Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Physica D 235 (2007) 72–81

PHYSICA D

www.elsevier.com/locate/physd

Method of Riemann surfaces in the study of supercavitating flow around two hydrofoils in a channel

Y.A. Antipov[a,*], V.V. Silvestrov[b]

[a] Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
[b] Department of Mathematics, Gubkin Russian State University of Oil and Gas, Moscow 119991, Russia

Available online 2 August 2007

Abstract

In the framework of the Tulin model of supercavitating flow, the problem of reconstructing the free surface of a channel and the shapes of the cavities behind two hydrofoils placed in an ideal fluid is solved in closed form. The conformal map that transforms a parametric plane with three cuts along the real axis into the triple-connected flow domain is found by quadratures. The use of the theory of Riemann surfaces (the Schottky doubles) enables the non-linear model problem to be reduced to two separate Riemann–Hilbert problems on a hyperelliptic surface of genus two. The solution to the first problem is a rational function with certain zeros and poles on a Riemann surface. The second problem is solved in terms of singular integrals with the Weierstrass kernel. The essential singularities of the solution at the infinite points of the surface due to a pole of the kernel are removed by solving a real analogue of the Jacobi inversion problem on the surface. The unknown parameters of the conformal map are recovered from a system of certain algebraic and transcendental equations.

Keywords: Supercavitating flow; Conformal map; Riemann surfaces; Riemann–Hilbert problem

1. Introduction

When an obstacle is placed stationary in a moving fluid the flow breaks away from the barrier along separating streamlines, and a wake (a "dead-zone") forms behind the obstacle. In the case of high-speed flow the wake becomes a vapor-filled cavity. For water under atmospheric pressure for example, this occurs when the speed is 100 f/s or more (lower speeds under reduced pressure) [1]. Cavitating flow has been studied intensively since marine engineers at the end of the 19th century became aware of the serious problem due to cavitation: an increase in pressure causes the cavities to collapse and release energy resulting in a force which may damage submarine propellers when they are operating at certain depths. Cavitation has long been of interest not only in the field of shipbuilding and hydraulic machinery, but also in chemical processing, nuclear physics and medicine (potential bioeffects of ultrasound caused by acoustic cavitation in blood vessels).

Modeling of cavitation is based on the results by Brillouin (1911) (see, e.g. [1]) who proved that the maximum velocity must be attained on the free surface and also that the boundary of a cavity is convex. Good references to the theory of cavitating flow around obstacles are [1–3]. Tulin [4] proposed a model of the cavitating flow which admits the presence of the singularity of the solution at the point say, C, where the two streamlines along the cavity attempt to close it, namely,

$$\log(\mathrm{d}w/\mathrm{d}z) \sim K(w-w_0)^{-1/2},\quad z\to C,\quad K=\text{const},\tag{1.1}$$

where $w_0 = w(C)$, $w = w(z)$ is the complex potential of the motion. This condition extends the class of solutions to the governing boundary-value problem which makes possible to reconstruct a flow that meets the condition

$$\oint_L \mathrm{d}z = 0,\tag{1.2}$$

and is therefore single-valued. Here L is the boundary of the cavity combined with the boundary of the hydrofoil.

* Corresponding author. Tel.: +1 225 578 1567; fax: +1 225 5784276.
E-mail address: antipov@math.lsu.edu (Y.A. Antipov).

0167-2789/$ - see front matter
doi:10.1016/j.physd.2007.04.013

For model problems of cavitating flow, numerical and analytical methods are employed. The former technique requires the solution of associated integral equations with the contour to be recovered. The analytical methods for the study of cavitating flow in simply- and double-connected domains are well developed [1,3,5]. These methods are based on the use of a Schwarz–Christoffel transformation to map a circle, a quadrant, or a plane with a cut (simply-connected flow), or an annulus, a rectangular, or a plane with two cuts on the real axis (double-connected flow) into the flow domain. A closed-form solution to the model problem can then be found in terms of elementary and elliptic functions for a simply- and double-connected flow, respectively.

In the case of free boundary problems in a triple-connected flow domain, the problem of fluid mechanics can be formulated [6] as the Hilbert boundary-value problem with a piecewise constant coefficient on three cuts along the real axis. The actual problem solved in [6] concerned non-cavitating flow of a fluid around a single foil in a half-plane with a free surface. A method of Riemann surfaces for an elasticity problem [7] on a system of cracks along an interface with mixed boundary conditions was extended for the problem of cavitating flow in the whole plane around three foils [8].

In this paper we model the steady flow of an ideal fluid in a channel with a free surface when two plates are held stationary. The main mathematical tool used is the method of the Riemann–Hilbert problem on a Riemann surface. This technique was applied and developed [9–11] for the solution of model problems of acoustic and electromagnetic scattering from a perforated sandwich panel and an anisotropic impedance half-plane. In Section 2 we formulate the problem using the non-linear model of cavitating flow by Tulin. Section 3 maps the problem into two Hilbert problems on three cuts on the real axis. One of the cuts is the image of the boundary of the channel (a portion of it is a free boundary and therefore is unknown). The other two cuts are the images of the cavities whose boundaries formed by the hydrofoils and the unknown convex contours. The derivative of the conformal map $z = f(\zeta)$ is represented as a quotient of two functions, $\mathrm{d}w/\mathrm{d}z$ and $\mathrm{d}w/\mathrm{d}\zeta$, where w is a complex potential of the flow. In Section 4 the function $\mathrm{d}w/\mathrm{d}\zeta$ is found as a rational function with certain zeros and poles on a hyperelliptic surface of genus two. To define the function $\omega(\zeta) = \log(V_\infty^{-1}\mathrm{d}w(z)/\mathrm{d}z)$ (V_∞ is the speed at infinity), in Section 5 we reduce the Hilbert problem on the three cuts with a piecewise constant coefficient to the Riemann–Hilbert problem on the Riemann surface introduced in Section 4. Its solution is found by quadratures in terms of singular integrals with the Weierstrass kernel. Initially, it has an inadmissible exponential growth at infinity. The conditions which make the solution bounded at infinity are written as the Jacobi inversion problem for hyperelliptic integrals. This non-linear problem requires finding two points on the surface and four integers. The solution of the Jacobi problem is found in closed form by reducing it to a system of two algebraic equations with the right-hand side expressed through the Riemann θ-function of the surface. Section 6 writes down additional conditions to be satisfied in order to fix 21 unknown real parameters. The system of equations for the unknowns consists of 8 linear and 13 non-linear relations which are algebraic and transcendental equations. Finally, equations of the free surface of the channel and the boundaries of the two cavities are found by quadratures.

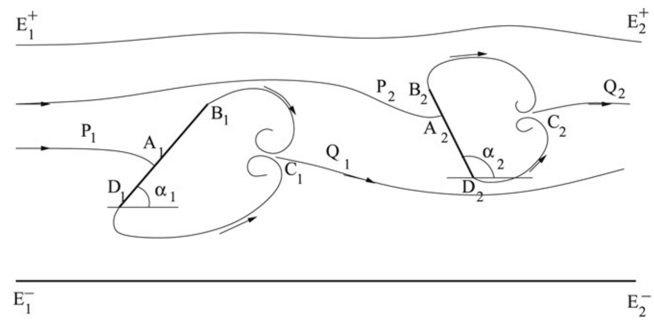

Fig. 1. Flow domain $\mathcal{D}$.

2. Formulation

Let two hydrofoils B_1D_1 and B_2D_2 (Fig. 1) be placed in an incompressible gravity-free fluid which is moving steadily and irrotationally in a channel. The bottom $\{-\infty < x < \infty, y = 0\}$ of the channel is solid, and its upper boundary is a free surface. Far away from the hydrofoils, the flow is uniform with velocity $\mathbf{v} = (V_\infty, 0)$ across the channel of depth h. It is assumed that at the ends B_j and D_j ($j = 1, 2$) the jets break away from the hydrofoils, and cavities $B_jC_jD_j$ ($j = 1, 2$) form behind the foils. The cavities are convex and bounded but not closed. The unknown boundaries of the cavities are streamlines. In the framework of the model considered, the velocity vector is constant and prescribed on the boundaries of the cavities (the constants are not necessarily the same). The loops $A_jB_jC_jD_jA_j$ are smooth in a neighborhood of the points B_j and D_j. At the stagnation points A_j (unknown *a priori*), the flow branches and the velocity vector vanishes.

