WEIGHT FUNCTIONS OF A CRACK IN A TWO-DIMENSIONAL MICROPOLAR SOLID

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Summary

Two models for a crack in an unbounded two-dimensional micropolar solid are analysed. The first one concerns plane-strain deformation, while the second problem assumes the conditions of antiplane-strain loading. In both cases, one of three modes is uncoupled, while the other two are coupled. For a semi-infinite crack, the models reduce to a scalar and an order-2 vector Riemann–Hilbert problems with a Hermitian matrix coefficient. In the plane-strain case, the problem is solved exactly, and the weight functions are derived by quadratures. In the antiplane-strain case, the matrix coefficient cannot be factorized by the methods available in the literature. For a finite antiplane-strain crack, an approximate solution is obtained by the method of orthogonal polynomials. The weight functions are found in a series form through the solution of an infinite system of linear algebraic equations. In both cases, the plane-strain and antiplane-strain micropolar theory, it is shown that if a certain micropolar parameter (the same for both theories) tends to zero, then the solution and the weight functions tend to those of classical elasticity. Numerical results for the weight functions are reported.

1. Introduction

The theory of couple–stress elasticity (also known as asymmetric elasticity) was initiated by E. Cosserat and F. Cosserat (**1**). According to this theory, an infinitesimal surface element is subject not only to the classical stresses $\lim_{\delta s\to 0}\delta\mathbf{p}/\delta s$ but also to the couple–stresses $\lim_{\delta s\to 0}\delta\mathbf{m}/\delta s$. Here, $\delta\mathbf{p}$ and $\delta\mathbf{m}$ are the force and the moment, respectively, acting on the surface element δs with an external normal $\mathbf{n}$. Another difference between the classical elasticity and the Cosserat theory is that the deformation of the body is described by two vectors, the displacement $\mathbf{u}$ and the rotation $\boldsymbol{\phi}$. The modern theory of the Cosserat medium including its linearized two-dimensional (2D) and three-dimensional (3D) versions was developed in (**2** to **6**). In the constrained couple–stress theory, the rotation vector is not free, $\boldsymbol{\phi} = \frac{1}{2}\operatorname{curl}\mathbf{u}$, while in the unconstrained couple–stress theory, the rotation and displacement vectors are independent. To distinguish the two cases, Eringen (**6**) suggested to call the unconstrained theory as micropolar elasticity. A review of the governing equations and derivation of two conservation laws of micropolar elasticity were given in Lubarda and Markenscoff (**7**).

In the framework of the first theory, the effect of couple–stresses on the stress concentration at the tip of a finite crack under conditions of plane strain subject to uniform uniaxial tension was studied by Sternberg and Muki (**8**). By using the method of Fredholm integral equations, they found that when the couple–stress parameter $l \to 0$, the stress intensity factor does not approach that obtained for the elastic case ($l = 0$). Both cases, the constrained couple–stress theory and micropolar elasticity, were analysed by Atkinson and Leppington (**9**) for a semi-infinite and a finite crack under

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conditions of plane strain. In the case of a semi-infinite crack, the boundary conditions were chosen to be $\sigma_y = -\tau_0 e^{x/a}$, $\tau_{yx} = 0$, $m_{yz} = 0$, $x < 0$, $y = 0^{\pm}$, and the problem was reduced to a system of two Wiener–Hopf equations. The system was decoupled, and the solution was found by quadratures. The problem of a finite crack was solved by the method of matched asymptotic expansions. For both theories, they showed that when a certain parameter tends to zero, then the energy release rate from a crack tip tends to the classical elastic value, while the stress intensity factor does not. That parameter was taken as the couple–stress parameter l (also used in Sternberg and Muki (**8**)) for the constrained couple–stress theory, and the parameter

$$s_1 = \frac{\gamma(\kappa + \mu)}{\kappa(\kappa + 2\mu)} \tag{1.1}$$

for the micropolar case. Here, γ, κ and μ are material constants. Another interesting limiting case $s \to \infty$ ($s = l$ in the the constrained couple–stress theory and $s = s_1$ in micropolar elasticity) is hard to implement if the solutions derived in Sternberg and Muki (**8**) and Atkinson and Leppington (**9**) are used, and it was discussed neither in Sternberg and Muki (**8**), nor in Atkinson and Leppington (**9**).

Asymptotics of the stresses, couple–stresses, displacements and rotations at the tip of a crack was studied in (**10**, **11**). The plane-strain problem of micropolar elasticity for a finite crack was reduced to a system of singular integral equations and solved numerically in Li and Lee (**12**). The Dirichlet, Neumann and mixed problems of plane-strain micropolar elasticity were analysed by the boundary integral equation method in Schiavone (**13**). Existence and uniqueness of a weak solution in a Sobolev space for a smooth curved crack in a micropolar medium were studied in Shmoylova and Potapenko (**14**).

