frac{1}{2}\oint_{\mathbf{a}_\nu^-} d[\hat{\omega}_\nu(r)]^2$$
$$= \tfrac{1}{2}[\hat{\omega}_\nu(r_\nu^{-+}) - \hat{\omega}_\nu(r_\nu^{--})][\hat{\omega}_\nu(r_\nu^{-+}) + \hat{\omega}_\nu(r_\nu^{--})] = \hat{\omega}_\nu(r_\nu^{-+}) - \tfrac{1}{2}$$
$$= \hat{\omega}_\nu(r_\nu^{++}) + \mathcal{B}_{\nu\nu} - \tfrac{1}{2}. \tag{5.17}$$

Notice that this formula is not reducible to the expression

$$\tfrac{1}{2} + \hat{\omega}_\nu(r_\nu^{++}) \quad \text{(modulo the periods)} \tag{5.18}$$

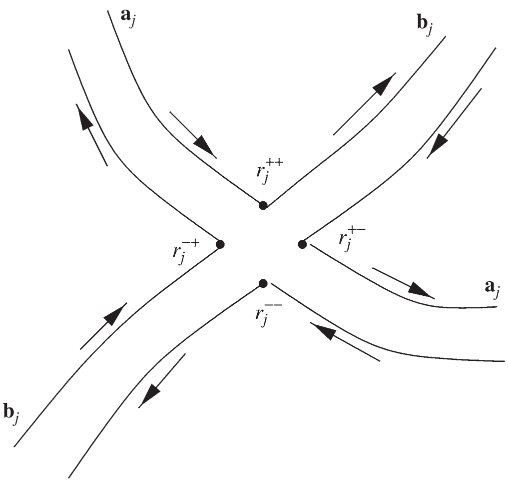

Fig. 5 Points $r_j^{++}, r_j^{+-}, r_j^{-+}$ and r_j^{--}

derived in (**19**, formula (5.15)). By substituting (5.17) into (5.16) and denoting $m_\nu = -(m'_\nu + m''_\nu) + 1$, $n_\nu = n'_\nu$, and

$$k_\nu = -\frac{1}{2} - \frac{1}{2}\mathcal{B}_{\nu\nu} - \sum_{j=1,\, j\neq\nu}^{h} \oint_{\mathbf{a}_j} \hat{\omega}_\nu^-(r)d\hat{\omega}_j(r), \tag{5.19}$$

the integral $\mathcal{J}_\nu$ is converted into

$$\mathcal{J}_\nu = e_\nu - k_\nu - \sum_{j=1}^{h} n_j \mathcal{B}_{\nu j} - m_\nu. \tag{5.20}$$

Comparing with (5.9) and putting $q_j = q'_j$ give the Jacobi inversion problem (5.1). Clearly, the Riemann constants (5.19) and the constants (5.7) defined in (**19**) are not the same. Indeed, the Jacobi problem in terms of the constants $k_\nu = k_\nu^{(z)}$ can be written as follows:

$$\sum_{j=1}^{h} \hat{\omega}_\nu(q_j) \equiv e_\nu - k_\nu^{(z)} + \mathcal{B}_{\nu\nu} \quad \text{(modulo the periods)}, \quad \nu = 1, 2, \ldots, h. \tag{5.21}$$

Equivalently,

$$\sum_{j=1}^{h} \hat{\omega}_\nu(q_j) = e_\nu - k_\nu^{(z)} - m_\nu - \sum_{j=1}^{j} (n_j - \delta_{\nu j})\mathcal{B}_{\nu j}, \quad \nu = 1, 2, \ldots, h. \tag{5.22}$$

Obviously, it is impossible to find integers m_ν and n_ν such that systems (5.1) and (5.22) are equivalent. This is the reason why use of the constants (5.7) for numerical computations gives fractional values for the constants m_ν and n_ν.

To simplify the expressions for the Riemann constants, evaluate the integral in (5.19). Rewrite it first as

$$\mathcal{K}_{\nu j} = \oint_{\mathbf{a}_j} \hat{\omega}_\nu^-(r)d\hat{\omega}_j(r) = \oint_{\mathbf{a}_j} \hat{\omega}_\nu^+(r)d\hat{\omega}_j(r) + \mathcal{B}_{\nu j}. \tag{5.23}$$

Let r^+, τ^+ and r^-, τ^- be points on the left and the right sides of the cut $[\zeta_{2j}, \zeta_{2j+1}] \subset \mathbb{C}_1$, respectively. Then

$$\hat{\omega}_\nu^+(r^\pm) = \int_{[\zeta_0,\zeta_{2j+1}]\subset\mathbb{C}_1} d\hat{\omega}_\nu(r) + \int_{[\zeta_{2j+1},r]\subset\mathbb{C}_1} d\hat{\omega}_\nu(\tau^\pm) = \hat{\omega}_\nu^+(\zeta_{2j+1}) \mp \tilde{\omega}_{\nu j}^+(r), \tag{5.24}$$

where

$$\tilde{\omega}_{\nu j}^+(r) = \int_{[r,\zeta_{2j+1}]\subset\mathbb{C}_1} d\hat{\omega}_\nu(\tau^+). \tag{5.25}$$

It follows from (5.24) and

$$\int_{[\zeta_{2j},\zeta_{2j+1}]} d\hat{\omega}_\nu(\tau^+) = \frac{1}{2}\oint_{\mathbf{a}_j} d\hat{\omega}_\nu(r) = 0, \quad \nu \neq j, \tag{5.26}$$

that

$$\hat{\omega}_\nu^+(r^+) + \hat{\omega}_\nu^+(r^-) = 2\hat{\omega}_\nu^+(\zeta_{2j+1}) = 2\hat{\omega}_\nu^+(\zeta_{2j}). \tag{5.27}$$

Hence

$$\mathcal{K}_{\nu j} = \mathcal{B}_{\nu j} + \hat{\omega}_\nu^+(\zeta_{2j}). \tag{5.28}$$

For a hyperelliptic surface $\mathcal{R}$ of genus h with the system of cross-sections as in Fig. 3, the integrals $\hat{\omega}_\nu^+(\zeta_{2j})$ can be calculated. By the Cauchy theorem,

$$\oint_{[\zeta_0,\zeta_1]} d\hat{\omega}_\nu(r) = -\sum_{j=1}^{h}\oint_{\mathbf{a}_j} d\hat{\omega}_\nu(r) = -1. \tag{5.29}$$

Therefore

$$\hat{\omega}_\nu^+(\zeta_2) = \left(\int_{[\zeta_0,\zeta_1]^+\subset\mathbb{C}_1} + \int_{[\zeta_1,\zeta_2]\subset\mathbb{C}_1}\right) d\hat{\omega}_\nu(r) = -\tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}_{\nu 1}. \tag{5.30}$$

Similarly, for $j = 2$

$$\hat{\omega}_\nu^+(\zeta_4) = \hat{\omega}_\nu^+(\zeta_2) + \tfrac{1}{2}\delta_{\nu 1} + \int_{[\zeta_3,\zeta_4]\subset\mathbb{C}_1} d\hat{\omega}_\nu(r) = -\tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}_{\nu 2} + \tfrac{1}{2}\delta_{\nu 1}. \tag{5.31}$$

For any $j \geqslant 1$, it may be deduced that

$$\hat{\omega}_\nu^+(\zeta_{2j}) = -\tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}_{\nu j} + \tfrac{1}{2}(\delta_{\nu 1} + \delta_{\nu 2} + \cdots + \delta_{\nu, j-1}). \tag{5.32}$$

By substituting (5.32) into (5.28) and (5.19), the Riemann constants k_ν become

$$k_\nu = -1 + \tfrac{1}{2}\nu - \frac{1}{2}\sum_{j=1}^{h}\mathcal{B}_{\nu j}. \tag{5.33}$$

Notice that the use of another set of the Riemann constants linked to the constants (5.33) by the relation

$$\hat{k}_\nu = k_\nu + \hat{m}_\nu + \hat{n}_\nu \sum_{j=1}^{h}\mathcal{B}_{\nu j} \tag{5.34}$$

(for example, $\hat{k}_\nu = -\frac{1}{2}\nu + \frac{1}{2}(\mathcal{B}_{\nu 1} + \cdots + \mathcal{B}_{\nu h})$) gives also a solution to the Jacobi problem. Here $\hat{m}_\nu$ and $\hat{n}_\nu$ are integers. In this case the Riemann θ-function should be taken in the form

$$\mathfrak{F}(q) = \theta(\hat{\omega}_1(q) - \hat{e}_1, \hat{\omega}_2(q) - \hat{e}_2, \ldots, \hat{\omega}_h(q) - \hat{e}_h), \quad \hat{e}_\nu = \hat{d}_\nu + \hat{k}_\nu. \tag{5.35}$$

This result follows from the periodicity properties of the Riemann θ-function

$$\theta(\hat{\omega}_1 + j_1, \hat{\omega}_\nu + j_2, \ldots, \hat{\omega}_h + j_h) = \theta(\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_h), \quad j_\nu = 0, 1,$$

$$\theta(\hat{\omega}_1 + \mathcal{B}_{\nu 1}, \hat{\omega}_2 + \mathcal{B}_{\nu 2}, \ldots, \hat{\omega}_h + \mathcal{B}_{\nu h}) = \mathrm{e}^{-\pi i \mathcal{B}_{\nu\nu} - 2\pi i \hat{\omega}_\nu} \theta(\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_h), \quad \nu = 1, 2, \ldots, h. \tag{5.36}$$

5.2 *Affixes of the points* q_1, q_2 *and* q_3

The problem of finding the three zeros of the θ-function

$$\mathfrak{F}(q) = \theta(\hat{\omega}_1(q) - e_1, \hat{\omega}_2(q) - e_2, \hat{\omega}_(q) - e_3) = \sum_{t_1, t_2, t_3 = -\infty}^{\infty} \mathcal{G}_{\mathbf{t}}(q) \tag{5.37}$$

for $h = 3$ can be reduced to a cubic equation. Here

$$\mathcal{G}_{\mathbf{t}}(q) = \exp\left\{\pi i \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} \mathcal{B}_{\mu\nu} t_\mu t_\nu + 2\pi i \sum_{\nu=1}^{3} t_\nu [\hat{\omega}_\nu(q) - e_\nu]\right\}, \quad \mathbf{t} = (t_1, t_2, t_3),$$

$$e_\nu = \hat{d}_\nu + k_\nu, \quad k_\nu = -\tfrac{1}{2}\nu + \tfrac{1}{2}(\mathcal{B}_{\nu 1} + \mathcal{B}_{\nu 2} + \mathcal{B}_{\nu 3}), \quad \nu = 1, 2, 3, \tag{5.38}$$

and the coefficients $\hat{d}_\nu$ are given by (4.24).