Under these assumptions, the model problem of fluid mechanics is reduced to that of finding a complex potential of the motion $w(z) = \phi + \mathrm{i}\psi$ in the 3-connected domain say, $\mathcal{D}$, occupied by the fluid (the physical domain), together with boundary conditions of the form

$$\operatorname{Im} w(z) = \begin{cases} W_0^\pm, & z \in E_1^\pm E_2^\pm \\ W_j, & z \in L_j, \end{cases} \qquad j = 1, 2,$$

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \begin{cases} V_\infty, & z \in E_1^+ E_2^+ \\ V_j, & z \in B_jC_jD_j, \end{cases} \qquad j = 1, 2,$$

$$\arg\frac{\mathrm{d}w}{\mathrm{d}z} = \begin{cases} 0, & z \in E_1^- E_2^- \\ -\alpha_j, & z \in A_jB_j \\ \pi - \alpha_j, & z \in A_jD_j, \end{cases} \qquad j = 1, 2. \tag{2.1}$$

Here $W_0^\pm$ and W_j ($j = 1, 2$) are some constants, $\mathrm{d}w/\mathrm{d}z = v_x - \mathrm{i}v_y$, v_x and v_y are the velocity components, V_j are positive constants defined by the Bernoulli equation

$$\frac{1}{2}(V_j^2 - V_\infty^2) + \frac{p_j - p_\infty}{\rho} = 0, \quad j = 1, 2, \tag{2.2}$$

p_∞ and p_j are the pressure at infinity and in the cavities,

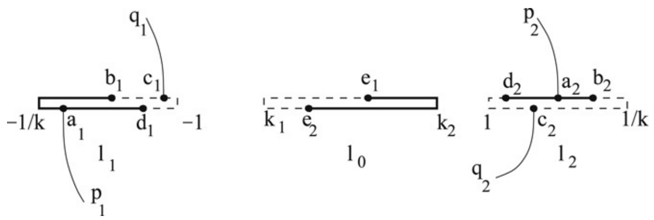

Fig. 2. Parametric ζ-plane with the cuts l_0, l_1, and l_2.

respectively, α_j are the angles the foils make with the real axis, $L_0 = E_1^- E_2^- E_2^+ E_1^+$ is the boundary of the channel, $E_j^- = \mp\infty + \mathrm{i}0$, $E_j^+ = \mp\infty + \mathrm{i}h$, $E_1^+ E_2^+$ is a free surface. The first equation in (2.1) reflects the fact that the stream function ψ is piecewise constant on the boundary of the flow that consists of L_0 and two lops $L_j = A_j B_j C_j D_j A_j$ $(j = 1, 2)$.

The complex potential w is an analytic function in the flow domain $\mathcal{D}$. It becomes a single-valued function in the domain $\mathcal{D}$ cut along the two stream lines $P_j A_j$ joining the critical points A_j $(j = 1, 2)$ with the infinite point $-\infty + \mathrm{i}y$, $0 \le y \le h$.

The problem of interest is non-linear first, for the shapes of the free surface of the channel, the boundaries of the cavities, and the position of the stagnation points A_j being unknown and second, for the boundary conditions (2.1) being non-linear.

3. Conformal mapping

Let $z = f(\zeta)$ be a conformal mapping of a parametric ζ-plane cut along three segments of the real axis $l_1 = [-1/k, 1]$, $l_0 = [k_1, k_2]$, and $l_2 = [1, 1/k]$ $(0 < k < 1$, $-1 < k_1 < k_2 < 1)$ onto the 3-connected flow domain $\mathcal{D}$. Such a map always exists [12]. It becomes unique if, in addition, it is required that the segments l_j are transformed into the loops L_j $(j = 0, 1, 2)$, and the images of the points a_j, b_j, c_j, d_j $(j = 1, 2)$, e_1, and e_2 are the points of the physical domain A_j, $B_j, C_j, D_j, -\infty + \mathrm{i}y$, and $\infty + \mathrm{i}y$ $(0 \le y \le h)$, respectively. None of the points of the real axis k_1, k_2, k, or the points a_j, b_j, c_j, d_j, and e_j $(j = 1, 2)$ can be prescribed arbitrary. They have to be found as a part of the solution to the problem. What is known however, is that these points lie on either side of the cuts: $a_j, b_j, c_j, d_j \in l_j$ and $e_1, e_2 \in l_0$, and they follow each other in clockwise direction (the same direction is chosen for the contours L_j in the flow domain $\mathcal{D}$). The image pattern of the flow domain by the inverse transform $\zeta = f^{-1}(z)$ is shown in Fig. 2. The solid line corresponds to the solid boundary of the flow, and the broken line is the image of the unknown curves.

It will be convenient to introduce a new function $\omega(\zeta)$,

$$\omega(\zeta) = \log \frac{\mathrm{d}w(z)}{V_\infty \mathrm{d}z}, \quad z = f(\zeta). \tag{3.1}$$

Then

$$\omega(\zeta) = \log \frac{V}{V_\infty} - \mathrm{i}\theta, \tag{3.2}$$

where $V = |\mathrm{d}w/\mathrm{d}z|$ is the speed, and $\theta = -\arg \mathrm{d}w/\mathrm{d}z$ is the angle that the velocity vector makes with the x-axis. The function $\omega(f^{-1}(z))$ is defined in the flow domain $\mathcal{D}$ cut along the streamlines joining the point $-\infty + \mathrm{i}y$ $(0 \le y \le h)$ with the points A_1 and A_2.

The boundary conditions (2.1) on the boundary of the flow transform into ones on the sides of the cuts. The free surface boundary condition becomes

$$\operatorname{Re}\omega(\zeta) = \begin{cases} 0, & \zeta \in e_2 e_1 \\ \log(V_j/V_\infty), & \zeta \in b_j c_j d_j, \end{cases} \quad j = 1, 2, \tag{3.3}$$

and the boundary condition on the solid part of the boundary can be written as

$$\operatorname{Im}\omega(\zeta) = \begin{cases} 0, & \zeta \in e_1 e_2 \\ -\alpha_j, & \zeta \in a_j b_j \\ \pi - \alpha_j, & \zeta \in d_j a_j, \end{cases} \quad j = 1, 2. \tag{3.4}$$

Since $\mathrm{d}w/\mathrm{d}z = 0$ at $z = A_j$ $(j = 1, 2)$, the new function $\omega(\zeta)$ has the logarithmic singularities at the points $\zeta = a_j$. In a neighborhood of the points c_j, the function $\omega(\zeta)$ has a singularity [4,5]

$$\omega(\zeta) \sim M_j'(w - w_j)^{-1/2}, \qquad z \to C_j, \quad M_j' \ne 0, \tag{3.5}$$

where $w_j = w(C_j)$, and a single branch of the square root is considered in the w-plane cut along the axis $\arg(w - w_j) = \pi$. A neighborhood $|w - w_j| < \varepsilon$ of the point $w = w_j$ cut along the line $\arg(w - w_j) = \pi$ corresponds to a neighborhood of the point $z = C_j$ which locally is a semi-disc. Therefore, $w - w_j \sim N_j'(z - C_j)^2$, $z \to C_j$, $N_j' \ne 0$. Next, $z - C_j \sim N_j''(\zeta - c_j)$, $z \to C_j$, $N_j'' \ne 0$. It follows from (3.5) now that the function $\omega(\zeta)$ has simple poles at the points $\zeta = c_j$

$$\omega(\zeta) \sim M_j''(\zeta - c_j)^{-1}, \quad \zeta \to c_j. \tag{3.6}$$

Here M_j'' are non-zero real constants and $j = 1, 2$.