Our goal of the paper was to derive the weight functions of a micropolar crack under conditions of plane strain and antiplane shear. The concept of weight functions in fracture was introduced by Bueckner, who found the weight functions for a semi-infinite and a penny-shaped cracks in a homogeneous elastic medium (**15**, **16**). For the elastic case, exact and approximate expressions for the weight functions are available for a variety of models including the exact formulas for 3D problems of a dynamic semi-infinite crack (**17**) and a static interfacial crack (**18**, **19**). The weight functions for a semi-infinite crack in a 2D poroelastic medium were derived in Craster and Atkinson (**20**).

In section 2, we analyse the problem for a semi-infinite crack subject to loading $\tau_{yx} = p_1(x)$, $\sigma_y = p_2(x)$, $m_{yz} = p_3(x)$. In contrast to (**8**, **9**), in our derivation, we do not employ the formulation in terms of the generalized Airy stress function (**21**). Instead, we apply the Fourier transform directly to the governing equilibrium equations of micropolar elasticity. This ultimately gives a representation of the solution that is easy to analyse when $s_1 \to \infty$. In this section, we reduce the problem to a vector Riemann–Hilbert problem (RHP). The mode-II is decoupled, while the other two equations are coupled. The coefficient of the RHP is a Hermitian matrix $G = ||g_{mn}||$ $(m, n = 1, 2)$, $g_{12} = \overline{g_{21}}$, that can be represented as $G = b_0 Q_0 + b_1 Q_1$ (b_0 and b_1 are Hölder functions and Q_0 and Q_1 are polynomial matrices). Such a matrix G always admits a closed-form factorization (for example, see (**22**)). The approach we use for the solution of the RHP is similar to the method proposed in Atkinson and Leppington (**9**). The only one difference is that we diagonalize the matrix coefficient and represent it in the form $G(\zeta) = T_0(\zeta)\Lambda(\zeta)T_1(\zeta)$ ($T_j(\zeta)$ are rational 2×2 matrices and $\Lambda(\zeta)$ is a diagonal matrix), while the method (**9**) triangularizes the coupled problem.

In section 2, we also derive exact formulas for the stress intensity factors and the four weight functions, the functions $W_{\mathrm{I},j}$ and $W_{\mathrm{VI},j}$ ($j =$ I, VI) associated with the modes I (σ_y) and VI (m_{yz}),