Consider the integral

$$\mathcal{N}_\nu = \frac{1}{2\pi i} \int_{\partial \hat{\mathcal{R}}} \tau^{-\nu} d \log \mathfrak{F}(r) \tag{5.39}$$

over the boundary $\partial\hat{\mathcal{R}}$ of the surface $\mathcal{R}$ cut along the cross-sections $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. By the logarithmic residue theorem,

$$\mathcal{N}_\nu = \sum_{j=1}^{3} \sigma_j^{-\nu} + \sum_{\mu=1}^{2} \left(\operatorname*{res}_{q=0_\mu} + \operatorname*{res}_{q=\infty_\mu}\right) \frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)}, \tag{5.40}$$

where $0_\mu = (0, (-1)^{\mu-1} p^{1/2}(0)) \in \mathbb{C}_\mu$, and $\infty_\mu \in \mathbb{C}_\mu$ ($\mu = 1, 2$) are the two infinite points of the surface. Without loss of generality, $\mathfrak{F}(0_\mu) \neq 0$, and $\mathfrak{F}(\infty_\mu) \neq 0$, $\mu = 1, 2$. Because of the asymptotics

$$\frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)} = O(\zeta^{-\nu-2}), \quad \zeta \to \infty, \quad \nu = 1, 2, 3, \tag{5.41}$$

the residues at the infinite points ∞_μ vanish. The same integral $\mathcal{N}_\nu$ may be represented as

$$\mathcal{N}_\nu = \sum_{j=1}^{3} \oint_{\mathbf{a}_j} t^{-\nu} d\hat{\omega}_j^{+}(r) = 2 \sum_{j=1}^{3} \sum_{l=1}^{3} M_{jl} \int_{[\zeta_{2j}\zeta_{2j+1}]} \frac{\tau^{l-\nu-1} d\tau}{p^{1/2}(\tau^{+})}, \tag{5.42}$$

where $\tau^{+} \in [\zeta_{2j}, \zeta_{2j+1}]^{+}$. Comparing the two expressions for $\mathcal{N}_\nu$ yields

$$\frac{1}{\sigma_1^{\nu}} + \frac{1}{\sigma_2^{\nu}} + \frac{1}{\sigma_3^{\nu}} = \varepsilon_\nu, \quad \nu = 1, 2, 3, \tag{5.43}$$

where

$$\varepsilon_\nu = 2 \sum_{j=1}^{3} \sum_{l=1}^{3} M_{jl} \int_{[\zeta_{2j}\zeta_{2j+1}]} \frac{\tau^{l-\nu-1} d\tau}{p^{1/2}(\tau^{+})} - \sum_{\mu=1}^{2} \operatorname*{res}_{q=0_\mu \in \mathbb{C}_\mu} \frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)}. \tag{5.44}$$

Note that the integral (5.39) is different from the one used in (**19**, **5**). The function τ^{ν} is replaced by the function $\tau^{-\nu}$. This change simplifies the task of elevating the coefficients ε_ν. Instead of the residues at the infinite points ∞_μ ($\mu = 1, 2$), it is required to find the residues at the two zeros of the surface $\mathcal{R}$.

The systems of algebraic equations (5.43) is converted into the cubic equation

$$(\varepsilon_1^3 - 3\varepsilon_1\varepsilon_2 + 2\varepsilon_3)\sigma^3 + 3(\varepsilon_2 - \varepsilon_1^2)\sigma^2 + 6\varepsilon_1\sigma - 6 = 0 \tag{5.45}$$

with complex coefficients. The three roots of this equation define the affixes of the zeros of the θ-function $\mathfrak{F}(q)$.

5.3 *Numerical procedure*

The first step of the procedure is to evaluate the matrix of B-periods of the canonical basis $\hat{\omega}$ by formulae (4.21) and (4.22). This requires the A- and B-periods of the basis ω:

$$A_{\nu j} = 2\int_{[\zeta_{2j}\zeta_{2j+1}]} \frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau^+)}, \quad j = 1, 2, 3,$$

$$B_{\nu 1} = -2\int_{[\zeta_1\zeta_2]} \frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)}, \quad B_{\nu 2} = B_{\nu 1} - 2\int_{[\zeta_3\zeta_4]} \frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)},$$

$$B_{\nu 3} = B_{\nu 2} - 2\int_{[\zeta_5\zeta_6]} \frac{\tau^{\nu-1}d\tau}{p^{1/2}(\tau)}, \quad \nu = 1, 2, 3. \tag{5.46}$$

These integrals are computed by the Gauss–Chebyshev quadrature formulae. The branch of the function $p^{1/2}(\zeta)$ in the cut ζ-plane (Fig. 2) has been fixed by the condition $p^{1/2}(\zeta) \sim \zeta^4$, $\zeta \to \infty$. For numerical purposes, this branch can algorithmically be described as follows. Define

$$y_m = \zeta_{2m+1} - \zeta_{2m} \quad (m = 0, 1, 2, 3), \quad \beta_m = \begin{cases} \tan^{-1}(\operatorname{Im} y_m / \operatorname{Re} y_m), & \operatorname{Re} y_m \neq 0, \\ \pi/2, & \operatorname{Re} y_m = 0, \end{cases}$$

$$\mu_j = \zeta - \zeta_j \quad (j = 0, 1, \ldots, 7), \quad \tau_j = \begin{cases} \tan^{-1}(\operatorname{Im} \mu_j / \operatorname{Re} \mu_j), & \operatorname{Re} \mu_j \neq 0, \\ \pi/2, & \operatorname{Re} \mu_j = 0,\ \operatorname{Im} \mu_j > 0, \\ -\pi/2, & \operatorname{Re} \mu_j = 0,\ \operatorname{Im} \mu_j < 0. \end{cases} \tag{5.47}$$

Then the branch $p^{1/2}(\zeta)$ behaves at infinity as ζ^4 if

$$p^{1/2}(\zeta) = \sqrt{\prod_{j=0}^{7} |\mu_j|}\, \exp\left\{\frac{i}{2}\sum_{j=0}^{7}\theta_j\right\}, \quad \theta_j = \arg \mu_j. \tag{5.48}$$

In the case $\beta_m < 0$ and $\zeta \notin \gamma_m$ the arguments θ_j are chosen to be $-\pi + \beta_m < \theta_j < \pi + \beta_m$ ($m = 0, 1, 2, 3$). This means that

$$\theta_j = \begin{cases} \tau_j, & \operatorname{Re} \mu_j \geqslant 0, \\ \tau_j + \pi, & \operatorname{Re} \mu_j < 0,\ \tau_j \leqslant \beta_m, \\ \tau_j - \pi, & \operatorname{Re} \mu_j < 0,\ \tau_j > \beta_m, \end{cases} \quad m = [j/2], \tag{5.49}$$

where $[j/2]$ is the integer part of $j/2$. If $\beta_m \geqslant 0$ and as before $\zeta \notin \gamma_m$, then $\beta_m < \theta_j < 2\pi + \beta_m$. Therefore,

$$\theta_j = \begin{cases} \tau_j + \pi, & \operatorname{Re} \mu_j < 0, \\ \tau_j + 2\pi, & \operatorname{Re} \mu_j \geqslant 0,\ \tau_j < \beta_m, \\ \tau_j, & \operatorname{Re} \mu_j \geqslant 0,\ \tau_j \geqslant \beta_m, \end{cases} \quad m = [j/2]. \tag{5.50}$$

The evaluation of the A-periods by formulae (5.46) requires defining $p^{1/2}(\tau^+)$, where $\tau^+ \in \gamma_m^+$. In this case for $\beta_m < 0$,

$$\theta_j = \begin{cases} \beta_m - \pi, & j = 0, 2, 4, 6, \\ \beta_m, & j = 1, 3, 5, 7, \end{cases} \quad m = [j/2], \tag{5.51}$$

and if $\beta_m \geqslant 0$, then

$$\theta_j = \begin{cases} \beta_m, & j = 0, 2, 4, 6, \\ \beta_m + \pi, & j = 1, 3, 5, 7, \end{cases} \quad m = [j/2]. \tag{5.52}$$

To compute the coefficients of the cubic equation (5.45) by formulae (5.44) one needs to evaluate the residues of the function $\zeta^{-\nu}\mathfrak{F}'(q)/\mathfrak{F}(q)$ $(\nu = 1, 2, 3)$ at the points $q = 0_\mu \in \mathbb{C}_\mu$, $\mu = 1, 2$. They are given by

$$\begin{aligned} \operatorname*{res}_{q=0_\mu\in\mathbb{C}_\mu} \frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)}\bigg|_{\nu=1} &= \frac{\mathfrak{F}'(q)}{\mathfrak{F}(q)}\bigg|_{q=0_\mu\in\mathbb{C}_\mu}, \\ \operatorname*{res}_{q=0_\mu\in\mathbb{C}_\mu} \frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)}\bigg|_{\nu=2} &= \left[\frac{\mathfrak{F}''(q)}{\mathfrak{F}(q)} - \left(\frac{\mathfrak{F}'(q)}{\mathfrak{F}(q)}\right)^2\right]\bigg|_{q=0_\mu\in\mathbb{C}_\mu}, \\ \operatorname*{res}_{q=0_\mu\in\mathbb{C}_\mu} \frac{\zeta^{-\nu}\mathfrak{F}'(q)}{\mathfrak{F}(q)}\bigg|_{\nu=3} &= \tfrac{1}{2}\left[\frac{\mathfrak{F}'''(q)}{\mathfrak{F}(q)} - \frac{3\mathfrak{F}''(q)\mathfrak{F}'(q)}{\mathfrak{F}^2(q)} + 2\left(\frac{\mathfrak{F}'(q)}{\mathfrak{F}(q)}\right)^3\right]\bigg|_{q=0_\mu\in\mathbb{C}_\mu}, \end{aligned} \tag{5.53}$$

where

$$\begin{aligned} \mathfrak{F}'(0_\mu) &= 2\pi i \sum_{t_1,t_2,t_3=-\infty}^{\infty} \sigma_{\mu\mathbf{t}}^{(1)} \mathcal{G}_{\mathbf{t}}(0_\mu), \\ \mathfrak{F}''(0_\mu) &= 2\pi i \sum_{t_1,t_2,t_3=-\infty}^{\infty} \left[2\pi i \left(\sigma_{\mu\mathbf{t}}^{(1)}\right)^2 + \sigma_{\mu\mathbf{t}}^{(2)}\right] \mathcal{G}_{\mathbf{t}}(0_\mu), \\ \mathfrak{F}'''(0_\mu) &= 2\pi i \sum_{t_1,t_2,t_3=-\infty}^{\infty} \left[-4\pi^2 \left(\sigma_{\mu\mathbf{t}}^{(1)}\right)^3 + 6\pi i \sigma_{\mu\mathbf{t}}^{(1)}\sigma_{\mu\mathbf{t}}^{(2)} + \sigma_{\mu\mathbf{t}}^{(3)}\right] \mathcal{G}_{\mathbf{t}}(0_\mu), \\ \sigma_{\mu\mathbf{t}}^{(s)} &= \sum_{\nu=1}^{3} t_\nu \hat{\omega}_\nu^{(s)}(0_\mu). \end{aligned} \tag{5.54}$$