The recovering of the mapping function $z = f(\zeta)$ is the key step in the definition of the free surfaces. Its derivative can be found from the relation

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}f}{\mathrm{d}\zeta} \tag{3.7}$$

provided the derivatives $\mathrm{d}w/\mathrm{d}\zeta$ and $\mathrm{d}w/\mathrm{d}z$ are known. Because of the link

$$\frac{\mathrm{d}w}{\mathrm{d}z} = V_\infty \mathrm{e}^{\omega(\zeta)}, \tag{3.8}$$

to find the derivative $\mathrm{d}w/\mathrm{d}z$, one needs to solve the Hilbert problem (3.3) and (3.4) for the function $\omega(\zeta)$.

As for the function $\mathrm{d}w/\mathrm{d}\zeta$, since $\operatorname{Im} w(f(\zeta))$ is a piecewise constant function on the sides of the cuts l_j $(j = 0, 1, 2)$, the imaginary part of the derivative $\mathrm{d}w/\mathrm{d}\zeta$ is equal to zero on l_j

$$\operatorname{Im}\frac{\mathrm{d}w}{\mathrm{d}\zeta} = 0, \quad \zeta \in l_0 \cup l_1 \cup l_2. \tag{3.9}$$

At the points a_j and c_j $(j = 1, 2)$, the function $\mathrm{d}w/\mathrm{d}\zeta$ has simple zeros if these points do not coincide with the ending points of the cuts. Indeed, according to the model, $\operatorname{Im} w(z)$ has constant values W_j on the curves $P_j A_j$, $A_j B_j C_j$, $A_j D_j C_j$, and $C_j Q_j$. These lines are the images of the lines $p_j a_j$, the corresponding parts of the cuts l_j, and the lines $c_j q_j$ in the parametric ζ-plane (Fig. 2). Since the curves $P_j A_j$ and $C_j Q_j$ are orthogonal to the curves $A_j B_j C_j$ and $A_j D_j C_j$, the curves

$p_j a_j$ and $q_j c_j$ are orthogonal to the cuts l_j $(j = 1, 2)$. Therefore, in a neighborhood of the points $\zeta = a_j$ and c_j,

$$w(f(\zeta)) = w(A_j) + K'_j(\zeta - a_j)^2 + \cdots, \quad \zeta \to a_j, \quad K'_j \neq 0,$$
$$w(f(\zeta)) = w(C_j) + K''_j(\zeta - c_j)^2 + \cdots, \quad \zeta \to c_j, \quad K''_j \neq 0, \tag{3.10}$$

and $\mathrm{d}w/\mathrm{d}\zeta = 0$ at $\zeta = a_j$ and $\zeta = c_j$, $j = 1, 2$. When the points a_j or c_j coincide with one of the branch points, the function $\mathrm{d}w/\mathrm{d}\zeta$ is bounded and not equal to zero.

The point $\zeta = \infty$ corresponds to a finite point of the flow domain. Therefore, in a neighborhood of the point $\zeta = \infty$, $w = w_0 + c\zeta^{-1} + \cdots$, $c =$ const. Clearly, the function $\mathrm{d}w/\mathrm{d}\zeta$ has a zero of the second order at the infinite point of the parametric ζ-plane.

Analyze next the behavior of the function $\mathrm{d}w/\mathrm{d}\zeta$ at the points e_1 and e_2. The function $z = f(\zeta)$ maps the ζ-plane with three cuts onto the domain $\mathcal{D}$. At $z \to \mp\infty + \mathrm{i}y$ $(0 \le y \le h)$, the flow domain $\mathcal{D}$ looks like a strip of width h. Therefore, locally,

$$f(\zeta) = \frac{(-1)^{j-1}h}{\sigma\pi}\log(\zeta - e_j) + O(1), \quad \zeta \to e_j, \quad j = 1, 2, \tag{3.11}$$

where $\sigma = 1$ if e_j does not coincide with the ends of the cut l_0, and $\sigma = 2$ otherwise. Then the function $\mathrm{d}w/\mathrm{d}\zeta$ has simple poles at the points e_1 and e_2,

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \frac{(-1)^{j-1}hV_\infty}{\sigma\pi}\frac{1}{\zeta - e_j} + O(1), \quad \zeta \to e_j, \quad j = 1, 2. \tag{3.12}$$

In a neighborhood of any ending point (call it $\hat{\zeta}$) of the cuts, if it coincides with none of the critical points a_j, c_j, or e_j, $f(\zeta) \sim K(\zeta - \hat{\zeta})^{1/2}$ $(K \neq 0)$. Therefore, in this case, $\mathrm{d}w/\mathrm{d}\zeta \sim \tilde{K}(\zeta - \hat{\zeta})^{-1/2}$, $\zeta \to \hat{\zeta}$, $\tilde{K}$ is a non-zero constant.

Thus, the functions $\mathrm{d}w/\mathrm{d}\zeta$ and $\omega(\zeta)$ solve the two Hilbert boundary-value problems (3.9), (3.3) and (3.4) in the classes of functions prescribed. The solution of these problems will be found in Sections 4 and 5 by a method that exploits the symmetry of the parametric ζ-plane, the fact that the imaginary part of the unknown function is equal to zero on the whole boundary (in the case of $\mathrm{d}w/\mathrm{d}\zeta$) or on its portion (in the case of $\omega(\zeta)$) and the theory of Riemann surfaces of algebraic functions. After the solution of the problems the mapping function $z = f(\zeta)$ is found by integration of the derivative $\mathrm{d}f/\mathrm{d}\zeta$.

4. Function $\mathrm{d}w/\mathrm{d}\zeta$

Let $\mathcal{R}$ be the hyperelliptic surface given by the algebraic equation

$$u^2 = p(\zeta), \quad p(\zeta) = (\zeta^2 - 1)(k^2\zeta^2 - 1)(\zeta - k_1)(\zeta - k_2). \tag{4.1}$$

The surface is formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex ζ-plane $\mathbb{C} \cup \{\infty\}$ cut along the segments l_j, $(j = 0, 1, 2)$. The positive sides l_j^+ of the cuts $l_j \subset \mathbb{C}_1$ are glued with the negative sides l_j^- of the cuts $l_j \subset \mathbb{C}_2$, and the sides $l_j^- \subset \mathbb{C}_1$ are glued with $l_j^+ \subset \mathbb{C}_2$. The function $u(\zeta)$ defined by (4.1) is single-valued on $\mathcal{R}$:

$$u = \begin{cases} p^{1/2}(\zeta), & \zeta \in \mathbb{C}_1 \\ -p^{1/2}(\zeta), & \zeta \in \mathbb{C}_2. \end{cases} \tag{4.2}$$

Here $p^{1/2}(\zeta)$ is a branch fixed by the condition

$$p^{1/2}(\zeta) \sim k\zeta^3, \quad \zeta \to \infty. \tag{4.3}$$

The pair $(\zeta, p^{1/2}(\zeta))$ denotes a point on $\mathbb{C}_1$ with affix ζ, whilst the notation $(\zeta, -p^{1/2}(\zeta))$ will be used for its counterpart on the second sheet.