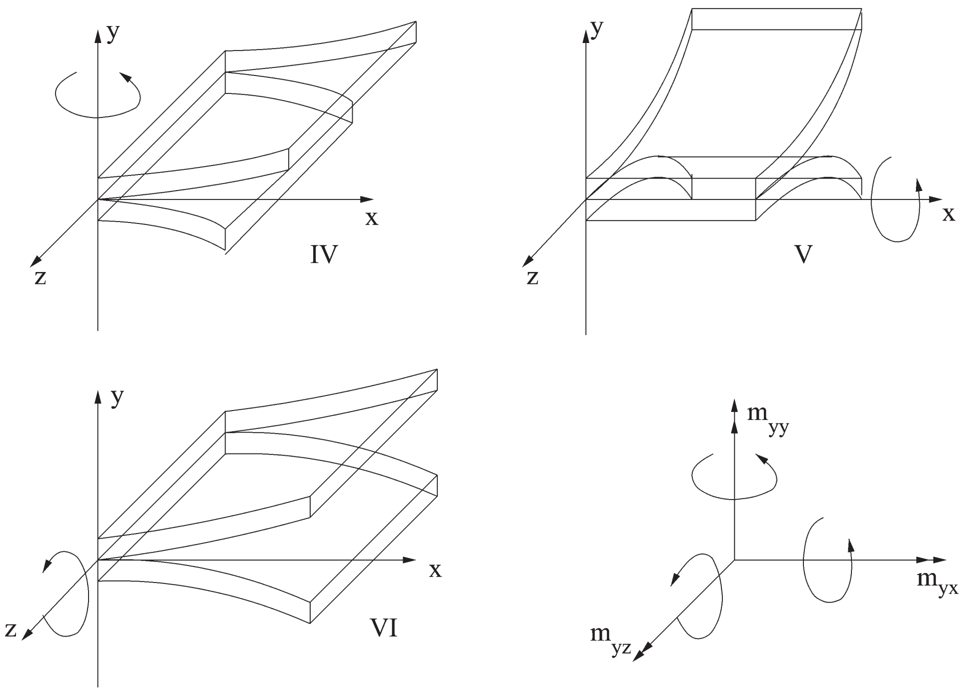

Fig. 1 The mode-IV, V and VI cracks

respectively (for the standard mode-I, II (τ_{yx}) and III (τ_{yz}) cracks, see the diagram (**23**, p. 59); for the other modes, see Fig. 1). We show that if the elastic parameter $\kappa \to 0^+$, and γ remains finite and non-zero (in this case, $s_1 \to +\infty$), then the equilibrium equations, the stress-strain relations and the solution of the RHP reduce to the classical equations and formulas of elasticity. As $\kappa \to 0^+$, the weight functions $W_{\mathrm{I,I}}$, W_{II} and $W_{\mathrm{VI,VI}}$ tend to $-\sqrt{2(\pi\xi)^{-1}}$, that is, the mode-I and mode-II weight function of elasticity. The other two functions $W_{\mathrm{I,VI}}$ and $W_{\mathrm{VI,I}}$ vanish. If $\gamma \to 0^+$ and $\kappa/\gamma \to 0^+$, then $W_{\mathrm{VI,VI}} \to 0$, and the only non-zero weight functions are $W_{\mathrm{I}} = W_{\mathrm{I,I}}$ and W_{II}. The asymptotic analysis is illustrated by numerical results. Section 3 contains the analysis of the antiplane-strain problem on a crack subject to the boundary conditions

$$\tau_{yz} = p_1(x), \quad m_{yx} = p_2(x), \quad m_{yy} = p_3(x), \quad 0 < x < a, \quad y = 0^{\pm}. \tag{1.2}$$

We have proved that the m_{yx}-mode (mode-V) is decoupled, while the τ_{yz}- and m_{yy}-modes (modes III and IV) are coupled. Therefore, the antiplane-strain problem for a crack in a plane subject to the boundary conditions (1.2) is equivalent to two boundary value problems for the upper half-plane. The first one,

$$\tau_{yz} = p_1(x), \quad m_{yx} = 0, \quad m_{yy} = p_3(x), \quad 0 < x < a, \quad y = 0,$$

$$u_z = 0, \quad m_{yx} = 0, \quad \phi_y = 0, \quad x < 0, \quad x > a \quad y = 0, \tag{1.3}$$

is associated with the modes III and IV, while the second problem,

$$m_{yx} = p_2(x), \quad 0 < x < a; \quad \phi_x = 0, \quad x < 0, \quad x > a, \quad y = 0,$$

$$\tau_{yz} = 0, \quad m_{yy} = 0, \quad -\infty < x < \infty, \quad y = 0, \tag{1.4}$$

recovers the mode-V solution. Notice that the numerical solution (**24**) relates to the antiplane-strain equations in the upper half-plane subject to the boundary conditions,

$$\tau_{yz} = p(x), \quad m_{yx} = 0, \quad m_{yy} = 0, \quad |x| < a, \quad y = 0,$$

$$u_z = 0, \quad \phi_x = 0, \quad \phi_y = 0, \quad |x| > a, \quad y = 0, \tag{1.5}$$

different from (1.3). Therefore, this boundary value problem does not model the antiplane-strain problem of micropolar elasticity for a crack, and that is why in the solution (**24**), all three modes, III, IV and V, are coupled. This also explains why in the case of loading (1.5), the mode-V stress intensity factor does not vanish (**24**), although it must identically be zero.

For a semi-infinite crack, the antiplane-strain problem reduces to a scalar RHP and an order-2 vector RHP. We solve the mode-V scalar RHP in closed form and find the mode-V weight function by two quadratures. Although, as in the plane-strain case, the matrix coefficient of the RHP for the other two modes is a Hermitian matrix, its structure is different, $G = b_0 Q_0 + b_1 Q_1 + b_2 Q_2$, where b_j are Hölder functions and Q_j are 2×2 polynomial matrices ($j = 0, 1, 2$). Such a matrix cannot be factorized in closed form by methods available in the literature. However, we have shown that the eigenvalues of the matrix are positive and it is positive definite. According to Shmul'yan (**25**) (see also (**26**)), the partial indices of a Hermitian positive definite matrix vanish, and the indices and the solution of the vector RHP are stable (**27**). We do not derive an approximate solution of the RHP associated with a semi-infinite crack and focus on the case of a finite crack. The problem is equivalent to a system of two singular integral equations with Cauchy and logarithmic kernels. The system is solved by the method of orthogonal polynomials. We derive the weight functions in a series form by using the generating function of the Chebyshev polynomials $U_n(x)$. The coefficients of the series solve a certain infinite system of linear algebraic equations. Again, as in the plane-strain case, if $\kappa \to 0^+$, then the mode-III and IV weight functions associated with the crack tips $x = 0$ and $x = a$ tend to the classical elastic weight functions $-\sqrt{2(a-\xi)(\pi a \xi)^{-1}}$ (for $x = 0$) and $-\sqrt{2\xi[\pi a(a-\xi)]^{-1}}$ (for $x = a$).