The derivatives $\hat{\omega}_\nu^{(s)}(0_\mu)$ of the abelian integrals $\hat{\omega}_\nu$ at the two zeros of the surface $\mathcal{R}$ are defined as follows:

$$\hat{\omega}_\nu'(0_\mu) = \frac{(-1)^{\mu-1}M_{\nu 1}}{p^{1/2}(0)},$$
$$\hat{\omega}_\nu''(0_\mu) = \frac{(-1)^{\mu-1}}{p^{1/2}(0)}\left[M_{\nu 2} - \frac{M_{\nu 1}p'(0)}{2p(0)}\right],$$
$$\hat{\omega}_\nu'''(0_\mu) = \frac{(-1)^{\mu-1}}{p^{1/2}(0)}\left\{2M_{\nu 3} - \frac{M_{\nu 2}p'(0)}{p(0)} - \frac{M_{\nu 1}}{2}\left[\frac{p''(0)}{p(0)} - \frac{3}{2}\left(\frac{p'(0)}{p(0)}\right)^2\right]\right\}, \quad (5.55)$$

where

$$\frac{p'(0)}{p(0)} = -\sum_{j=0}^{7}\frac{1}{\zeta_j}, \quad \frac{p''(0)}{p(0)} = \left(\sum_{j=0}^{7}\frac{1}{\zeta_j}\right)^2 - \sum_{j=0}^{7}\frac{1}{\zeta_j^2}. \quad (5.56)$$

To find a root $\sigma_1 = x_1 + ix_2$ of equation (5.45) the following procedure is applied. Equation (5.45) is written as $H_1 + iH_2 = 0$, where

$$H_1 = x_1^3 - 3x_1x_2^2 + a_1(x_1^2 - x_2^2) - 2a_2x_1x_2 + b_1x_1 - b_2x_2 + c_1,$$
$$H_2 = -x_2^3 + 3x_1^2x_2 + a_2(x_1^2 - x_2^2) + 2a_1x_1x_2 + b_1x_2 + b_2x_1 + c_2,$$
$$a_1 + ia_2 = 3(\varepsilon_2 - \varepsilon_1^2)\varepsilon_*^{-1}, \quad b_1 + ib_2 = 6\varepsilon_1\varepsilon_*^{-1},$$
$$c_1 + ic_2 = -6\varepsilon_*^{-1}, \quad \varepsilon_* = \varepsilon_1^3 - 3\varepsilon_1\varepsilon_2 + 2\varepsilon_3. \quad (5.57)$$

The Newton–Raphson iteration for evaluation of x_1 and x_2

$$x_j^{(n)} = x_j^{(n-1)} - \frac{H_{1,1}H_j + (-1)^j H_{1,2}H_{3-j}}{\Delta},$$
$$\Delta = H_{1,1}^2 + H_{1,2}^2, \quad H_{1,j} = \frac{\partial H_1}{\partial x_j}, \quad j = 1, 2, \quad (5.58)$$

is employed. The standard iteration formula becomes simpler because of the relations $H_{2,2} = H_{1,1}$ and $H_{2,1} = -H_{1,2}$. The other two roots of equation (5.45) are found from the associated quadratic equation.

The numerical algorithm proposed can be tested by checking numerically the following properties.

(i) The matrix $\mathcal{B}$ of B-periods is symmetric and $\operatorname{Im}\mathcal{B}$ is a positive definite matrix.
(ii) The Riemann θ-function meets the conditions (5.36) ($h = 3$).
(iii) Among each pair of points $(\sigma_j, p^{1/2}(\sigma_j)) \in \mathbb{C}_1$ and $(\sigma_j, -p^{1/2}(\sigma_j)) \in \mathbb{C}_2$ ($j = 1, 2, 3$) there is one and only one point q_j such that $\mathfrak{F}(q_j) = 0$.
(iv) The solution n_ν ($\nu = 1, 2, 3$) of the algebraic system (5.4) and the numbers m_ν defined by (5.5) are *integers*.
(v) The points q_j and the integers m_j and n_j satisfy the conditions (4.16), and therefore the solution $X(\zeta, u)$ is bounded at infinity.
(vi) The solution of the Jacobi problem is independent of the paths Ξ_j, $j = 1, 2, 3$. It depends on the location of the free points r_1, r_2, r_3, p_{11} and p_{21}. However, the solution of Problem 2.2 is independent of the choice of these points provided the paths Ξ_j ($j = 1, 2, 3$) do not intersect the **a**- and **b**-loops and the contour $\mathcal{L}$.

All these properties of the solution have been successfully verified.

For the parameters ρ_μ and δ_j chosen to be

$$\rho_\mu = \mu + 1 \quad (\mu = 1, 2), \qquad \delta_j = \zeta_0 - ij \quad (j = 1, 2, 3), \tag{5.59}$$

and for the set of parameters (3.22) it turns out that $n_\nu = m_\nu = 0$, $\nu = 1, 2, 3$,

$$\sigma_1 = 0{\cdot}3104 - 0{\cdot}4492i, \quad \sigma_2 = -2{\cdot}5062 - 1{\cdot}2753i, \quad \sigma_3 = -4{\cdot}6000 + 2{\cdot}0691i.$$

The points q_1 and q_3 lie on the first sheet $\mathbb{C}_1$, and $q_2 \in \mathbb{C}_2$. If $\beta = \frac{1}{8}\pi$ and the other parameters are the same, then $n_\nu = m_\nu = 0$, $\nu = 1, 2, 3$, $q_1 \in \mathbb{C}_1$, $q_2 \in \mathbb{C}_2$, $q_3 \in \mathbb{C}_2$, and their affixes are

$$\sigma_1 = 0{\cdot}5112 - 0{\cdot}2745i, \quad \sigma_2 = 6{\cdot}5251 - 3{\cdot}8299i, \quad \sigma_3 = -15{\cdot}8337 + 10{\cdot}4951i.$$

It is clear that if $|\zeta|$ is large, then even a small error of computation in the affixes of the points q_j may produce a big error for the values of the function $X(\zeta, u)$. To overcome this difficulty, in the case $|\zeta| > 1$, formula (4.12) for the function $\chi_2(\zeta)$ is replaced by

$$\chi_2(\zeta) = \frac{1}{\zeta^3}\left\{\frac{1}{4\pi i}\int_0^1 [\log l_1(\sqrt{\tau}) - \log l_2(\sqrt{\tau})]\frac{\tau^3 d\tau}{p^{1/2}(\tau)(\tau-\zeta)} + \frac{1}{2}\int_1^{\rho_1}\frac{\tau^3 d\tau}{p^{1/2}(\tau)(\tau-\zeta)}\right.$$
$$\left. + \frac{1}{2}\int_1^{\rho_2}\frac{\tau^3 d\tau}{p^{1/2}(\tau)(\tau-\zeta)} + \frac{1}{2}\sum_{j=1}^{3}\left(\int_{\Xi_j} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{\tau^3 d\tau}{\eta(\tau)(\tau-\zeta)}\right\}. \tag{5.60}$$

Notice the above formula is not an asymptotic relation but an exact equality. The right-hand side of the second formula in (4.12) coincides with (5.60) provided the integers n_j and m_j and the points q_j $(j = 1, 2, 3)$ solve the Jacobi problem (4.23). Formula (5.60) guarantees the stability of the numerical values of the function $\exp\{u(\zeta)\chi_2(\zeta)\}$ for large $|\zeta|$ and makes the numerical algorithm efficient for numerical purposes.

6. Solution to the physical problem and definition of the constants

To proceed now with the general solution of Problem 4.1, notice that the function $X(\zeta, u)$ defined by (4.7) has simple zeros at the points $p_{11} = (\rho_1, z_1)$ and $q_j = (\sigma_j, u_j)$ $(j = 1, 2, 3)$, where $u_j = (-1)^{\mu-1}p^{1/2}(\sigma_j)$, $\mu = 1$ if $q_j \in \mathbb{C}_1$ and $\mu = 2$ if $q_j \in \mathbb{C}_2$. Therefore, the general solution may have removable singularities at these points. Due to the presence of the prescribed poles of the functions $\phi_\mu^\pm(s)$ $(\mu = 1, 2)$ at $s = \theta_0$ and $s = \pm 2\pi - \theta_0$, the functions $F_{\mu\pm}(w)$ have simple poles at the points $w = i\tan(\theta_0 \pm \pi)/4$ and $w = -i\tan(\theta_0 \pm \pi)/4$. Thus, the general solution of the problem (4.3) on the surface $\mathcal{R}$ has simple poles at the points of the surface with affixes $\alpha_{1\pm} = -\tan^2\frac{1}{4}(\pi \pm \theta_0)$. Next, the functions $\phi_1^\pm(s)$ and $\phi_2^\pm(s)$ are analytic at $s = \frac{1}{2}\pi$, an inner point of the strips $\Pi^\pm$. This point is the image of the infinite points ∞_μ $(\mu = 1, 2)$ of the Riemann surface $\mathcal{R}$. Therefore, the functions $F_\pm(\zeta, u)$ are bounded at the infinite points. By the generalized Liouville theorem applied to the analytic continuations of the functions $[X^+(\tau, \eta)]^{-1}F_\pm^+(\tau, \eta)$ and $[X^-(\tau, \eta)]^{-1}F_\pm^-(\tau, \eta)$ from the contour $\mathcal{L}$ to the surface $\mathcal{R}$,

$$F_\pm(\zeta, u) = X(\zeta, u)R_\pm(\zeta, u), \tag{6.1}$$

where

$$R_{\pm}(\zeta,u)=R_{1\pm}(\zeta)+u(\zeta)R_{2\pm}(\zeta),$$

$$R_{1\pm}(\zeta)=\sum_{j=1}^{3}\frac{C_j^{\pm}u_j}{\zeta-\sigma_j}+\frac{C_{12}^{\pm}}{\zeta-\alpha_{1\pm}}+\frac{C_{14}^{\pm}z_1}{\zeta-\rho_1}+C_{15}^{\pm},$$

$$R_{2\pm}(\zeta)=\sum_{j=1}^{3}\frac{C_j^{\pm}}{\zeta-\sigma_j}+\sum_{j=0}^{7}\frac{C_{j+4}^{\pm}}{\zeta-\zeta_j}+\frac{C_{13}^{\pm}}{\zeta-\alpha_{1\pm}}+\frac{C_{14}^{\pm}}{\zeta-\rho_1}, \tag{6.2}$$

and $C_j^{\pm}$ ($j=1,2,\dots,15$) are arbitrary constants to be defined. From (6.2) it is seen that the rational functions $R_{\pm}(\zeta,u)$ have simple poles at the points $q_j=(\sigma_j,u_j)$, and they are bounded at the points $(\sigma_j,-u_j)$. Because of the factor $u(\zeta)$ in the first formula (6.2), the points ζ_j ($j=0,1,\dots,7$) are simple poles of the functions $F_{\pm}(\zeta,u)$ on the Riemann surface $\mathcal{R}$. The solution has prescribed poles at the points $(\alpha_{1\pm},\,p^{1/2}(\alpha_{1\pm}))$ and $(\alpha_{1\pm},-p^{1/2}(\alpha_{1\pm}))$. The point $(\rho_1,\,p^{1/2}(\rho_1))\in\mathbb{C}_1$ is a removable singular point for the functions $F_{\pm}(\zeta,u)$. This is because $z_1=p^{1/2}(\rho_1)$, and the function $X(\zeta,u)$ has a simple zero at the point $(\rho_1,\,p^{1/2}(\rho_1))$. The point $(\rho_1,-p^{1/2}(\rho_1))\in\mathbb{C}_2$ is a regular point.