Notice that the sides of the cuts l_j $(j = 0, 1, 2)$ form the symmetry line for the surface $\mathcal{R}$ which splits the surface into two symmetrical halves, and the surface itself is therefore the Schottky double. Since $\operatorname{Im} \mathrm{d}w/\mathrm{d}\zeta = 0$ on l_j $(j = 0, 1, 2)$ the function $\mathrm{d}w/\mathrm{d}\zeta$ can be analytically continued by symmetry on the whole Riemann surface. Then

$$(\mathrm{d}w/\mathrm{d}\zeta)^+ - (\mathrm{d}w/\mathrm{d}\zeta)^- = 0, \quad (\zeta, u) \in (l_0 \cup l_1 \cup l_2) \subset \mathcal{R}. \tag{4.4}$$

By the generalized Liouville theorem, the solution of this simplest form of the Riemann–Hilbert problem is a rational function say, $R(\zeta, u)$, on the Riemann surface $\mathcal{R}$. If the points a_j, c_j do not coincide with the branch points of the surface $\mathcal{R}$, this function has simple zeros at the points $(a_j, u(a_j))$ and $(c_j, u(c_j))$ which simultaneously belong to the first and second sheet of the surface. Notice that the function $\mathrm{d}w/\mathrm{d}\zeta$ is bounded and not equal to zero at any point of the set $\{a_1, a_2, c_1, c_2\}$ if that point coincides with one of the branch points of the surface. The rational function $R(\zeta, u)$ on the surface has zeros of the second order at the infinite points $(\infty, +\infty)$ and $(\infty, -\infty)$ of the surface $\mathcal{R}$. At the points e_j (but not at the points $\bar{e}_j$) $(j = 1, 2)$ this function has simple poles and

$$\operatorname*{res}_{\zeta = e_j} R(\zeta, u) = (-1)^{j-1}\frac{hV_\infty}{\sigma\pi}. \tag{4.5}$$

Notice that the number of poles and zeros (counting multiplicities) of this function is the same.

Consider the most interesting and realistic case when none of the points a_j, c_j, e_j $(j = 1, 2)$ coincide with the branch points of the surface. Then $\sigma = 1$ and the function $R(\zeta, u)$ must have simple zeros at the points $(a_j, u(a_j))$ and $(c_j, u(c_j))$. The most general form of the rational function $R(\zeta, u)$ on the surface $\mathcal{R}$ is

$$R(\zeta, u) = R_1(\zeta) + \frac{R_2(\zeta)}{u}, \tag{4.6}$$

where the functions $R_j(\zeta)$ have simple poles at the points e_1 and e_2. At infinity, $R_1(\zeta) = O(\zeta^{-2})$ and $R_2(\zeta) = O(\zeta)$. The functions with these properties have the form

$$R_1(\zeta) = \left(\frac{1}{\zeta - e_1} - \frac{1}{\zeta - e_2}\right)N_0,$$

$$R_2(\zeta) = \mathrm{i}N_1 + \mathrm{i}N_2\zeta + \frac{N_3}{\zeta - e_1} + \frac{N_4}{\zeta - e_2}, \tag{4.7}$$

where N_j $(j = 0, 1, \ldots, 4)$ are arbitrary constants. It is necessary to impose the following four conditions

$$\operatorname*{res}_{\zeta=e_j}\left[R_1(\zeta) + \frac{R_2(\zeta)}{u(\zeta)}\right] = (-1)^{j-1}\frac{hV_\infty}{\sigma\pi}.$$
$$\operatorname*{res}_{\zeta=\bar{e}_j}\left[R_1(\zeta) + \frac{R_2(\zeta)}{u(\zeta)}\right] = 0, \tag{4.8}$$

which define the constants N_0, N_3, and N_4

$$N_0 = \frac{hV_\infty}{2\pi}, \qquad N_3 = \frac{hV_\infty}{2\pi}u(e_1), \qquad N_4 = -\frac{hV_\infty}{2\pi}u(e_2). \tag{4.9}$$

Substitute now (4.7) and (4.9) into (4.6) and take (ζ, u) as a point on the first sheet $\mathbb{C}_1$. This defines the expression of the derivative $\mathrm{d}w/\mathrm{d}\zeta$ on the parametric ζ-plane

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\mathrm{i}[N_1 + N_2\zeta + r(\zeta)]}{p^{1/2}(\zeta)}, \quad \zeta \in \mathbb{C}, \tag{4.10}$$

where

$$r(\zeta) = -\frac{\mathrm{i}hV_\infty}{2\pi}\left(\frac{p^{1/2}(\zeta) + p^{1/2}(e_1)}{\zeta - e_1} - \frac{p^{1/2}(\zeta) + p^{1/2}(e_2)}{\zeta - e_2}\right). \tag{4.11}$$

Now, the function $\mathrm{d}w/\mathrm{d}\zeta$ has to be equal to zero at the point a_j and c_j $(j = 1, 2)$. The first two conditions define uniquely the real constants N_1 and N_2

$$N_1 = \frac{a_2 r(a_1) - a_1 r(a_2)}{a_1 - a_2},$$
$$N_2 = \frac{r(a_2) - r(a_1)}{a_1 - a_2}. \tag{4.12}$$

The other two conditions $\mathrm{d}w/\mathrm{d}\zeta(c_j) = 0$ $(j = 1, 2)$ give two real non-linear equations for the parameters a_j, c_j, and e_j $(j = 1, 2)$

$$N_1 + N_2 c_j + r(c_j) = 0, \quad j = 1, 2. \tag{4.13}$$

5. Riemann–Hilbert problem on a Riemann surface

5.1. Formulation

The Hilbert problem (3.3) and (3.4) for the function $\omega(\zeta)$ may be formulated as a boundary-value problem on the Riemann surface of the algebraic function (4.1). Introduce the function

$$\Phi(\zeta, u) = \begin{cases} \omega(\zeta), & (\zeta, u) \in \mathbb{C}_1 \\ \overline{\omega(\bar{\zeta})}, & (\zeta, u) \in \mathbb{C}_2. \end{cases} \tag{5.1}$$

This function satisfies the symmetry condition

$$\overline{\Phi(\zeta_*, u_*)} = \Phi(\zeta, u), \tag{5.2}$$

where $(\zeta_*, u_*) = (\bar{\zeta}, -u(\bar{\zeta}))$ is the point symmetrical to a point $(\zeta, u(\zeta))$ with respect to the lines $l_0 \cup l_1 \cup l_2$ along which the two halves of the surface are glued. Thus, if $(\zeta, u) \in \mathbb{C}_1$, then $(\zeta_*, u_*) \in \mathbb{C}_2$. On the symmetry lines, the boundary values of the function $\Phi(\zeta, u)$ are

$$\Phi^+(\xi, v) = \omega(\xi), \qquad \Phi^-(\xi, v) = \overline{\omega(\xi)},$$
$$(\xi, v) \in l_1 \cup l_0 \cup l_2, \qquad v = u(\xi). \tag{5.3}$$

It is clear from inspection of the boundary conditions (3.3) and (3.4) that the function $\Phi(\zeta, u)$ is continuous through the curve e_1e_2 (a part of the contour l_0) and is discontinuous through the contours l_1, l_2 and the rest e_2e_1 of the contour l_0. Thus, the function $\Phi(\zeta, u)$ solves the following Riemann–Hilbert problem on the Riemann surface $\mathcal{R}$.

Find all the functions $\Phi(\zeta, u)$ analytic in $\mathcal{R} \setminus (l_0 \cup l_1 \cup l_2)$, Hölder-continuous up to the boundary $l_0 \cup l_1 \cup l_2$ with boundary values satisfying the relations

$$\Phi^+(\xi, v) - \Phi^-(\xi, v) = \begin{cases} 0, & (\xi, v) \in e_1e_2 \\ -2\mathrm{i}\alpha_j, & (\xi, v) \in a_jb_j \\ 2\mathrm{i}(\pi - \alpha_j), & (\xi, v) \in d_ja_j, \end{cases}$$
$$j = 1, 2,$$
$$\Phi^+(\xi, v) + \Phi^-(\xi, v) = \begin{cases} 0, & (\xi, v) \in e_2e_1 \\ 2\log(V_j/V_\infty), & (\xi, v) \in b_jc_jd_j, \end{cases}$$
$$j = 1, 2, \tag{5.4}$$

and the symmetry condition (5.2). The function $\Phi(\zeta, u)$ has logarithmic singularities at the points $(a_j, u(a_j))$ and poles of the first order at the points $(c_j, u(c_j))(j = 1, 2)$. It is bounded at the points $(b_j, u(b_j))$, $(d_j, u(d_j))$, both infinite points $(\infty, \pm\infty)$, and $\Phi(e_j, u(e_j)) = 0$, $j = 1, 2$.