2. Plane-strain problem of micropolar elasticity

2.1 *Formulation*

The problem to be analysed is that of a semi-infinite crack $\{0 < x < \infty \; y = 0^\pm\}$ in an otherwise unbounded micropolar solid. The crack faces are subjected to plane-strain loading

$$\tau_{yx} = p_1(x), \quad \sigma_y = p_2(x), \quad m_{yz} = p_3(x), \quad 0 < x < \infty, \quad y = 0^\pm. \tag{2.1}$$

If $-\infty < x \leqslant 0$, the displacements u_x and u_y and the microrotation ϕ_z are continuous through the line $y = 0$,

$$u_x(x,0^+) - u_x(x,0^-) = 0, \quad u_y(x,0^+) - u_y(x,0^-) = 0,$$

$$\phi_z(x,0^+) - \phi_z(x,0^-) = 0, \quad -\infty < x \leqslant 0, \tag{2.2}$$

and they are discontinuous if $0 < x < \infty$. The stresses (force–stresses) σ_x, σ_y, τ_{xy} and τ_{yx} and the couple–stresses m_{xz} and m_{yz} satisfy the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\tau_{yx}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} = 0. \tag{2.3}$$

The force– and couple–stresses are expressed through the displacements u_x and u_y and the micro-rotation ϕ_z as follows:

$$\sigma_x = (\kappa + \lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y}, \quad \sigma_y = (\kappa + \lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_x}{\partial x},$$

$$\tau_{xy} = (\kappa + \mu)\frac{\partial u_y}{\partial x} + \mu \frac{\partial u_x}{\partial y} - \kappa \phi_z, \quad \tau_{yx} = (\kappa + \mu)\frac{\partial u_x}{\partial y} + \mu \frac{\partial u_y}{\partial x} + \kappa \phi_z,$$

$$m_{xz} = \gamma \frac{\partial \phi_z}{\partial x}, \quad m_{yz} = \gamma \frac{\partial \phi_z}{\partial y}. \tag{2.4}$$

Here, λ, μ, γ and κ are material constants such that $\gamma > 0$, $\kappa > 0$, $\kappa + 2\mu > 0$ and $\kappa + 2\mu + 3\lambda > 0$ (**28**, **29**). As $\gamma \to 0^+$ and $\kappa \to 0^+$, the parameters λ and μ tend to the Lamé constants, and the governing equations become those of classical elasticity.

Equations (2.3) and (2.4) can be rearranged to give

$$(\kappa + \lambda + 2\mu)\frac{\partial^2 u_x}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u_y}{\partial x \partial y} + (\kappa + \mu)\frac{\partial^2 u_x}{\partial y^2} + \kappa \frac{\partial \phi_z}{\partial y} = 0,$$

$$(\kappa + \lambda + 2\mu)\frac{\partial^2 u_y}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_x}{\partial x \partial y} + (\kappa + \mu)\frac{\partial^2 u_y}{\partial x^2} - \kappa \frac{\partial \phi_z}{\partial x} = 0,$$

$$\gamma \left(\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} \right) + \kappa \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) - 2\kappa \phi_z = 0. \tag{2.5}$$

2.2 *Vector RHP*

On applying the Fourier transform to (2.5)

$$\begin{pmatrix} \hat{u}_x \\ \hat{u}_y \\ \hat{\phi}_z \end{pmatrix} (\zeta; y) = \int_{-\infty}^{\infty} \begin{pmatrix} u_x \\ u_y \\ \phi_z \end{pmatrix} (x, y) e^{i\zeta x} dx, \tag{2.6}$$

we obtain the following system of ordinary differential equations

$$(\kappa+\mu)\hat{u}_x''-\zeta^2(\kappa+\lambda+2\mu)\hat{u}_x-i\zeta(\lambda+\mu)\hat{u}_y'+\kappa\hat{\phi}_z'=0,$$

$$-i\zeta(\lambda+\mu)\hat{u}_x'+(\kappa+\lambda+2\mu)\hat{u}_y''-\zeta^2(\kappa+\mu)\hat{u}_y+i\zeta\kappa\hat{\phi}_z=0,$$

$$-\kappa\hat{u}_x'-i\zeta\kappa\hat{u}_y+\gamma\hat{\phi}_z''-(\zeta^2\gamma+2\kappa)\hat{\phi}_z=0. \tag{2.7}$$

Suppose first that $y>0$. The general solution of the system is derived in the form

$$\hat{u}_x=\left[iC_1\operatorname{sgn}\zeta-\left(\frac{3\kappa_2-\lambda}{\zeta(2\kappa_2-\kappa_1)}-y\operatorname{sgn}\zeta\right)iC_2\right]e^{-|\zeta|y}+\frac{\gamma r}{\kappa_1}C_3e^{-ry},$$
$$\hat{u}_y=(C_1+C_2y)e^{-|\zeta|y}-\frac{i\zeta\gamma}{\kappa_1}C_3e^{-ry},\quad \hat{\phi}_z=-\frac{2i\kappa_2\operatorname{sgn}\zeta}{2\kappa_2-\kappa_1}C_2e^{-|\zeta|y}+C_3e^{-ry}, \tag{2.8}$$

for $0<y<\infty$. Here, C_1, C_2 and C_3 are free constants to be determined,

$$r=\sqrt{\zeta^2+\rho^2},\quad \rho=\sqrt{\frac{\kappa(\kappa+2\mu)}{\gamma(\kappa+\mu)}}>0,\quad \kappa_1=\kappa+2\mu>0,\quad \kappa_2=\kappa+\lambda+2\mu>0. \tag{2.9}$$