The functions $F_{\pm}(\zeta,u)$ have to be bounded at the infinite points ∞_μ ($\mu=1,2$). Therefore

$$\lim_{\zeta\to\infty}\zeta^{\nu}R_{2\pm}(\zeta)=0,\quad \nu=1,2,3. \tag{6.3}$$

Equivalently, these three conditions may be written as

$$\sum_{j=1}^{3}C_j^{\pm}\sigma_j^{\nu-1}+\sum_{j=0}^{7}C_{j+4}^{\pm}\zeta_j^{\nu-1}+C_{13}^{\pm}\alpha_{1\pm}^{\nu-1}+C_{14}^{\pm}\rho_1^{\nu-1}=0,\quad \nu=1,2,3. \tag{6.4}$$

It will be convenient to use the following representations for the rational functions $R_{1\pm}(\zeta)$ and $R_{2\pm}(\zeta)$:

$$R_{1\pm}(\zeta)=\sum_{j=1}^{15}C_j^{\pm}\varphi_{j\pm}(\zeta),\quad R_{2\pm}(\zeta)=\sum_{j=1}^{15}C_j^{\pm}\psi_{j\pm}(\zeta), \tag{6.5}$$

where

$$\varphi_{j\pm}(\zeta)=\frac{u_j}{\zeta-\sigma_j},\quad \psi_{j\pm}(\zeta)=\frac{1}{\zeta-\sigma_j},\quad j=1,2,3,$$

$$\varphi_{j\pm}(\zeta)=0,\quad \psi_{j\pm}(\zeta)=\frac{1}{\zeta-\zeta_{j-4}},\quad j=4,5,\dots,11,$$

$$\varphi_{12\pm}(\zeta)=\psi_{13\pm}(\zeta)=\frac{1}{\zeta-\alpha_{1\pm}},\quad \psi_{12\pm}(\zeta)=\varphi_{13\pm}(\zeta)=0,$$

$$\varphi_{14\pm}(\zeta)=\frac{z_1}{\zeta-\rho_1},\quad \psi_{14\pm}(\zeta)=\frac{1}{\zeta-\rho_1},\quad \varphi_{15\pm}(\zeta)=1,\quad \psi_{15\pm}(\zeta)=0. \tag{6.6}$$

Here $z_1=p^{1/2}(\rho_1)$.

Because of the poles of the function $X(\zeta, u)$ at the points $r_\nu = (\delta_\nu, \upsilon_\nu) \in \mathbb{C}_1$ ($\nu = 1, 2, 3$) and $p_{21} = (\rho_2, -z_2) \in \mathbb{C}_2$, the general solution (6.1) has inadmissible poles at these points. The following conditions remove the singularities:

$$\sum_{j=1}^{15} [\varphi_{j\pm}(\delta_\nu) + \upsilon_\nu \psi_{j\pm}(\delta_\nu)] C_j^{\pm} = 0, \quad \nu = 1, 2, 3,$$

$$\sum_{j=1}^{15} [\varphi_{j\pm}(\rho_2) - z_2 \psi_{j\pm}(\rho_2)] C_j^{\pm} = 0. \tag{6.7}$$

To derive the other conditions for the constants $C_j^{\pm}$ ($j = 1, 2, \ldots, 15$), expressions for the functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$ are needed. Inverting the relations (3.9) and using (3.15), (4.2), (4.7), (4.12), and (6.1) yield

$$\Phi_1^{\pm}(s) = F_{\pm}(\zeta, u) + F_{\pm}(\zeta, -u),$$

$$\Phi_2^{\pm}(s) = -f_1^{\pm}(s)[F_{\pm}(\zeta, u) + F_{\pm}(\zeta, -u)] + f_{\pm}^{1/2}(s)[F_{\pm}(\zeta, u) - F_{\pm}(\zeta, -u)], \tag{6.8}$$

where

$$F_{\pm}(\zeta, u) = \mathrm{e}^{\chi_1(\zeta) + u(\zeta)\chi_2(\zeta)} [R_{1\pm}(\zeta) + u(\zeta) R_{2\pm}(\zeta)]. \tag{6.9}$$

The above formulae may be rewritten in the form

$$\Phi_1^{\pm}(s) = 2\hat{\chi}_1(\zeta)[R_{1\pm}(\zeta)\cosh\hat{\chi}_2(\zeta) + p^{1/2}(\zeta) R_{2\pm}(\zeta)\sinh\hat{\chi}_2(\zeta)],$$

$$\Phi_2^{\pm}(s) = -f_1^{\pm}(s)\Phi_1^{\pm}(s) + 2 f_{\pm}^{1/2}(s)\hat{\chi}_1(\zeta)[R_{1\pm}(\zeta)\sinh\hat{\chi}_2(\zeta) + p^{1/2}(\zeta) R_{2\pm}(\zeta)\cosh\hat{\chi}_2(\zeta)], \tag{6.10}$$

where

$$\hat{\chi}_1(\zeta) = \mathrm{e}^{\chi_1(\zeta)}, \quad \hat{\chi}_2(\zeta) = p^{1/2}(\zeta)\chi_2(\zeta). \tag{6.11}$$

By substituting (6.5) into (6.10), expressions with explicit dependence on the constants $C_j^{\pm}$ are obtained:

$$\Phi_1^{\pm}(s) = 2\hat{\chi}_1(\zeta) \sum_{j=1}^{15} C_j^{\pm} [\varphi_{j\pm}(\zeta)\cosh\hat{\chi}_2(\zeta) + p^{1/2}(\zeta)\psi_{j\pm}(\zeta)\sinh\hat{\chi}_2(\zeta)],$$

$$\begin{aligned}\Phi_2^{\pm}(s) = 2\hat{\chi}_1(\zeta) \sum_{j=1}^{15} C_j^{\pm} \Big\{ &[-f_1^{\pm}(s)\varphi_{j\pm}(\zeta) + f_{\pm}^{1/2}(s) p^{1/2}(\zeta)\psi_{j\pm}(\zeta)]\cosh\hat{\chi}_2(\zeta) \\ &+ [-f_1^{\pm}(s) p^{1/2}(\zeta)\psi_{j\pm}(\zeta) + f_{\pm}^{1/2}(s)\varphi_{j\pm}(\zeta)]\sinh\hat{\chi}_2(\zeta)\Big\}.\end{aligned} \tag{6.12}$$

The functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$ have to meet the four conditions (2.17) which give the next four equations for the constants $C_j^{\pm}$ ($j = 1, 2, \ldots, 15$):

$$\begin{aligned}\sum_{j=1}^{15} \Big\{ &\xi'_{\nu\pm}\varphi_{j\pm}(\hat{\xi}_\nu^{\pm}) - \xi''_{\nu\pm} p^{1/2}(\hat{\xi}_\nu^{\pm})\psi_{j\pm}(\hat{\xi}_\nu^{\pm}) \\ &+ [\xi'_{\nu\pm} p^{1/2}(\hat{\xi}_\nu^{\pm})\psi_{j\pm}(\hat{\xi}_\nu^{\pm}) - \xi''_{\nu\pm}\varphi_{j\pm}(\hat{\xi}_\nu^{\pm})]\tanh\hat{\chi}_2(\hat{\xi}_\nu^{\pm})\Big\} C_j^{\pm} = 0, \quad \nu = 1, 2, 3, 4,\end{aligned} \tag{6.13}$$

where

$$\hat{\xi}_\nu^{\pm} = -\cot^2 \tfrac{1}{4}(\pi \mp \xi_\nu^{\pm}), \quad \xi'_{\nu\pm} = -\sin \xi_\nu^{\pm} \pm \eta_2^{\pm} \sin\beta + \cos \xi_\nu^{\pm} \cos\beta f_1^{\pm}(\xi_\nu^{\pm}),$$

$$\xi''_{\nu\pm} = \cos \xi_\nu^{\pm} \cos\beta f_{\pm}^{1/2}(\xi_\nu^{\pm}), \quad \nu = 1, 2, 3, 4. \tag{6.14}$$

The function $\Phi_2^{\pm}(s)$ defined by (6.12) has two simple poles at the points $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. To remove these singularities the following conditions have to be imposed:

$$\operatorname*{res}_{s=\frac{1}{2}\pi} \Phi_2^{\pm}(s) = 0, \qquad \operatorname*{res}_{s=-\frac{1}{2}\pi} \Phi_2^{\pm}(s) = 0. \tag{6.15}$$

Because of the symmetry property $\Phi_2^{\pm}(s) = \Phi_2^{\pm}(\pm 2\pi - s)$, the function $\Phi_2^{\pm}(s)$ is regular at the points $s = \pm\frac{3}{2}\pi$ and $s = \pm\frac{5}{2}\pi$ provided it meets the conditions (6.15). Equations (6.15) can be rewritten in the form

$$\sum_{j=1}^{15} \Big\{ \beta' \varphi_{j\pm}(\hat{\zeta}_\nu) + \beta'' \sin\beta p^{1/2}(\hat{\zeta}_\nu) \psi_{j\pm}(\hat{\zeta}_\nu) + \Big[\beta' p^{1/2}(\hat{\zeta}_\nu)\psi_{j\pm}(\hat{\zeta}_\nu) + \beta'' \sin\beta \varphi_{j\pm}(\hat{\zeta}_\nu)\Big] \tanh \hat{\chi}_2(\hat{\zeta}_\nu) \Big\} C_j^{\pm} = 0, \quad \nu = 1, 2, \tag{6.16}$$

where

$$\hat{\zeta}_1 = -\cot^2 \tfrac{1}{8}\pi, \quad \hat{\zeta}_2 = -\tan^2 \tfrac{1}{8}\pi,$$

$$\beta' = \frac{1 - g_0 \sin^2\beta}{\sin\beta}, \quad \beta'' = \sqrt{f^*\left(-\tfrac{1}{2}\pi\right)} = \sqrt{f^*\left(\tfrac{1}{2}\pi\right)} = \frac{64 p^{1/2}(\hat{\zeta}_1)}{p^{1/2}(1)(\hat{\zeta}_1 - 1)^4}. \tag{6.17}$$

The last two conditions for the constants can be obtained by fixing the residues of the functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$ at the geometric optics pole $s = \theta_0$:

$$\operatorname*{res}_{s=\theta_0} \Phi_1^{\pm}(s) = e_1\left(-\sin\theta_0 \pm \frac{\sin\beta}{\eta_1^{\pm}}\right) + e_2 \cos\theta_0 \cos\beta,$$

$$\operatorname*{res}_{s=\theta_0} \Phi_2^{\pm}(s) = e_2\left(-\sin\theta_0 \pm \eta_2^{\pm} \sin\beta\right) - e_1 \cos\theta_0 \cos\beta. \tag{6.18}$$

Evaluating the residue of the function $(\zeta - \alpha_{1\pm})^{-1}$ at the point $s = \theta_0$ yields

$$\theta'_{\pm} = \operatorname*{res}_{s=\theta_0} \frac{1}{\zeta - \alpha_{1\pm}} = \mp \frac{2\sin^3 \frac{1}{4}(\pi \mp \theta_0)}{\cos \frac{1}{4}(\pi \mp \theta_0)}. \tag{6.19}$$