5.2. Factorization

Let $\mathcal{L} = b_1c_1d_1 \cup e_2e_1 \cup b_2c_2d_2$. To solve the problem (5.4), consider first the homogeneous Riemann–Hilbert problem

$$X^+(\xi, v) = -X^-(\xi, v), \quad (\xi, v) \in \mathcal{L}. \tag{5.5}$$

By the Sokhotski–Plemelj formulas the function

$$\chi_0(\zeta, u) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \log(-1)\mathrm{d}W\right\}$$
$$= \left[\frac{(\zeta - d_1)(\zeta - d_2)(\zeta - e_1)}{(\zeta - b_1)(\zeta - b_2)(\zeta - e_2)}\right]^{1/4} \times \exp\left\{\frac{u}{4}\int_{\mathcal{L}} \frac{\mathrm{d}\xi}{(\xi - \zeta)v}\right\} \tag{5.6}$$

satisfies the boundary condition (5.5). Here $\mathrm{d}W$ is the Weierstrass kernel

$$\mathrm{d}W = \frac{1}{2}\left(1 + \frac{u}{v}\right)\frac{\mathrm{d}\xi}{\xi - \zeta}, \qquad u = u(\zeta), \qquad v = u(\xi). \tag{5.7}$$

Also, it is directly verified that the function $\chi_0(\zeta, u)$ meets the symmetry condition (5.2). However, at the points $(\infty, \pm\infty)$ it has essential singularities. This is because of the second order

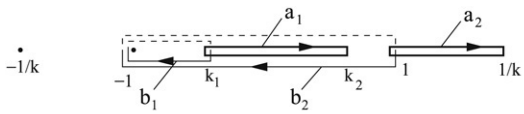

Fig. 3. Canonical cross-sections $\mathbf{a}_j$ and $\mathbf{b}_j$ $(j = 1, 2)$.

pole of the kernel $\mathrm{d}W$ at both infinite points of the surface. To remove these essential singularities, analyze the function

$$\chi_0(\zeta, u)\chi_1(\zeta, u)\overline{\chi_1(\zeta_*, u_*)}, \tag{5.8}$$

where

$$\chi_1(\zeta, u) = \exp\left\{-\sum_{j=1}^{2}\left(\int_{\gamma_j} + m_j \oint_{\mathbf{a}_j} + n_j \oint_{\mathbf{b}_j}\right)\mathrm{d}W\right\}. \tag{5.9}$$

Here $\mathbf{a}_j$ and $\mathbf{b}_j$ $(j = 1, 2)$ form a system of canonical cross-sections of the surface $\mathcal{R}$. The contours $\mathbf{a}_1$ and $\mathbf{a}_2$ are smooth loops which lie on both sheets of the surface and coincide with the banks of the cuts l_0 and l_2, respectively (Fig. 3). The positive direction is that which leaves the sheet $\mathbb{C}_1$ on the left. The loop $\mathbf{b}_1$ consists of the segments $[k_1, -1] \subset \mathbb{C}_1$ and $[-1, k_1] \subset \mathbb{C}_2$. The contour $\mathbf{b}_2$ consists of the segments $[1, -1] \subset \mathbb{C}_1$ and $[-1, 1] \subset \mathbb{C}_2$. In Fig. 3, the part of the contours $\mathbf{b}_j$ which lies on the first sheet is shown as the solid lines. The broken line corresponds to the part lying on $\mathbb{C}_2$. The loops $\mathbf{a}_j$ and $\mathbf{b}_\nu$ $(\nu \neq j)$ do not intersect. The cross-section $\mathbf{a}_j$ intersects $\mathbf{b}_j$ from left to the right, and there is only one point of intersection.

The contours γ_1 and γ_2 are continuous curves with the same starting point $(-1/k, 0)$. The ending points $q_j = (\zeta_j, u_j) \in \mathcal{R}$ $(u_j = u(\zeta_j))$ are to be determined. These points may lie on either sheet of the surface. The contours γ_j cannot intersect the canonical cross-sections. The parameters m_j and n_j are integers and will be found later on.

Now, formulas (5.6) and (5.9) can be combined to give an expression for the function (5.8)

$$\chi_0(\zeta, u)\chi_1(\zeta, u)\overline{\chi_1(\zeta_*, u_*)} = \left[\frac{(\zeta - d_1)(\zeta - d_2)(\zeta - e_1)}{(\zeta - b_1)(\zeta - b_2)(\zeta - e_2)}\right]^{1/4} \times \frac{(\zeta + 1/k)^2\chi(\zeta, u)}{\sqrt{(\zeta - \zeta_1)(\zeta - \bar{\zeta}_1)(\zeta - \zeta_2)(\zeta - \bar{\zeta}_2)}}, \tag{5.10}$$

where

$$\chi(\zeta, u) = \exp\left\{\frac{u}{4}\int_{\mathcal{L}} \frac{1}{v}\frac{\mathrm{d}\xi}{\xi - \zeta} - u\sum_{j=1}^{2}\left(\int_{\Gamma_j}\frac{1}{2v}\frac{\mathrm{d}\xi}{\xi - \zeta} + 2m_j\int_{\mathbf{a}_j^+}\frac{\mathrm{d}\xi}{p^{1/2}(\xi)(\xi - \zeta)}\right)\right\}. \tag{5.11}$$

Here the contours Γ_j are curves symmetrical with respect to the real line, lying on one of the sheets of the surface, starting at the points $(\overline{\zeta}_j, \overline{u}_j)$ and ending at (ζ_j, u_j). The contours $\mathbf{a}_1^+$ and $\mathbf{a}_2^+$ are the upper banks of the cuts l_0 and l_2, respectively. Notice that the integrals over the loops $\mathbf{b}_j$ vanish in formula (5.11) because of the symmetry.

It will be convenient to take the solution of the factorization problem (5.5) as

$$X(\zeta, u) = \frac{(\zeta - d_1)^{1/4}(\zeta - d_2)^{1/4}(\zeta - b_1)^{3/4}(\zeta - b_2)^{3/4}(\zeta - e_1)^{1/4}(\zeta - e_2)^{3/4}}{\sqrt{(\zeta - \zeta_1)(\zeta - \bar{\zeta}_1)(\zeta - \zeta_2)(\zeta - \bar{\zeta}_2)}}\chi(\zeta, u). \tag{5.12}$$

Let β be one of the points b_j, d_j, or e_j. Analysis of the singular integrals in (5.11) in a neighborhood of that point indicates that

$$X(\zeta, u) \sim \Lambda_\beta(\zeta - \beta)^{1/2}, \quad \zeta \to \beta, \ \Lambda_\beta = \text{const} \neq 0, \tag{5.13}$$

while at the point $\zeta = \bar{\beta}$ that lies on the opposite side of the corresponding cut,

$$X(\zeta, u) \sim \begin{cases} \tilde{\Lambda}_{b_j}(\zeta - b_j), & \zeta \to \bar{b}_j \\ \tilde{\Lambda}_{d_j}, & \zeta \to \bar{d}_j \\ \tilde{\Lambda}_{e_1}, & \zeta \to \bar{e}_1 \\ \tilde{\Lambda}_{e_2}(\zeta - e_2), & \zeta \to \bar{e}_2, \end{cases} \qquad \tilde{\Lambda}_\beta = \text{const} \neq 0, \quad j = 1, 2. \tag{5.14}$$

At the points $q_j = (\zeta_j, u_j)$, the factorization function $X(\zeta, u)$ has simple poles and it is bounded at the points $(\overline{\zeta}_j, \overline{u}_j)$.