The next stage is to extend the boundary conditions (2.1) for $y=0^+$ to the whole real axis and apply the Fourier transform. With the aid of (2.4), we find

$$\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\\ \hat{p}_3\end{pmatrix}(\zeta)=\begin{pmatrix}\kappa+\mu & 0 & 0\\ 0 & \kappa_2 & 0\\ 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}\hat{u}_x'\\ \hat{u}_y'\\ \hat{\phi}_z'\end{pmatrix}(\zeta;0^+)+\begin{pmatrix}0 & -i\zeta\mu & \kappa\\ -i\zeta\lambda & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\hat{u}_x\\ \hat{u}_y\\ \hat{\phi}_z\end{pmatrix}(\zeta;0^+), \tag{2.10}$$

where

$$\hat{p}_j(\zeta)=\Phi_j^-(\zeta)+\hat{p}_j^+(\zeta),\quad \hat{p}_j^+(\zeta)=\int_0^\infty p_j(x)e^{i\zeta x}dx,\quad j=1,2,3,$$

$$\Phi_1^-(\zeta)=\int_{-\infty}^0\tau_{yx}(x,0)e^{i\zeta x}dx,\quad \Phi_2^-(\zeta)=\int_{-\infty}^0\sigma_y(x,0)e^{i\zeta x}dx,$$

$$\Phi_3^-(\zeta)=\int_{-\infty}^0 m_{yz}(x,0)e^{i\zeta x}dx. \tag{2.11}$$

On substituting the expressions (2.8) into the boundary conditions (2.10) and solving the system for the constants C_1, C_2 and C_3, we obtain

$$C_1=\frac{i}{\kappa_1(2\kappa_2-\kappa_1)\zeta\Delta(\zeta)}\big[(-\kappa_1^2+2\gamma\kappa_2\zeta^2)r\hat{p}_1(\zeta)-2i\kappa_2\operatorname{sgn}\zeta(|\zeta|^3\gamma-\kappa_1 r)\hat{p}_2(\zeta)$$
$$+\kappa_1|\zeta|(\kappa_1|\zeta|-2\kappa_2 r)\hat{p}_3(\zeta)\big],$$
$$C_2=\frac{i}{\zeta\Delta(\zeta)}\big[-|\zeta|r\hat{p}_1(\zeta)+i\zeta r\hat{p}_2(\zeta)+(|\zeta|-r)\zeta^2\hat{p}_3(\zeta)\big],$$
$$C_3=-\frac{i}{\gamma(2\kappa_2-\kappa_1)\Delta(\zeta)}\big[2i\gamma\kappa_2|\zeta|\hat{p}_1(\zeta)+2\gamma\kappa_2\zeta\hat{p}_2(\zeta)-i\kappa_1(2\kappa_2-\kappa_1)\hat{p}_3(\zeta)\big],$$

where

$$\Delta(\zeta) = \kappa_1 r - \frac{2\gamma\kappa_2\zeta^2}{2\kappa_2 - \kappa_1}(|\zeta| - r). \tag{2.12}$$

This gives, with reference to (2.8), the Fourier images of the two displacements and the microrotation in terms of the functions $\hat{p}_j(\zeta)$, $j = 1, 2, 3$. On putting $y = 0$, we get

$$\begin{aligned}
\hat{u}_x(\zeta; 0^+) &= \frac{i}{\kappa_1(2\kappa_2 - \kappa_1)\zeta\Delta(\zeta)}\big\{2i \operatorname{sgn}\zeta\,\kappa_1\kappa_2 r\,\hat{p}_1(\zeta) + [\kappa_1^2 r - 2\gamma\kappa_2\zeta^2(r - |\zeta|)]\hat{p}_2(\zeta) \\
&\quad + 2i\kappa_1\kappa_2\zeta(r - |\zeta|)\hat{p}_3(\zeta)\big\}, \\
\hat{u}_y(\zeta; 0^+) &= \frac{i}{\kappa_1(2\kappa_2 - \kappa_1)\zeta\Delta(\zeta)}\big\{[2\gamma\kappa_2\zeta^2(r - |\zeta|) - \kappa_1^2 r]\hat{p}_1(\zeta) + 2i \operatorname{sgn}\zeta\,\kappa_1\kappa_2 r\,\hat{p}_2(\zeta) \\
&\quad - 2\kappa_1\kappa_2|\zeta|(r - |\zeta|)\hat{p}_3(\zeta)\big\}, \\
\hat{\phi}_z(\zeta; 0^+) &= \frac{i}{\gamma(2\kappa_2 - \kappa_1)\Delta(\zeta)}\big\{[2i\gamma\kappa_2(r - |\zeta|)\hat{p}_1(\zeta) + 2\gamma\kappa_2 \operatorname{sgn}\zeta(r - |\zeta|)\hat{p}_2(\zeta) \\
&\quad + i[\kappa_1(2\kappa_2 - \kappa_1) + 2\gamma\kappa_2|\zeta|(r - |\zeta|)]\hat{p}_3(\zeta)\big\}.
\end{aligned} \tag{2.13}$$