The conditions (6.18) become therefore

$$C_{12}^{\pm} + p^{1/2}(\alpha_{1\pm}) \tanh \hat{\chi}_2(\alpha_{1\pm}) C_{13}^{\pm} = -\frac{1}{2\theta'_{\pm}\hat{\chi}_1(\alpha_{1\pm}) \cosh \hat{\chi}_2(\alpha_{1\pm})}\left[e_1\left(\sin\theta_0 \mp \frac{1}{\eta_1^{\pm}}\sin\beta\right) - e_2\cos\theta_0\cos\beta\right],$$

$$[f_{\pm}^{1/2}(\theta_0) \tanh \hat{\chi}_2(\alpha_{1\pm}) - f_1^{\pm}(\theta_0)]C_{12}^{\pm} + [f_{\pm}^{1/2}(\theta_0) - f_1^{\pm}(\theta_0) \tanh \hat{\chi}_2(\alpha_{1\pm})] p^{1/2}(\alpha_{1\pm}) C_{13}^{\pm} = -\frac{1}{2\theta'_{\pm}\hat{\chi}_1(\alpha_{1\pm}) \cosh \hat{\chi}_2(\alpha_{1\pm})}[e_2\left(\sin\theta_0 \mp \eta_2^{\pm}\sin\beta\right) + e_1\cos\theta_0\cos\beta]. \tag{6.20}$$

The solution of the problem is defined by quadratures (2.4) in terms of the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ expressed through the functions $\Phi_1^\pm(s)$ and $\Phi_2^\pm(s)$ and the integrals (4.12) by formulae (2.9), (2.10), (6.12) (6.11) and (6.5). The constants $C_j^\pm$ ($j = 0, 1, \dots, 15$) solve the two separate systems of linear algebraic equations (6.4), (6.7), (6.13), (6.16) and (6.20) which can be written in the form

$$\sum_{j=1}^{15} a_{\nu j}^\pm C_j^\pm = h_\nu^\pm, \quad \nu = 1, 2, \dots, 15. \tag{6.21}$$

It turns out, for some physical parameters, the coefficients $a_{\nu j}^\pm$ might be badly scaled. For numerics, the system (6.21) is rescaled

$$\sum_{j=1}^{15} \tilde{a}_{\nu j}^\pm C_j^\pm = \tilde{h}_\nu^\pm, \quad \nu = 1, 2, \dots, 15, \tag{6.22}$$

where

$$\tilde{a}_{\nu j}^\pm = 10^{s_\nu^\pm} a_{\nu j}^\pm, \quad \tilde{h}_\nu = 10^{s_\nu^\pm} h_\nu^\pm, \quad s_\nu^\pm = [-\lg \hat{a}_\nu^\pm], \quad \nu = 1, 2, \dots, 15. \tag{6.23}$$

For the parameters (3.22) and (5.59), $-4 \leqslant s_\nu^+ \leqslant 0$, $-5 \leqslant s_\nu^- \leqslant 0$.

Because of the relations $\hat{\xi}_\nu^+ = \mathcal{T}\hat{\xi}_\nu^-$ ($\nu = 1, 2, 3, 4$), $\alpha_{1+} = \mathcal{T}\alpha_{1-}$, and $\theta'_+ = -\mathcal{T}\theta'_-$, the transformation $\mathcal{T}$ maps the constants C_j^+ into the constants C_j^-: $\mathcal{T}C_j^+ = C_j^-$ ($j = 1, \dots, 15$).

7. Normal incidence

In the case $\beta = \frac{1}{2}\pi$, the representations (2.10) become

$$\mathcal{S}_\mu^\pm(s) = -\frac{\Phi_\mu^\pm(s)}{\sin s \mp \hat{\eta}_\mu^\pm}, \quad \mu = 1, 2. \tag{7.1}$$

The matrices $\mathbf{G}^\pm(\sigma)$ are diagonal, and Problem 2.2 is uncoupled

$$\Phi_\mu^\pm(\sigma) = G_{\mu\mu}(\sigma)\Phi_\mu^\pm(\sigma - 4\pi), \quad \sigma \in \Omega^\pm,$$
$$\Phi_\mu^\pm(s) = \Phi_\mu^\pm(\pm 2\pi - s), \quad s \in \Pi^\pm, \quad \mu = 1, 2, \tag{7.2}$$

where

$$G_{\mu\mu}(\sigma) = \frac{(\sin\sigma + \hat{\eta}_\mu^+)(\sin\sigma + \hat{\eta}_\mu^-)}{(\sin\sigma - \hat{\eta}_\mu^+)(\sin\sigma - \hat{\eta}_\mu^-)}. \tag{7.3}$$

As $s \to \infty$ ($\operatorname{Re} s$ is finite), $\Phi_\mu^\pm(s) = O(e^{|s|})$. The functions $\Phi_\mu^\pm(s)$ have to satisfy the conditions (2.17) which read

$$\Phi_\mu^\pm(\xi_{\mu\nu}^\pm) = 0, \quad \mu, \nu = 1, 2, \tag{7.4}$$

where $\sin\xi_{\mu\nu}^\pm = \pm\hat{\eta}_\mu^\pm$ and $\operatorname{Re}\xi_{\mu\nu}^\pm \in (-\pi, \pi)$. If the impedances $\eta_1^\pm$ and $\eta_2^\pm$ are real and positive, then $\xi_{\mu 1}^\pm = \pm\sin^{-1}\hat{\eta}_\mu^\pm$, $\xi_{\mu 2}^\pm = \pm\pi - \xi_{\mu 1}^\pm$. Otherwise $\xi_{\mu\nu}^\pm$ are those numbers

$$\xi_{\mu\nu}^\pm = -i\log\left(\pm i\hat{\eta}_\mu^\pm - (-1)^\nu\sqrt{1 - (\hat{\eta}_\mu^\pm)^2}\right) + 2\pi m, \quad m = 0, \pm 1, \dots, \tag{7.5}$$

which lie in the strip $-\pi < \operatorname{Re} s < \pi$.

By the mapping (3.13), the difference equations (7.2) reduce to the scalar Riemann–Hilbert problems on the segment $(-1, 1)$

$$F^+_{\mu\pm}(t) = l_\mu(t)F^-_{\mu\pm}(t), \quad t \in (-1, 1),$$
$$F_{\mu\pm}(w) = F_{\mu\pm}(-w), \quad w = w_\pm =\in \mathbb{C} \setminus [-1, 1], \quad \mu = 1, 2. \tag{7.6}$$

where

$$l_\mu(t) = G_{\mu\mu}(\sigma), \quad \Phi^\pm_\mu(s) = F_{\mu\pm}(w), \quad w = -i \tan \tfrac{1}{4}(s \pm \pi). \tag{7.7}$$

The functions $F_{\mu\pm}(w)$ grow at the ends

$$F_{\mu\pm}(w) = O(|w \mp 1|^{-2}), \quad w \to \pm 1. \tag{7.8}$$

The coefficients $l_\mu(t)$ have the following properties:

$$l_\mu(t)l_\mu(-t) = 1, \quad l_\mu(0) = l_\mu(\pm 1) = 1,$$
$$[\arg l_\mu(t)]_{[-1,0]} = [\arg l_\mu(t)]_{[0,1]} = -2\pi. \tag{7.9}$$

Here $[g(t)]_{[a,b]}$ is the increment of a function $g(t)$ as t traverses the contour $[a, b]$ in the positive direction. Choose $\arg l_\mu(0) = 0$. Then $\arg l_\mu(\pm 1) = \mp 2\pi$. Factorize next the coefficients $l_\mu(t)$:

$$l_\mu(t) = \frac{X^+_\mu(t)}{X^-_\mu(t)}, \quad t \in (-1, 1), \tag{7.10}$$

where $X^\pm_\mu(t) = X_\mu(t \pm i0)$, and

$$X_\mu(w) = (w^2 - 1)^{-1} \exp\left\{ \frac{1}{2\pi i} \int_{-1}^{1} \frac{\log l_\mu(t)}{t - w} dt \right\}$$
$$= (w^2 - 1)^{-1} \exp\left\{ \frac{1}{\pi i} \int_{0}^{1} \frac{\log l_\mu(t)}{t^2 - w^2} t dt \right\}. \tag{7.11}$$

Analysis of the Cauchy integrals (7.11) shows that

$$X_\mu(w) \sim A^\pm_\mu (w \mp 1)^{-2}, \quad w \to \pm 1, \quad A^\pm_\mu = \text{const}. \tag{7.12}$$

Thus, the functions $X_\mu(w)$ are even and have the asymptotics at the ends $w = \pm 1$ required by the class of solutions. The generalized Liouville theorem yields the solution

$$F_{\mu\pm}(w) = C^\pm_\mu X_\mu(w) \left(\frac{1}{w^2 - \hat{w}^2_\pm} + d^\pm_{\mu 1} + d^\pm_{\mu 2} w^2 \right), \quad w \in \mathbb{C}, \quad \mu = 1, 2, \tag{7.13}$$

where $C^\pm_\mu$ and $d^\pm_{\mu\nu}$ $(\mu = 1, 2, \nu = 0, 1, 2)$ are arbitrary constants. The functions $F_{\mu\pm}(w)$ are even, bounded at infinity, have the asymptotics (7.8) at the ends $w = \pm 1$ and possess the geometrical optics poles at the points $w = \pm \hat{w}_\pm$, $\hat{w}_\pm = -i \tan \frac{1}{4}(\theta_0 \pm \pi)$. The conditions (7.4) give the expressions for the constants $d^{\pm}_{\mu 1}$ and $d^{\pm}_{\mu 2}$

$$d^{\pm}_{\mu 1} = \frac{(\hat{\xi}^{\pm}_{\mu 1})^2 \hat{d}^{\pm}_{\mu 2} - (\hat{\xi}^{\pm}_{\mu 2})^2 \hat{d}^{\pm}_{\mu 1}}{(\hat{\xi}^{\pm}_{\mu 2})^2 - (\hat{\xi}^{\pm}_{\mu 1})^2}, \quad d^{\pm}_{\mu 2} = \frac{\hat{d}^{\pm}_{\mu 1} - \hat{d}^{\pm}_{\mu 2}}{(\hat{\xi}^{\pm}_{\mu 2})^2 - (\hat{\xi}^{\pm}_{\mu 1})^2}, \tag{7.14}$$

where

$$\hat{\xi}^{\pm}_{\mu\nu} = -i \tan \frac{\xi^{\pm}_{\mu\nu} \pm \pi}{4}, \quad \hat{d}^{\pm}_{\mu\nu} = \frac{1}{(\hat{\xi}^{\pm}_{\mu\nu})^2 - \hat{\omega}^2_{\pm}}. \tag{7.15}$$

The physical conditions (2.5) and the invariance of the solution with respect to the transformation $\theta \to -\theta, \theta_0 \to -\theta_0$, and $\eta^{\pm}_{\mu} \to \eta^{\mp}_{\mu}$ $(\mu = 1, 2)$ define the constants $C^{\pm}_{\mu}$

$$C^{+}_{\mu} = -\frac{e_\mu(\sin\theta_0 - \hat{\eta}^{+}_{\mu})}{X_\mu(\hat{\omega}_+)\theta'_+}, \quad C^{-}_{\mu} = -\frac{e_\mu(\sin\theta_0 + \hat{\eta}^{-}_{\mu})}{X_\mu(\hat{\omega}_-)\theta'_-}, \quad \mu = 1, 2, \tag{7.16}$$

where $\theta'_{\pm}$ are given by (6.19). Formula (7.16) indicates that $\mathcal{T} C^{+}_{\mu} = C^{-}_{\mu}$, $\mu = 1, 2$.