At present, the function $\chi(\zeta, u)$ and therefore the function $X(\zeta, u)$ have an inappropriate exponential growth at infinity caused by the pole of the second order at infinity of the kernel $\mathrm{d}W$. In order to get a solution of a finite order at infinity, expand the function $\chi(\zeta, u)$ as $\zeta \to \infty$,

$$X(\zeta, u) = \zeta\exp\left\{-\sum_{\nu=1}^{2}\frac{u}{\zeta^\nu}\left[\frac{1}{4}\int_{\mathcal{L}}\frac{\xi^{\nu-1}\mathrm{d}\xi}{v} - \sum_{j=1}^{2}\left(\int_{\Gamma_j}\frac{\xi^{\nu-1}\mathrm{d}\xi}{2v} + 2m_j\int_{\mathbf{a}_j^+}\frac{\xi^{\nu-1}\mathrm{d}\xi}{p^{1/2}(\xi)}\right)\right]\right\} + O(\zeta), \quad \zeta \to \infty. \tag{5.15}$$

Because of the symmetry of the contours Γ_j, the requirement of the canonical function to have a finite order at infinity enables us to write the following system of non-linear equations

$$\operatorname{Im}\left\{\frac{1}{4}\int_{\mathcal{L}}\frac{\xi^{\nu-1}\mathrm{d}\xi}{v} - \sum_{j=1}^{2}\left[\int_{\gamma_j}\frac{\xi^{\nu-1}\mathrm{d}\xi}{v} + 2m_j\int_{\mathbf{a}_j^+}\frac{\xi^{\nu-1}\mathrm{d}\xi}{p^{1/2}(\xi)}\right]\right\} = 0, \quad \nu = 1, 2. \tag{5.16}$$

This system presents a real analogue of the Jacobi inversion problem on the Riemann surface $\mathcal{R}$. To reduce the system (5.16) to the classical inversion problem, notice that

$$\operatorname{Re}\left(\int_{\mathcal{L}}\frac{\xi^{\nu-1}\mathrm{d}\xi}{v}\right) = 0,$$

$$2\int_{\mathbf{a}_j^+}\frac{\xi^{\nu-1}\mathrm{d}\xi}{p^{1/2}(\xi)} = \oint_{\mathbf{a}_j}\mathrm{d}\omega_\nu = A_{\nu j}, \quad \operatorname{Re}A_{\nu j} = 0,$$

$$\oint_{\mathbf{b}_j}\mathrm{d}\omega_\nu = B_{\nu j}, \quad \operatorname{Im}B_{\nu j} = 0, \tag{5.17}$$

where $d\omega_\nu$ are abelian differentials of the first kind of the surface $\mathcal{R}$:

$$d\omega_\nu = \frac{\xi^{\nu-1}d\xi}{v}, \quad \nu = 1, 2, \tag{5.18}$$

and $A_{\nu j}$ and $B_{\nu j}$ are the A- and B-periods of the abelian integrals

$$\omega_\nu = \omega_\nu(\zeta, u) = \int_{(-1/k,0)}^{(\zeta,u)} \frac{\xi^{\nu-1}d\xi}{v}, \quad \nu = 1, 2. \tag{5.19}$$

Now, the imaginary part of the system

$$\frac{1}{4}\int_{\mathcal{L}} \frac{\xi^{\nu-1}d\xi}{v} - \sum_{j=1}^{2}\left(\int_{\gamma_j} \frac{\xi^{\nu-1}d\xi}{v} + m_j A_{\nu j} + n_j B_{\nu j}\right) = 0, \quad \nu = 1, 2 \tag{5.20}$$

coincides with the system (5.16). On the other hand, it can be written as the classical Jacobi problem

$$\sum_{j=1}^{2}\left[\omega_\nu(q_j) + m_j A_{\nu j} + n_j B_{\nu j}\right] = \eta_\nu, \quad \nu = 1, 2, \tag{5.21}$$

where

$$\eta_\nu = \frac{1}{4}\int_{\mathcal{L}} \frac{\xi^{\nu-1}d\xi}{v}. \tag{5.22}$$

The solution to the inversion problem (5.21) with respect to the two points $q_j = (\zeta_j, u_j) \in \mathcal{R}$ and the four integers m_j and n_j ($j = 1, 2$) (the function $\chi(\zeta, u)$ in (5.11) is independent of the constants n_j) makes the function $\chi(\zeta, u)$ bounded at infinity.

5.3. Jacobi inversion problem

Following [9,11] normalize first the basis of abelian integrals

$$\hat{\omega} = T\omega, \quad T = A^{-1}, \tag{5.23}$$

where $A = \{A_{\nu j}\}$ ($\nu, j = 1, 2$). The A- and B-periods of the canonical basis $\hat{\omega}$ are

$$\mathcal{A}_{\nu j} = \oint_{\mathbf{a}_j} d\hat{\omega}_\nu = \delta_{\nu j},$$

$$\mathcal{B}_{\nu j} = \oint_{\mathbf{b}_j} d\hat{\omega}_\nu = \sum_{m=1}^{2} T_{\nu m} B_{mj}. \tag{5.24}$$

Here $\delta_{\nu j}$ is the Kronecker symbol. The matrix $\mathcal{B} = \{\mathcal{B}_{\nu j}\}$ ($\nu, j = 1, 2$) is symmetric, and $\operatorname{Im}\mathcal{B}$ is positive definite. For the canonical basis, the inversion problem (5.21) becomes

$$\sum_{j=1}^{2}\hat{\omega}_\nu(q_j) = g_\nu - \kappa_\nu - \sum_{j=1}^{2} n_j \mathcal{B}_{\nu j} - m_\nu \equiv g_\nu - \kappa_\nu, \quad \text{(modulo the periods)}, \quad \nu = 1, 2, \tag{5.25}$$

where

$$g_\nu = \sum_{j=1}^{2} T_{\nu j}\eta_j + \kappa_\nu, \quad \nu = 1, 2, \tag{5.26}$$

and κ_1 and κ_2 are the Riemann constants for the chosen system of **a** and **b** cross-sections computed in [11]

$$\kappa_\nu = \frac{1}{2}(-\nu + \mathcal{B}_{\nu 1} + \mathcal{B}_{\nu 2}). \tag{5.27}$$

The affixes ζ_1 and ζ_2 of the points q_1 and q_2 on the surface $\mathcal{R}$ coincide with the two zeros of the Riemann θ-function

$$\begin{aligned}\mathfrak{F}(q) &= \theta(\hat{\omega}_1(q) - g_1, \hat{\omega}_2(q) - g_2) \\ &= \sum_{t_1,t_2=-\infty}^{\infty} \exp\left\{\pi i \sum_{\mu=1}^{2}\sum_{\nu=1}^{2} \mathcal{B}_{\mu\nu} t_\mu t_\nu + 2\pi i \sum_{\nu=1}^{2} t_\nu[\hat{\omega}_\nu(q) - g_\nu]\right\}.\end{aligned} \tag{5.28}$$

Since the matrix $\operatorname{Im}\mathcal{B}$ is positive definite the double series (5.28) converges exponentially. The affixes of the points q_1 and q_2 solve the system of two equations [13]

$$\zeta_1^\nu + \zeta_2^\nu = \varepsilon_\nu, \quad \nu = 1, 2, \tag{5.29}$$

where

$$\varepsilon_\nu = 2\sum_{j=1}^{2}\sum_{m=1}^{2} T_{jm}\int_{\mathbf{a}_j^+} \frac{\tau^{m+\nu-1}d\tau}{p^{1/2}(\tau)} - \sum_{\mu=1}^{2} \operatorname*{res}_{q=\infty_\mu \in \mathbb{C}_\mu} \frac{\zeta^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)}. \tag{5.30}$$

Here $\infty_\mu = (\infty, (-1)^{\mu-1}\infty) \in \mathbb{C}_\mu$ ($\mu = 1, 2$) are the two infinite points of the Riemann surface.

By solving (5.29),

$$\zeta_j = \frac{\varepsilon_1 - (-1)^j\sqrt{2\varepsilon_2 - \varepsilon_1^2}}{2}, \quad j = 1, 2. \tag{5.31}$$

Since $\operatorname{Re}\mathcal{B}_{\nu j} = 0$, the integers m_ν are given by [13,11]

$$m_\nu = \operatorname{Re}(\hat{g}_\nu), \quad \hat{g}_\nu = g_\nu - \kappa_\nu - \sum_{j=1}^{2}\hat{\omega}_\nu(q_j), \tag{5.32}$$

and the integers n_ν solve the system of linear algebraic equations

$$\sum_{j=1}^{2} n_j \operatorname{Im}(\mathcal{B}_{\nu j}) = \operatorname{Im}(\hat{g}_\nu), \quad \nu = 1, 2. \tag{5.33}$$

This completes the solution to the Jacobi problem (5.25) and therefore (5.21). We emphasize that the function $\chi(\zeta, u)$ is independent of the integers n_ν.