The expressions for the functions $\hat{u}_x(\zeta; 0^-)$, $\hat{u}_y(\zeta; 0^-)$ and $\phi_z(\zeta; 0^-)$ can be obtained from (2.13) by replacing $|\zeta|$ and r by $-|\zeta|$ and $-r$, respectively.

To satisfy the conditions of continuity (2.2) for the displacements u_x and u_y and the rotation ϕ_z for negative x, we subtract the expressions $\hat{u}_x(\zeta; 0^-)$, $\hat{u}_y(\zeta; 0^-)$ and $\phi_z(\zeta; 0^-)$ from the ones in (2.13). The new relations may be written in the form of the following vector RHP on the real axis L:

$$\mathbf{\Phi}^+(\zeta) = d(\zeta)G(\zeta)[\mathbf{\Phi}^-(\zeta) + \hat{\mathbf{p}}^+(\zeta)], \quad \zeta \in L, \tag{2.14}$$

where

$$G(\zeta) = \begin{pmatrix} r/|\zeta| & 0 & 0 \\ 0 & r/|\zeta| & -i\zeta\,(1 - r/|\zeta|) \\ 0 & i\zeta\,(1 - r/|\zeta|) & \kappa_0^2 - \zeta^2\,(1 - r/|\zeta|) \end{pmatrix},$$

$$\mathbf{\Phi}^+(\zeta) = \int_0^\infty \begin{pmatrix} u_x(x, 0^+) - u_x(x, 0^-) \\ u_y(x, 0^+) - u_y(x, 0^-) \\ \phi_z(x, 0^+) - \phi_z(x, 0^-) \end{pmatrix} e^{i\zeta x} dx,$$

$$\kappa_0 = \sqrt{\frac{\kappa_1(2\kappa_2 - \kappa_1)}{2\gamma\kappa_2}}, \quad d(\zeta) = -\frac{4\kappa_2}{(2\kappa_2 - \kappa_1)\Delta(\zeta)}, \tag{2.15}$$

the parameter κ_0 is real and positive ($\kappa_1 > 0$, $\kappa_2 > 0$, $\gamma > 0$, $2\kappa_2 - \kappa_1 > 0$), and the components of the vectors $\mathbf{\Phi}^-(\zeta)$ and $\hat{\mathbf{p}}^+(\zeta)$ are given by (2.11).

2.3 *Mode-II crack*

The structure of the matrix $G(\zeta)$ indicates that the τ_{yx}-mode (mode-II) is uncoupled, while the σ_y-mode (mode-I) and the m_{yz}-mode (mode-VI) are coupled. The scalar RHP for the determination of the Fourier transform $\Phi_1^+(\zeta)$ of the function $[u_x](x) = u_x(x, 0^+) - u_x(x, 0^-)$,

$$\Phi_1^+(\zeta) = \frac{d(\zeta)r}{|\zeta|}[\Phi_1^-(\zeta) + \hat{p}_1^+(\zeta)], \quad \zeta \in L, \tag{2.16}$$

can be solved in a standard way. Split the coefficient of the problem as

$$\frac{d(\zeta)r}{|\zeta|} = -\frac{\gamma_0}{|\zeta|\delta_0(\zeta)}, \tag{2.17}$$

where

$$\gamma_0 = \frac{4}{\gamma(2\kappa_0^2 + \rho^2)}, \quad \delta_0(\zeta) = \left(1 + \frac{\rho^2}{2\kappa_0^2}\right)^{-1}\left[1 + \frac{\zeta^2}{\kappa_0^2}\left(1 - \frac{|\zeta|}{r}\right)\right]. \tag{2.18}$$

The function $\delta_0(\zeta)$ possesses the following properties: $\delta_0(\zeta) = 1 + O(\zeta^{-2})$ as $\zeta \to \pm\infty$ and $\delta_0(\zeta) = \left(1 + \frac{1}{2}\rho^2/\kappa_0^2\right)^{-1} + O(\zeta^2)$ as $\zeta \to 0$. Since it is real and even on the real axis, we factorize it as

$$\delta_0(\zeta) = \frac{\chi_0^+(\zeta)}{\chi_0^-(\zeta)}, \quad \zeta \in L, \tag{2.19}$$

where $\chi_0^\pm(\zeta) = \chi_0(\zeta \pm i0)$, $\zeta \in L$, and

$$\chi_0(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln\delta_0(\zeta_0)d\zeta_0}{\zeta_0 - \zeta}\right\} = \exp\left\{\frac{\zeta}{\pi i}\int_0^{\infty}\frac{\ln\delta_0(\zeta_0)d\zeta_0}{\zeta_0^2 - \zeta^2}\right\}, \quad \zeta \in \mathbb{C}^\pm, \tag{2.20}$$

$\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and lower half-planes, respectively, and $\ln\delta_0(\zeta)$ is the branch fixed by the condition $\ln 1 = 0$.