8. High frequency asymptotics. Numerical results

High frequency asymptotics of the electrical and magnetic field can be represented in the form

$$\begin{aligned} E_z &\sim E^i_z + E^r_z + E^s_z + E^d_z, \\ H_z &\sim H^i_z + H^r_z + H^s_z + H^d_z, \quad k\rho \to \infty, \end{aligned} \tag{8.1}$$

where E^i_z and H^i_z are the incident waves, E^r_z and H^r_z are the reflected waves, E^s_z and H^s_z are the surface waves, and E^d_z and H^d_z are the diffracted waves. These waves may be recovered by applying the steepest descent method to the Sommerfeld integrals (2.4). Analysis of the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ shows that the geometrical optics poles should be found among the roots of the equation

$$\tan^2 \frac{s \pm \pi}{4} = \tan^2 \frac{\theta_0 \pm \pi}{4}, \tag{8.2}$$

which lie in the strip $-\pi + \theta < \operatorname{Re} s < \pi + \theta$. The pole $s = \theta_0$ gives rise to the incident waves (2.1). The next two poles $s = -2\pi - \theta_0$ and $s = 2\pi - \theta_0$ reproduce the reflected waves provided that the angle of incidence θ_0 and the angle of observation θ satisfy the conditions $-\pi < \theta_0 < 0$ and $-\pi < \theta < -\pi - \theta_0$ in the case $s = -2\pi - \theta_0$, and $0 < \theta_0 < \pi$ and $\pi - \theta_0 < \theta < \pi$ in the case $s = 2\pi - \theta_0$. By the Cauchy theorem the reflected waves become

$$\begin{pmatrix} E^r_z \\ H^r_z \end{pmatrix} = \mathrm{e}^{-ikz\cos\beta + ik\rho\sin\beta\cos(\theta+\theta_0)} \left[\begin{pmatrix} R^-_e \\ R^-_h \end{pmatrix} \omega_\theta(-\pi, -\pi - \theta_0) + \begin{pmatrix} R^+_e \\ R^+_h \end{pmatrix} \omega_\theta(\pi - \theta_0, \pi) \right], \tag{8.3}$$

where

$$\omega_\theta(a, b) = \begin{cases} 1, & \theta \in [a, b], \\ 0, & \theta \notin [a, b]. \end{cases} \tag{8.4}$$

The reflection coefficients R_e and R_h are defined by

$$\begin{pmatrix} R_e^{\pm} \\ R_h^{\pm} \end{pmatrix} = \frac{1}{2\Gamma_{\theta_0}(1/\eta_1^+, \eta_2^+)} \begin{pmatrix} \sin\theta_0 + \eta_2^+ \sin\beta & -\cos\theta_0\cos\beta \\ \cos\theta_0\cos\beta & \sin\theta_0 + 1/\eta_1^+ \sin\beta \end{pmatrix} \operatorname*{res}_{s=-\theta_0\pm 2\pi} \begin{pmatrix} \Phi_1^+(s) \\ \Phi_2^+(s) \end{pmatrix}$$
$$+ \frac{1}{2\Gamma_{\theta_0}(-1/\eta_1^-, -\eta_2^-)} \begin{pmatrix} \sin\theta_0 - \eta_2^- \sin\beta & -\cos\theta_0\cos\beta \\ \cos\theta_0\cos\beta & \sin\theta_0 - 1/\eta_1^- \sin\beta \end{pmatrix}$$
$$\times \operatorname*{res}_{s=-\theta_0\pm 2\pi} \begin{pmatrix} \Phi_1^-(s) \\ \Phi_2^-(s) \end{pmatrix}. \tag{8.5}$$

To evaluate the residues

$$\operatorname*{res}_{s=-\theta_0-2\pi} \begin{pmatrix} \Phi_1^+(s) \\ \Phi_2^+(s) \end{pmatrix} \quad \text{and} \quad \operatorname*{res}_{s=-\theta_0+2\pi} \begin{pmatrix} \Phi_1^-(s) \\ \Phi_2^-(s) \end{pmatrix}, \tag{8.6}$$

the vector functions $\mathbf{\Phi}^+(s)$ and $\mathbf{\Phi}^-(s)$ should be continued analytically into the strips $-5\pi < \operatorname{Re} s < -\pi$ and $\pi < \operatorname{Re} s < 5\pi$, respectively. This can be done by the relations

$$\mathbf{\Phi}^+(s) = [\mathbf{G}^+(s)]^{-1}\mathbf{\Phi}^+(s+4\pi), \quad -5\pi < \operatorname{Re} s < -\pi.$$
$$\mathbf{\Phi}^-(s) = \mathbf{G}^-(s)\mathbf{\Phi}^-(s-4\pi), \quad \pi < \operatorname{Re} s < 5\pi. \tag{8.7}$$

Clearly, the residues in (8.5) can be expressed explicitly through the angles θ_0 and β and the parameters $\eta_\mu^{\pm}$, e_1, and e_2. To find them, there is no need to solve Problem 2.2 since the residues (2.5) are prescribed.

Because of the conditions (2.17) the points $s = -\theta + \xi_j^{\pm}$ ($j = 1, 2, 3, 4$) are removable points of the functions $\mathcal{S}_1(s+\theta)$ and $\mathcal{S}_2(s+\theta)$. However, outside the strip $-\pi < \operatorname{Re}(s+\theta) < \pi$, the points $s = -\theta + \xi_j^{\pm} + 2\pi m$ ($m = \pm 1, \pm 2, \ldots$) are simple poles of the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$. The poles $s = -\theta + \xi_j^{\pm} + 2\pi$ and $-\theta + \xi_j^{\pm} - 2\pi \in \tilde{\Pi}$ may give rise to the surface waves. The domain $\tilde{\Pi}$ is a curved strip. The right-hand boundary is described by the equation $\operatorname{Re} s = \pi + \operatorname{gd}(\operatorname{Im} s)\operatorname{sgn}(\operatorname{Im} s)$, where $\operatorname{gd} x$ is the Gudermann function $\operatorname{gd} x = \cos^{-1}(1/\cosh x)$. This curve is symmetric with respect to the point $s = \pi$. Its starting and ending points are $s = \frac{1}{2}\pi - i\infty$ and $s = \frac{3}{2}\pi + i\infty$, respectively. The lines $\operatorname{Re} s = \frac{1}{2}\pi$ and $\operatorname{Re} s = \frac{3}{2}\pi$ are the asymptotes of the lower and upper parts of the path. The left-hand boundary of the domain $\tilde{\Pi}$ is symmetric to the the right-hand side with respect to the point $s = 0$. Since $\theta \in (-\pi, \pi)$ and $\operatorname{Re} \xi_j^{\pm} \in (0, \pi)$ ($j = 1, 2, 3, 4$) for $\operatorname{Re} \eta_\mu^{\pm} > 0$, the surface waves are given by

$$\begin{pmatrix} E_z^s \\ H_z^s \end{pmatrix} = \sum_{j=1}^{4} \left[\begin{pmatrix} W_{ej}^{+-} \\ W_{hj}^{+-} \end{pmatrix} \omega_\theta(-\pi, -\pi + g_j^+) + \begin{pmatrix} W_{ej}^{++} \\ W_{hj}^{++} \end{pmatrix} \omega_\theta(\pi + g_j^+, \pi) \right]$$
$$\times\, \mathrm{e}^{-ikz\cos\beta + ik\rho\sin\beta\cos(\xi_j^+ - \theta)} + \sum_{j=1}^{4} \left[\begin{pmatrix} W_{ej}^{--} \\ W_{hj}^{--} \end{pmatrix} \omega_\theta(-\pi, -\pi + g_j^-) \right.$$
$$\left. + \begin{pmatrix} W_{ej}^{-+} \\ W_{hj}^{-+} \end{pmatrix} \omega_\theta(\pi + g_j^-, \pi) \right] \mathrm{e}^{-ikz\cos\beta + ik\rho\sin\beta\cos(\xi_j^- - \theta)}, \tag{8.8}$$

where

$$W_{ej}^{+\pm} = \frac{1}{2\tilde{\Gamma}_j^+}[(-\sin\xi_j^+ + \eta_2^+ \sin\beta)\Phi_1^+(\xi_j^+ \pm 2\pi) - \cos\xi_j^+ \cos\beta\Phi_2^+(\xi_j^+ \pm 2\pi)],$$

$$W_{hj}^{+\pm} = \frac{1}{2\tilde{\Gamma}_j^+}\left[\left(-\sin\xi_j^+ + \frac{1}{\eta_1^+}\sin\beta\right)\Phi_2^+(\xi_j^+ \pm 2\pi) + \cos\xi_j^+ \cos\beta\Phi_1^+(\xi_j^+ \pm 2\pi)\right],$$

$$W_{ej}^{-\pm} = \frac{1}{2\tilde{\Gamma}_j^-}[(-\sin\xi_j^- - \eta_2^- \sin\beta)\Phi_1^-(\xi_j^- \pm 2\pi) - \cos\xi_j^- \cos\beta\Phi_2^-(\xi_j^- \pm 2\pi)],$$

$$W_{hj}^{-\pm} = \frac{1}{2\tilde{\Gamma}_j^-}\left[\left(-\sin\xi_j^- - \frac{1}{\eta_1^-}\sin\beta\right)\Phi_2^-(\xi_j^- \pm 2\pi) + \cos\xi_j^- \cos\beta\Phi_1^-(\xi_j^- \pm 2\pi)\right],$$

$$g_j^\pm = \operatorname{Re}\xi_j^\pm - \operatorname{gd}(\operatorname{Im}\xi_j^\pm)\operatorname{sgn}(\operatorname{Im}\xi_j^\pm),$$

$$\tilde{\Gamma}_j^\pm = \frac{d}{ds}\Gamma_s\left(\mp\frac{1}{\eta_1^\pm}, \mp\eta_2^\pm\right)\bigg|_{s=\xi_j^\pm}. \tag{8.9}$$

The steepest descent method applied to (2.4) for $k\rho \to \infty$ yields the diffracted field

$$\begin{pmatrix} E_z^d \\ H_z^d \end{pmatrix} = \frac{e^{-ik(z\cos\beta+\rho\sin\beta)}}{\sqrt{k\rho}}\begin{pmatrix} D_e(\theta) \\ D_h(\theta) \end{pmatrix}, \tag{8.10}$$

where the diffraction coefficients $D_e(\theta)$ and $D_h(\theta)$ are expressed through the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ as follows:

$$D_\mu^*(\theta) = \frac{e^{-i\pi/4}}{\sqrt{2\pi\sin\beta}\,e_1}[\mathcal{S}_\mu(\theta-\pi) - \mathcal{S}_\mu(\theta+\pi)], \quad \mu = 1, 2, \tag{8.11}$$

and $D_1^* = e_1^{-1}D_e$ and $D_2^* = Z_0 e_1^{-1} D_h$. Since the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ have a simple pole at the point $s = \theta_0$, the diffraction coefficients are infinite at the points $\theta = \pi - |\theta_0|$ and $\theta = -\pi + |\theta_0|$. To evaluate the diffraction coefficients by formulae (8.11), one needs to use formulae (2.9), (2.10), and the analytical continuation (8.7). If $s = \xi_j^\pm$ $(j = 1, 2, 3, 4)$ is a real zero of the function $\Gamma_s(\mp 1/\eta_1^\pm, \mp\eta_2^\pm)$, and $-\pi + \xi_j^\pm \in (-\pi, 0)$ $(\pi + \xi_j^\pm \in (0, \pi))$, then the diffraction coefficients are infinite at the point $\theta = -\pi + \xi_j^\pm$ $(\theta = \pi + \xi_j^\pm)$. Since $\mathcal{T}C_j^+ = C_j^-$ $(j = 1, \dots, 15)$, analysis of formulae (2.9), (2.10), (2.12) and (8.11) shows that the diffraction coefficients are invariant with respect to the transformation $\mathcal{T}$ defined by (2.8).