5.4. Function dw/dz

The definition of the function dw/dz requires solving the Riemann–Hilbert problem (5.4). Using the factorization (5.5) write the boundary condition (5.4) as

$$\frac{\Phi^+(\xi, v)}{X^+(\xi, v)} - \Psi^+(\xi, v) = \frac{\Phi^-(\xi, v)}{X^-(\xi, v)} - \Psi^-(\xi, v), \quad (\xi, v) \in \mathcal{L}, \tag{5.34}$$

where $\Psi^{\pm}(\xi, v)$ are the boundary values of the function

$$\Psi(\zeta, u) = \Psi_0(\zeta) + u\Psi_1(\zeta),$$

$$\Psi_\nu(\zeta) = \frac{1}{2}\sum_{j=1}^{2}\left(-\frac{\alpha_j}{\pi}\int_{d_j a_j b_j} + \int_{d_j a_j} + \frac{\log(V_j/V_\infty)}{\pi i}\int_{b_j c_j d_j}\right)\frac{d\xi}{X^+(\xi, v)(\xi-\zeta)v^\nu}, \quad \nu = 0, 1. \tag{5.35}$$

By the generalized Liouville theorem, the general solution of the problem (5.34) has the form

$$\Phi(\zeta, u) = X(\zeta, u)\{\Psi_0(\zeta) + \Omega_0(\zeta) + u[\Psi_1(\zeta) + \Omega_1(\zeta)]\}, \tag{5.36}$$

where $\Omega_j(\zeta) (j = 0, 1)$ are certain rational functions on the complex ζ-plane. The function $X(\xi, u)$ has simple zeros at the points $\bar{b}_j$ $(j = 1, 2)$, $\bar{e}_2$ and the solution itself has simple poles at the points c_1 and c_2. The functions $X(\zeta, u)$ and $u(\zeta)$ grow at infinity as ζ and ζ^3, respectively. Therefore, the functions $\Omega_0(\zeta)$ and $\Omega_1(\zeta)$ are given by

$$\Omega_0(\zeta) = \sum_{j=0}^{2} M_j\zeta^j + \sum_{j=1}^{2}\left(\frac{iM_{j+2}u(c_j)}{\zeta - c_j} - \frac{iM_{j+4}u(b_j)}{\zeta - b_j}\right) - \frac{iM_7 u(e_2)}{\zeta - e_2},$$

$$\Omega_1(\zeta) = \sum_{j=1}^{2}\left(\frac{iM_{j+2}}{\zeta - c_j} + \frac{iM_{j+4}}{\zeta - b_j}\right) + \frac{iM_7}{\zeta - e_2}, \tag{5.37}$$

where M_j $(j = 0, 1, \ldots, 7)$ are arbitrary real constants.

The factorization function $X(\zeta, u)$ has simple poles at the points $q_j = (\zeta_j, u_j)$. To make these points removable singularities for the function $\Phi(\zeta, u)$, the following two complex conditions need to be satisfied

$$\Psi_0(\zeta_j) + \Omega_0(\zeta_j) + u_j[\Psi_1(\zeta_j) + \Omega_1(\zeta_j)] = 0, \quad j = 1, 2. \tag{5.38}$$

At infinity, after the solution of the Jacobi inversion problem, since the factorization function $X(\zeta, u)$ has a simple pole, the function $\Phi(\zeta, u)$ has a pole of the third order. To write down the conditions which eliminate these singularities, expand the functions $u(\zeta)$, $\Omega_0(\zeta)$, $\Omega_1(\zeta)$, and $\Psi_1(\zeta)$ at infinity

$$u(\zeta) = k\zeta^3 + K_1\zeta^2 + K_2\zeta + \cdots,$$
$$\Omega_0(\zeta) = M_2\zeta^2 + M_1\zeta + M_0 + \cdots,$$
$$\Omega_1(\zeta) = \frac{\Omega_{10}}{\zeta} + \frac{\Omega_{11}}{\zeta^2} + \frac{\Omega_{12}}{\zeta^3} + \cdots,$$
$$\Psi_1(\zeta) = \frac{\Psi_{10}}{\zeta} + \frac{\Psi_{11}}{\zeta^2} + \frac{\Psi_{12}}{\zeta^3} + \cdots, \tag{5.39}$$

where

$$K_1 = -\frac{k}{2}(k_1 + k_2), \quad K_2 = -\frac{k}{4}\left(\frac{k_1^2 + k_2^2}{2} + 2 + \frac{2}{k^2} - k_1 k_2\right),$$

$$\Omega_{1\nu} = i\sum_{j=1}^{2}(c_j^\nu M_{j+2} + b_j^\nu M_{j+4}) + iM_7 e_2^\nu, \quad \nu = 0, 1, 2,$$

$$\Psi_{1\nu} = -\frac{1}{2}\sum_{j=1}^{2}\left(-\frac{\alpha_j}{\pi}\int_{d_j a_j b_j} + \int_{d_j a_j} + \frac{\log(V_j/V_\infty)}{\pi i}\int_{b_j c_j d_j}\right)\frac{\xi^\nu d\xi}{X^+(\xi, v)v}, \quad \nu = 0, 1, 2. \tag{5.40}$$

The solution to the Riemann–Hilbert problem (5.4) is finite at infinity provided the following three complex conditions are met

$$M_2 + k(\Psi_{10} + \Omega_{10}) = 0,$$
$$M_1 + K_1(\Psi_{10} + \Omega_{10}) + k(\Psi_{11} + \Omega_{11}) = 0,$$
$$M_0 + K_2(\Psi_{10} + \Omega_{10}) + K_1(\Psi_{11} + \Omega_{11}) + k(\Psi_{12} + \Omega_{12}) = 0. \tag{5.41}$$

At the points $\zeta = e_j$ $(j = 1, 2)$ the function $\Phi(\zeta, u)$ vanishes as required. It follows from this fact that the speed of the fluid far away from the foils approaches the given value V_∞. The function $\Phi(\zeta, u)$ is bounded at the points b_j and d_j, and it has the logarithmic singularity at the points $\zeta = a_j$.

6. Definition of the velocity and the free boundaries

The solution of the Riemann–Hilbert problem (5.4) defines the velocity of the fluid $\mathbf{v} = (v_x, v_y)$ by

$$v_x - iv_y = \frac{dw}{dz} = V_\infty \exp\{\Phi(\zeta, u)\}, \quad (\zeta, u) \in \mathbb{C}_1. \tag{6.1}$$

The solution found possesses 21 unknown real constants. Among them, there are ten images a_j, b_j, c_j, d_j, and e_j $(j = 1, 2)$ of the points A_j, B_j, C_j, D_j, and $E_j^{\pm}$, respectively. The three parameters k, k_1, and k_2 of the conformal map $z = f(\zeta)$ are also to be determined. The last group of the unknowns are the eight real constants M_j $(j = 0, 1, \ldots, 7)$ in the representation (5.36) of the solution to the Riemann–Hilbert problem (5.4). For their definition, we have already written two real conditions (4.13), two complex equations (5.38) and three complex conditions (5.41). Thus, in total, there are 12 real equations.