Factorize next the function $|\zeta|$ as $|\zeta| = s^+(\zeta)s^-(\zeta)$, $\zeta \in L$, and introduce the Cauchy integral

$$\Psi_0^\pm(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\chi_0^-(\zeta_0)\hat{p}_1^+(\zeta_0)d\zeta_0}{s^-(\zeta_0)(\zeta_0 - \zeta)}, \quad \zeta \in \mathbb{C}^\pm \setminus L, \tag{2.21}$$

whose boundary values $\Psi_0^\pm(\zeta) = \Psi_0(\zeta \pm i0)$, $\zeta \in L$, due to the Sohkotski–Plemelj formulas, have the property

$$\Psi_0^+(\zeta) - \Psi_0^-(\zeta) = \frac{\chi_0^-(\zeta)\hat{p}_1^+(\zeta)}{s^-(\zeta)}, \quad \zeta \in L. \tag{2.22}$$

Here, $s^\pm(\zeta) = \sqrt{|\zeta|}\exp\{i\theta_\pm/2\}$, $0 < \theta_+ < \pi$, and $-\pi < \theta_- < 0$. Referring to the continuity principle and the Liouville theorem, we deduce the formulas

$$\Phi_1^+(\zeta) = -\frac{\gamma_0\Psi_0^+(\zeta)}{s^+(\zeta)\chi_0^+(\zeta)}, \quad \zeta \in \mathbb{C}^+ \quad \text{and} \quad \Phi_1^-(\zeta) = \frac{s^-(\zeta)\Psi_0^-(\zeta)}{\chi_0^-(\zeta)}, \quad \zeta \in \mathbb{C}^-. \tag{2.23}$$

2.4 *Modes I and VI*

In order to determine the other two components of the vectors $\mathbf{\Phi}^{\pm}(\zeta)$, $\Phi_2^{\pm}(\zeta)$ and $\Phi_3^{\pm}(\zeta)$, we need to solve the following vector RHP:

$$\boldsymbol{\phi}^{+}(\zeta) = d(\zeta)A(\zeta)[\boldsymbol{\phi}^{-}(\zeta) + \tilde{\mathbf{p}}^{+}(\zeta)], \quad \zeta \in L, \tag{2.24}$$

where

$$A(\zeta) = \begin{pmatrix} r/|\zeta| & -i\zeta\,(1 - r/|\zeta|) \\ i\zeta\,(1 - r/|\zeta|) & \kappa_0^2 - \zeta^2\,(1 - r/|\zeta|) \end{pmatrix}, \tag{2.25}$$

and

$$\boldsymbol{\phi}^{\pm}(\zeta) = \begin{pmatrix} \Phi_2^{\pm}(\zeta) \\ \Phi_3^{\pm}(\zeta) \end{pmatrix}, \quad \tilde{\mathbf{p}}^{+}(\zeta) = \begin{pmatrix} \hat{p}_2^{+}(\zeta) \\ \hat{p}_3^{+}(\zeta) \end{pmatrix}. \tag{2.26}$$

To factorize the matrix $A(\zeta)$, it is advisable to split it first as $A(\zeta) = R(\zeta)A^{\circ}(\zeta)$, where

$$R(\zeta) = \begin{pmatrix} 0 & -i\zeta \\ i\zeta & \kappa_0^2 - \zeta^2 \end{pmatrix}, \quad A^{\circ}(\zeta) = I_2 - \frac{r}{|\zeta|\zeta^2}\begin{pmatrix} \kappa_0^2 & i\kappa_0^2\zeta \\ -i\zeta & \zeta^2 \end{pmatrix} \tag{2.27}$$

and $I_2 = \mathrm{diag}\{1, 1\}$. The next stage is to find the eigenvalues of the matrix $A^{\circ}(\zeta)$,

$$\lambda_1 = 1, \quad \lambda_2(\zeta) = 1 - \frac{(\zeta^2 + \kappa_0^2)r}{|\zeta|\zeta^2}, \tag{2.28}$$

and diagonalize the matrix $A^{\circ}(\zeta)$. Finally, we obtain the following splitting of the matrix coefficient $A(\zeta)$ of the RHP (2.24):

$$A(\zeta) = \frac{1}{\zeta^2 + \kappa_0^2}\begin{pmatrix} -i\zeta & -\zeta^2 \\ \kappa_0^2 & i\zeta^3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(\zeta) \end{pmatrix}\begin{pmatrix} i\zeta & \kappa_0^2 \\ 1 & i\zeta \end{pmatrix}. \tag{2.29}$$