When the upper and lower sides of the screen have the same impedances $\eta_\mu^\pm = \eta_\mu$, and $\beta = \frac{1}{2}\pi$, formula (8.11) becomes simpler:

$$D_\mu^*(\theta) = \frac{e^{-i\pi/4}}{2\sqrt{2\pi}\,e_1}\left[\frac{\Phi_\mu^+(\theta-\pi)}{-\sin\theta - \hat{\eta}_\mu} + \frac{\Phi_\mu^-(\theta-\pi)}{-\sin\theta + \hat{\eta}_\mu} + \frac{\Phi_\mu^+(\theta+\pi)}{\sin\theta + \hat{\eta}_\mu} + \frac{\Phi_\mu^-(\theta+\pi)}{\sin\theta - \hat{\eta}_\mu}\right], \quad \mu = 1, 2. \tag{8.12}$$

Here $\Phi_\mu^-(\theta+\pi) = G_{\mu\mu}(\theta+\pi)\Phi_\mu^-(\theta-3\pi)$ if $0 < \theta < \pi$, and $\Phi_\mu^+(\theta-\pi) = \Phi_\mu^+(\theta+3\pi)/G_{\mu\mu}(\theta-\pi)$ if $-\pi < \theta < 0$. Since $C_\mu^+(\theta_0) = C_\mu^-(-\theta_0)$ and

$$\Phi_1^\pm(\theta-\pi) = \Phi_1^\mp(-\theta+\pi), \quad \Phi_1^\pm(\theta+\pi) = \Phi_1^\mp(-\theta-\pi), \tag{8.13}$$

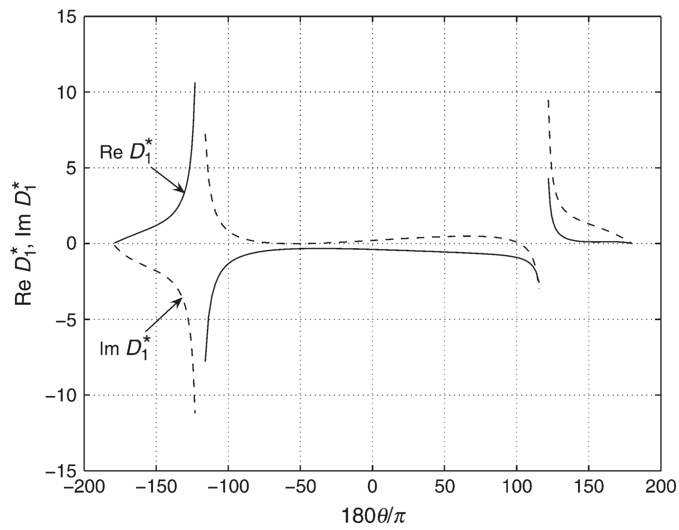

Fig. 6 Normal incidence. The real and imaginary parts of the diffraction coefficient $D_1^*(\theta)$ for $\theta_0 = \frac{1}{3}\pi$, $\eta_1^+ = 1 - i$, $\eta_2^+ = 0{\cdot}1 - i$, $\eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

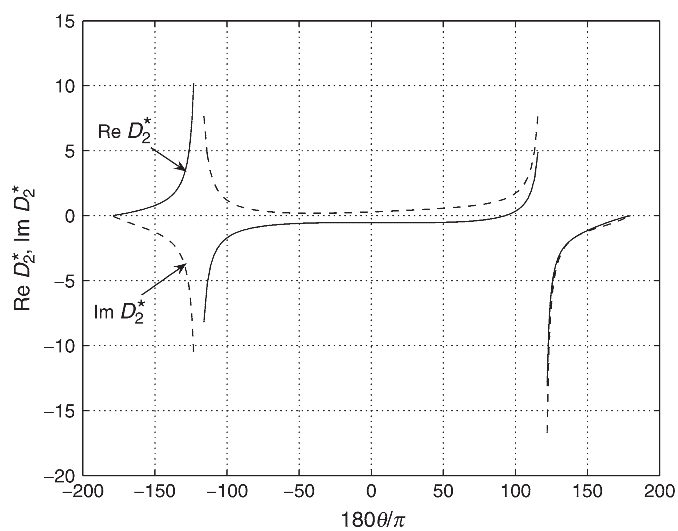

Fig. 7 Normal incidence. The real and imaginary parts of the diffraction coefficient $D_2^*(\theta)$ for $\theta_0 = \frac{1}{3}\pi$, $\eta_1^+ = 1 - i$, $\eta_2^+ = 0{\cdot}1 - i$, $\eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

the diffraction coefficients are invariant with respect to the transformation $\theta \to -\theta$ and $\theta_0 \to -\theta_0$: $D_\mu(\theta, \theta_0) = D_\mu(-\theta, -\theta_0)$. For the case of normal incidence ($\beta = \frac{1}{2}\pi$) numerical calculations for the diffraction coefficients $D_1^*(\theta)$ and $D_2^*(\theta)$ are illustrated in Figs 6 and 7 for the impedance parameters (3.22), $\theta_0 = \frac{1}{3}\pi$, and $e_* = e_1/e_2 = 1$. Numerical values are the same if θ, θ_0, and $\eta_j^\pm$

($j = 1, 2$) are replaced by $-\theta$, $-\theta_0$, and $\eta_j^{\mp}$. As $\theta = \pm\pi$, the diffraction coefficients D_μ^* ($\mu = 1, 2$) vanish. As θ approaches the angles $-\frac{2}{3}\pi$ and $\frac{2}{3}\pi$, that is, the angles $\mp\pi \pm\theta_0$, the real and imaginary parts of the diffraction coefficients grow to infinity.

The skew incidence case is presented in Figs 8 to 10 for the same impedance parameters when $\theta_0 = \frac{1}{3}\pi$, $e_* = 1$, and $\beta = \frac{1}{4}\pi$. The numerical values are symmetric with respect to the transformation $\mathcal{T}$. The surface waves poles are not close to the real axis, and the graphs are smooth curves. However, for some sets of parameters these poles could be real or close to the real axis. For example, for the impedance parameters (3.22) and $\beta = \frac{1}{3}\pi$, the zeros $\xi_j^{\pm}$ of the of the functions $\Gamma_s(\mp 1/\eta_1^{\pm}, \mp\eta_2^{\pm})$ are

$$\begin{aligned} &\xi_1^+ = 2{\cdot}7363 - 0{\cdot}6508i, \quad \xi_2^+ = 0{\cdot}1342, -1{\cdot}0348i,\\ &\xi_3^+ = 0{\cdot}4053 + 0{\cdot}6508i, \quad \xi_4^+ = 3{\cdot}007 + 1{\cdot}0348i,\\ &\xi_1^- = -2{\cdot}5114 + 1{\cdot}3157i, \quad \xi_2^- = -0{\cdot}4547 + 0{\cdot}0129i,\\ &\xi_3^- = -0{\cdot}6302 - 1{\cdot}3157i, \quad \xi_4^- = -2{\cdot}6869 - 0{\cdot}0129i. \end{aligned}$$

Because the imaginary part of the zeros ξ_2^- and ξ_4^- is small, the graphs of the real and imaginary parts of the diffraction coefficients change sharply (Fig. 11) in small neighbourhoods of the points $\pi - \operatorname{Re}\xi_2^-$ and $\pi - \operatorname{Re}\xi_4^-$, namely as $\theta = 154^\circ$ and $\theta = 26^\circ$. It is known (**21**) that (8.10) is valid for large $k\rho$ and in those regions where the reflected and surface waves are not present. For angles θ close to these critical values, the total field should be evaluated separately by using methods of the uniform geometrical theory of diffraction (**20**, **21**).

In Fig. 12, the real and imaginary parts of the diffraction coefficient D_1^* are given for $\beta = 0{\cdot}49\pi$. It is seen that as $\beta \to \frac{1}{2}\pi$, the skew incidence coefficient D_1^* approaches the values of the normal incidence diffraction coefficient D_1^*. The same tendency is observed for the second coefficient D_2^*.

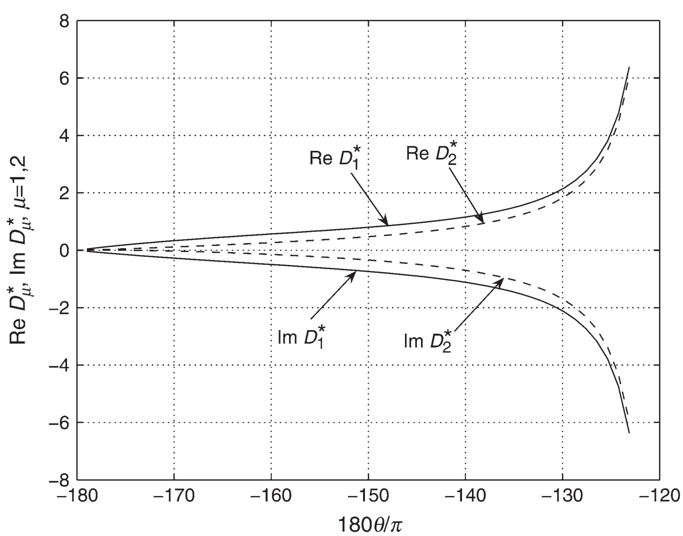

Fig. 8 Skew incidence. The real and imaginary parts of the diffraction coefficients $D_1^*(\theta)$ and $D_2^*(\theta)$ for $-\pi < \theta < -\pi + \theta_0$, $\theta_0 = \frac{1}{3}\pi$, $\beta = \frac{1}{4}\pi$, $\eta_1^+ = 1 - i$, $\eta_2^+ = 0{\cdot}1 - i$, $\eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

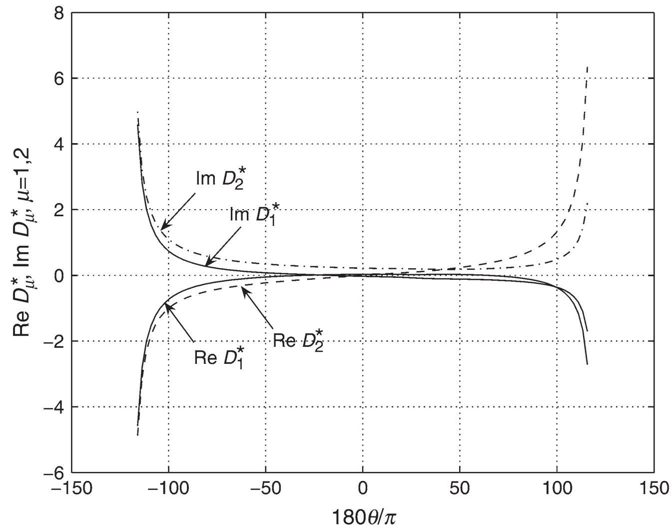

Fig. 9 Skew incidence. The real and imaginary parts of the diffraction coefficients $D_1^*(\theta)$ and $D_2^*(\theta)$ for $-\pi + \theta_0 < \theta < \pi - \theta_0, \theta_0 = \frac{1}{3}\pi, \beta = \frac{1}{4}\pi, \eta_1^+ = 1 - i, \eta_2^+ = 0.1 - i, \eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$.