We should also add the following five real conditions which specify the geometry of the problem. First,

$$\int_{d_j a_j b_j}\left|\frac{df}{d\zeta}\right||d\zeta| = \lambda_j, \quad j = 1, 2, \tag{6.2}$$

which define the prescribed lengths λ_1 and λ_2 of the foils. Here the derivative $df/d\zeta$ is expressed through the functions $dw/d\zeta$ and dw/dz as

$$\frac{df}{d\zeta} = \frac{dw/d\zeta}{dw/dz}. \tag{6.3}$$

Next, we specify the two distances μ_1 and μ_2 from the ending points D_j to the bottom of the channel by

$$\operatorname{Im}\int_{e_0}^{d_j}\frac{df}{d\zeta}d\zeta = \mu_j, \quad j = 1, 2, \tag{6.4}$$

where e_0 is an arbitrary fixed point on the portion e_1e_2 of the cut l_0. If it turns out that $k_2 \in e_1e_2$, then k_2 can be chosen as the point e_0.

The fifth condition describes the distance μ between the two ending points D_1 and D_2 of the hydrofoils

$$\mathrm{Re} \int_{d_1}^{d_2} \frac{\mathrm{d}f}{\mathrm{d}\zeta} \mathrm{d}\zeta = \mu. \tag{6.5}$$

We emphasize that, in general, the conformal map $z = f(\zeta)$ is a multi-valued function. It becomes a one-to-one map if

$$\oint_{l_j} \frac{\mathrm{d}f}{\mathrm{d}\zeta} \mathrm{d}\zeta = 0, \quad j = 1, 2. \tag{6.6}$$

These two complex equations add four real conditions for the unknown parameters. Thus, in total we have 21 real conditions for the same number of real constants. Note that the system of these equations can be split into a set of eight linear equations for the constants M_j $(j = 0, 1, \ldots, 7)$ and a system of 13 non-linear (algebraic and transcendental) equations for the parameters a_j, b_j, c_j, d_j, e_j $(j = 1, 2)$, k, k_1, and k_2. These non-linear equations can be solved numerically.

Finally, we determine the equations for the unknown *a priori* boundaries of the cavities behind the hydrofoils and the free surface of the channel. Integrating the function (6.3) gives the boundaries of the cavities

$$z(\tau) = \int_{b_j}^{\tau} \frac{\mathrm{d}w(f(\zeta))/\mathrm{d}\zeta}{\mathrm{d}w/\mathrm{d}z} \mathrm{d}\zeta + B_j, \quad \tau \in b_jc_j,\ z \in B_jC_j,$$
$$j = 1, 2, \tag{6.7}$$

and

$$z(\tau) = \int_{d_j}^{\tau} \frac{\mathrm{d}w(f(\zeta))/\mathrm{d}\zeta}{\mathrm{d}w/\mathrm{d}z} \mathrm{d}\zeta + D_j, \quad \tau \in c_jd_j,\ z \in C_jD_j,$$
$$j = 1, 2, \tag{6.8}$$

whilst letting $\tau \in e_2e_1$ by integration we recover the free surface of the channel

$$z(\tau) = \int_{b}^{\tau} \frac{\mathrm{d}w(f(\zeta))/\mathrm{d}\zeta}{\mathrm{d}w/\mathrm{d}z} \mathrm{d}\zeta + B, \quad \tau \in e_2e_1,\ z \in E_2^+E_1^+. \tag{6.9}$$

The integrals (6.7) and (6.8) are taken either over the banks of the corresponding cuts, or over smooth curves on the cut ζ-plane $\mathbb{C} \setminus (l_0 \cup l_1 \cup l_2)$. The contour of integration $b\tau$ in (6.9) is a smooth curve on $\mathbb{C} \setminus (l_0 \cup l_1 \cup l_2)$ which do not intersect the images of the cuts P_jA_j (the complex potential w is a single-valued analytic function in the flow domain $\mathcal{D}$ cut along the streamlines P_jA_j joining the stagnation points A_j and the infinite point $-\infty + \mathrm{i}y$, $0 \le y \le h$), and b is the image of a point $B \in \mathcal{D}$. In the case shown in Fig. 1 B_2 can be taken as the the point B.

7. Conclusions

In the present paper, in the framework of the Tulin's model of cavitating flow the steady streaming flow in a channel with a free surface around two hydrofoils (plates) with cavities of unknown shape behind them has been analyzed. By a conformal map of a parametric complex plane cut along three cuts on the real axis onto the triple-connected flow domain, the model problem has been reduced to two Riemann–Hilbert problems on a Schottky double which is a hyperelliptic surface of genus two. The solution of these problems was found to be necessary in order to recover the actual form of the conformal map. The solution to the first problem turned out to be a rational function on the Riemann surface and was constructed explicitly without quadratures. The second problem required the use of singular integrals with the Weierstrass kernel. Due to the second order pole of the kernel, in general, the solution had an essential singularity at the infinite points of the surface. The procedure of elimination of the essential singularities of the solution led to the classical Jacobi inversion problem of genus two which was solved explicitly.

Final equations for the velocity, the free surface of the channel, cavities and the conformal mapping itself possessed 21 unknown real parameters. To recover them the same number of real conditions (13 non-linear and 8 linear equations) have been found. The non-linear system of algebraic and transcendental equations can be separated from the linear system.

It is worth pointing out some limitations and possible extensions to the work done in the paper. First, for the method presented it is essential to be able to map a parametric plane cut along n segments *on the same straight line* into the n-connected flow domain. If $1 \le n \le 3$, then such a map always exists [12]. If $n \ge 4$, in general, it does not. For this more general case, it is hoped to develop another technique based on the theory of the Riemann–Hilbert problem for symmetric automorphic functions. Second, the paper concerns the supercavitating *steady* flow around hydrofoils. In the *transient* case for not too rapidly varying flows, it is possible (at least approximately) to assume that the free boundaries of the cavities are streamlines [14] and modify the procedure of the paper accordingly.

Acknowledgments

This work was partly funded by NSF through grant DMS0707724, Louisiana Board of Regents through grant LEQSF(2005-07)-ENH-TR-09 and Russian Foundation for Basic Research through grant 07-01-00038.

References

[1] G. Birkhoff, E.H. Zarantello, Jets, Wakes, and Cavities, Academic Press, New York, 1957.
[2] P. Eisenberg, M.P. Tulin, Cavitation, in: V.L. Streeter (Ed.), Handbook of Fluid Dynamics, McGraw-Hill, New York, 1961, pp. 12-1–12-46.
[3] M.I. Gurevich, The Theory of Jets in an Ideal Fluid, Pergamon Press, Oxford, 1966.
[4] M.P. Tulin, Supercavitating flows — small perturbation theory, J. Ship Res. 7 (3) (1963–1964) 16–37.
[5] A.G. Terent'ev, Non-linear theory of cavitational flow around obstacles, Fluid Dynam. 11 (1976) 142–145.
[6] G.P. Cherepanov, Flow of an ideal fluid with free surface in doubly connected and triply connected regions, J. Appl. Math. Mech. 27 (1964) 1117–1124.
[7] E.I. Zverovich, A mixed problem of elasticity theory for the plane with cuts that lie on the real axis, in: Proc. Symp. Continuum Mechanics

and Related Problems of Analysis, vol. 1, Mecniereba, Tbilisi, 1973, pp. 103–114 (in Russian).
[8] V.V. Silvestrov, Analytical solution of the problem of the cavitating flow around a system of plates by the method of Riemann surfaces, in: Dynamics of Continua with Free Boundaries, Cheboksary, 1996, pp. 206–221 (in Russian).
[9] Y.A. Antipov, V.V. Silvestrov, Factorization on a Riemann surface in scattering theory, Quart. J. Mech. Appl. Math. 55 (2002) 607–654.
[10] Y.A. Antipov, V.V. Silvestrov, Vector functional-difference equation in electromagnetic scattering, IMA J. Appl. Math. 69 (2004) 27–69.
[11] Y.A. Antipov, V.V. Silvestrov, Electromagnetic scattering from an anisotropic impedance half-plane at oblique incidence: The exact solution, Quart. J. Mech. Appl. Math. 59 (2006) 211–251.
[12] R. Courant, Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, Interscience Publishers, New York, 1950.
[13] E.I. Zverovich, Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces, Russian Math. Surveys 26 (1971) 117–192.
[14] D. Gilbarg, Unsteady flows with free boundaries, Z. Angew. Math. Phys. 3 (1952) 34–42.