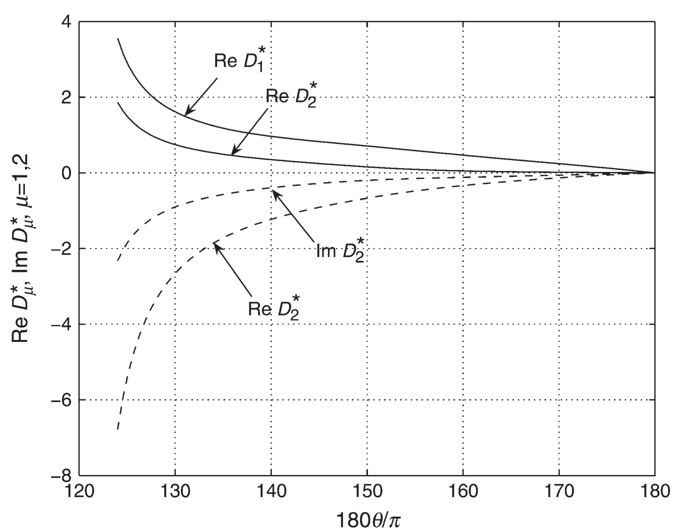

Fig. 10 Skew incidence. The real and imaginary parts of the diffraction coefficients $D_1^*(\theta)$ and $D_2^*(\theta)$ for $\pi - \theta_0 < \theta < \pi, \theta_0 = \frac{1}{3}\pi, \beta = \frac{1}{4}\pi, \eta_1^+ = 1 - i, \eta_2^+ = 0{\cdot}1 - i, \eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

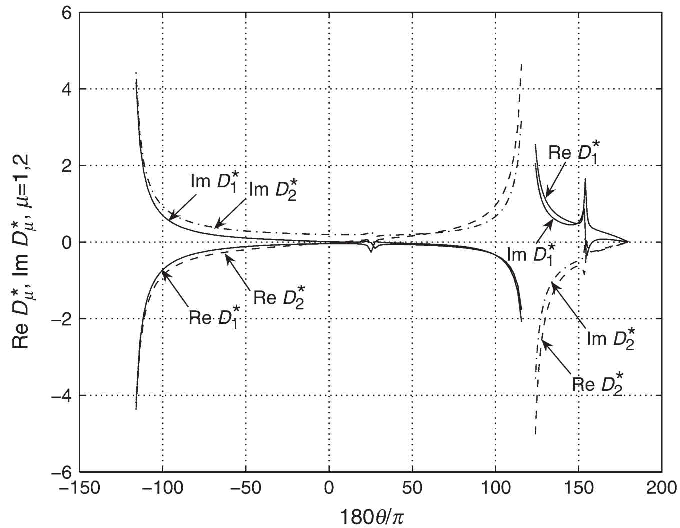

Fig. 11 Skew incidence. The real and imaginary parts of the diffraction coefficients $D_1^*(\theta)$ and $D_2^*(\theta)$ for $-\pi + \theta_0 < \theta < \pi$, $\theta_0 = \frac{1}{3}\pi$, $\beta = \frac{1}{3}\pi$, $\eta_1^+ = 1 - i$, $\eta_2^+ = 0{\cdot}1 - i$, $\eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

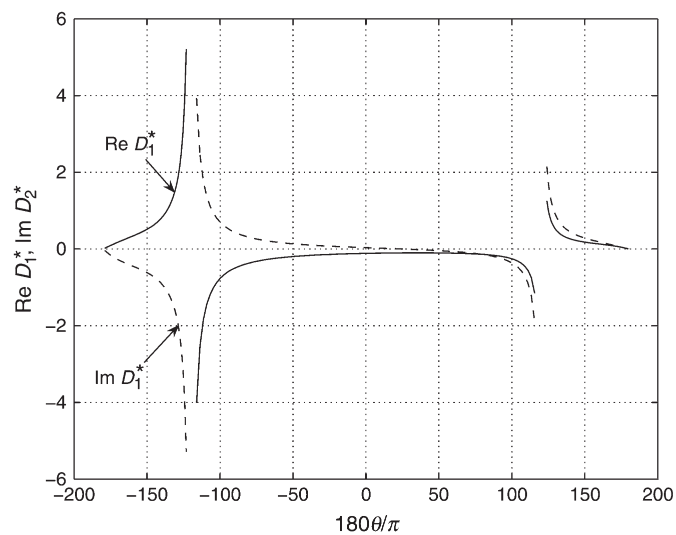

Fig. 12 Skew incidence. The real and imaginary parts of the diffraction coefficient $D_1^*(\theta)$ for $\theta_0 = \frac{1}{3}\pi$, $\beta = 0{\cdot}49\pi$, $\eta_1^+ = 1 - i$, $\eta_2^+ = 0{\cdot}1 - i$, $\eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

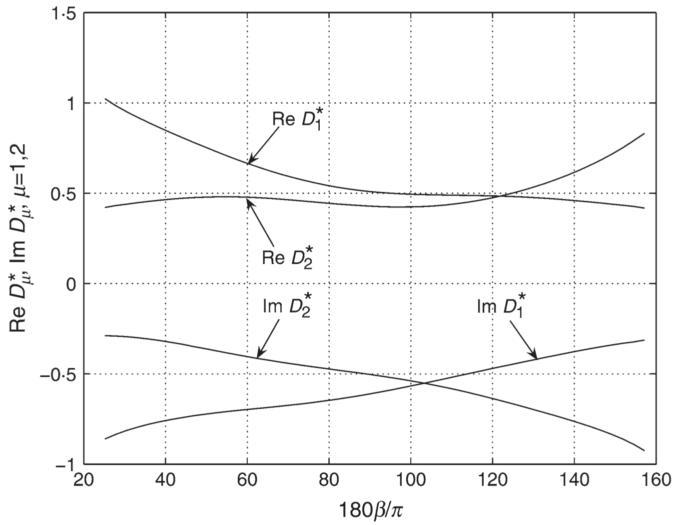

Fig. 13 Skew incidence. The real and imaginary parts of the diffraction coefficient $D_1^*(\theta)$ and $D_2^*(\theta)$ for $\theta_0 = \frac{1}{3}\pi, \theta = -\frac{5}{6}\pi, \eta_1^+ = 1 - i, \eta_2^+ = 0{\cdot}1 - i, \eta_1^- = 2 - i$, and $\eta_2^- = 1 + i$

For $\beta \to \frac{1}{2}\pi$, the branch points ζ_j approach the images of the second-order zeros of the function $f^*(s) = (\cos^2 s + g_0 - 1)^2$ $(\beta = \frac{1}{2}\pi)$. For example, for $\beta = 0{\cdot}499\pi$, the branch points are

$$\zeta_0 = -1{\cdot}1414 - 2{\cdot}0527i, \quad \zeta_1 = -1{\cdot}0124, -2{\cdot}0090i,$$
$$\zeta_2 = 0{\cdot}0166 - 0{\cdot}1168i, \quad \zeta_3 = 0{\cdot}0271 - 0{\cdot}1167i, \text{‘}$$
$$\zeta_4 = -0{\cdot}2069 + 0{\cdot}3721i, \quad \zeta_5 = -0{\cdot}2000 + 0{\cdot}3970i,$$
$$\zeta_6 = 1{\cdot}8877 + 8{\cdot}1293i, \quad \zeta_7 = 1{\cdot}1927 + 8{\cdot}3925i.$$

Notice that the point $\zeta = 0$ is to the right of the segment $\zeta_3\zeta_4$. As $\beta \to \frac{1}{2}\pi$, the accuracy of the numerical solution to the Jacobi problem increases. For $\beta \to 0$ or $\beta \to \pi$, the distance between the branch points ζ_0 and ζ_7 grows. For $\beta = \frac{1}{10}\pi$ and the impedance parameters (3.22), for example, $|\zeta_7 - \zeta_0| = 39{\cdot}654$. The values of the function $p^{1/2}(\zeta)$ become large even for small values of ζ. Therefore, the solution of the problem requires higher accuracy for small or close to π incidence angles. The dependence of the coefficients D_1^* and D_2^* on the angle of incidence β is shown in Fig. 13.

9. Conclusion

Scattering of a plane wave at skew incidence from an anisotropic impedance half-plane for two different matrix impedances $\mathbf{Z}^+$ and $\mathbf{Z}^-$ is a long-standing problem of diffraction theory (**13**). Only some particular cases were studied in the literature. It turns out, using a new technique by Antipov and Silvestrov (**6**), the problem in its general formulation with arbitrary complex entries of the impedance matrices is exactly solvable. The key step of the method is to formulate the governing system of difference equations as a scalar Riemann–Hilbert problem on two unit segments of a hyperelliptic surface of genus 3. Its solution has been found in terms of singular integrals with the Weierstrass kernel on a Riemann surface. Initially, the solution had an essential singularity at infinity and possessed some free parameters. The singularity has been removed by fixing these parameters which solve the associated Jacobi inversion problem for the surface. Regardless of which analogue of the Cauchy kernel on the Riemann surface is chosen, in general, the solution of the Jacobi problem cannot be bypassed. In this paper, a constructive numerical procedure for the inversion problem for a surface of any finite genus has been proposed and described in detail. This technique made it possible to recover the total electric and magnetic fields and compute the edge diffraction coefficients. The solution has been found in closed form in terms of a finite number of quadratures. It possesses three transcendents which are the roots of a certain cubic equation with known complex coefficients and also 15 constants which solve a non-homogeneous system of 15 linear equations. It has been proved that the solution is invariant with respect to the symmetric transformation of the angles θ, θ_0 and β and the impedance parameters $\eta_j^{\pm}$ ($j = 1, 2$). As the angle of incidence β approaches $\frac{1}{2}\pi$, the diffraction coefficients approach the limiting values for the case of normal incidence.

The technique employed here can be applied for more complicated boundary-value problems formulated either as a system of the Maliuzhinets difference equations, or a system of the Wiener–Hopf functional equations. If, ultimately, the problem reduces to a scalar Riemann–Hilbert problem on a *hyperelliptic* surface of high genera, then the procedure proposed should be modified accordingly. The method of the paper has the potential to be extended to those systems of the Maliuzhinets and the Wiener–Hopf equations which are equivalent to a scalar Riemann–Hilbert problem on *n-sheeted* ($n \geqslant 3$) Riemann surfaces.

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