This representation, after rearrangement, transforms the boundary condition (2.24) as

$$\begin{pmatrix} \left(\kappa_0^2 + \frac{1}{2}\rho^2\right) i\zeta & \kappa_0^2 + \frac{1}{2}\rho^2 \\ \kappa_0^2 & i\zeta \end{pmatrix}\boldsymbol{\phi}^{+}(\zeta) = -\frac{2}{\gamma}\begin{pmatrix} \frac{1}{r\delta_0(\zeta)} & 0 \\ 0 & \frac{\delta_1(\zeta)}{|\zeta|\delta_0(\zeta)} \end{pmatrix}\begin{pmatrix} i\zeta & \kappa_0^2 \\ 1 & i\zeta \end{pmatrix}[\boldsymbol{\phi}^{-}(\zeta) + \tilde{\mathbf{p}}^{+}(\zeta)], \tag{2.30}$$

for $\zeta \in L$, where $\delta_0(\zeta)$ was given in (2.18) and

$$\delta_1(\zeta) = \frac{(\zeta^2 + \kappa_0^2)r - \zeta^2|\zeta|}{\left(\kappa_0^2 + \rho^2/2\right) r}. \tag{2.31}$$

Since the function $\delta_1(\zeta)$ is even, $\delta_1(\zeta) = 1 + O(\zeta^{-2})$ as $\zeta \to \pm\infty$ and $\delta_1(\zeta) \sim \left(1 + \frac{1}{2}\rho^2/\kappa_0^2\right)^{-1}$ as $\zeta \to 0$, it can be factorized as

$$\delta_1(\zeta) = \frac{\chi_1^+(\zeta)}{\chi_1^-(\zeta)}, \quad \zeta \in L, \tag{2.32}$$

where $\chi_1^\pm(\zeta) = \chi_1(\zeta \pm i0)$, $\zeta \in L$, and

$$\chi_1(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln \delta_1(\zeta_0)d\zeta_0}{\zeta_0 - \zeta}\right\}, \quad \zeta \in \mathbb{C}^\pm. \tag{2.33}$$

The function $r = r(\zeta)$ is factorized explicitly in the form $r(\zeta) = r^+(\zeta)r^-(\zeta)$, $\zeta \in L$, where $r^\pm(\zeta) = (\zeta \pm i\rho)^{1/2}$ are analytic functions in the half-planes $\mathbb{C}^\pm$, and $\arg(\zeta + i\rho) \in [0, \pi]$, $\arg(\zeta - i\rho) \in [-\pi, 0]$. We introduce next the vector function

$$\mathbf{\Psi}(\zeta) = -\frac{1}{\gamma\pi i}\int_{-\infty}^{\infty} \begin{pmatrix} \frac{i\zeta_0}{r^-(\zeta_0)} & \frac{\kappa_0^2}{r^-(\zeta_0)} \\ \frac{1}{s^-(\zeta_0)\chi_1^-(\zeta_0)} & \frac{i\zeta_0}{s^-(\zeta_0)\chi_1^-(\zeta_0)} \end{pmatrix} \frac{\chi_0^-(\zeta_0)\tilde{\mathbf{p}}^+(\zeta_0)d\zeta_0}{\zeta_0 - \zeta} \tag{2.34}$$

and rewrite the boundary condition (2.30) in the form

$$\chi_0^+(\zeta)\begin{pmatrix} \left(\kappa_0^2 + \frac{\rho^2}{2}\right)i\zeta r^+(\zeta) & \left(\kappa_0^2 + \frac{\rho^2}{2}\right)r^+(\zeta) \\ \frac{\kappa_0^2 s^+(\zeta)}{\chi_1^+(\zeta)} & \frac{i\zeta s^+(\zeta)}{\chi_1^+(\zeta)} \end{pmatrix}\boldsymbol{\phi}^+(\zeta) - \mathbf{\Psi}^+(\zeta)$$
$$= -\frac{2}{\gamma}\chi_0^-(\zeta)\begin{pmatrix} \frac{i\zeta}{r^-(\zeta)} & \frac{\kappa_0^2}{r^-(\zeta)} \\ \frac{1}{s^-(\zeta)\chi_1^-(\zeta)} & \frac{i\zeta}{s^-(\zeta)\chi_1^-(\zeta)} \end{pmatrix}\boldsymbol{\phi}^-(\zeta) - \mathbf{\Psi}^-(\zeta), \quad \zeta \in L. \tag{2.35}$$

On applying the Liouville theorem, we obtain that the left- and right-hand sides of (2.35) equal a constant vector $\mathbf{D} = (D_1, D_2)^\top$. Thus, the vector functions $\boldsymbol{\phi}^\pm(\zeta)$ are defined by

$$\boldsymbol{\phi}^+(\zeta) = -\frac{2\chi_1^+(\zeta)}{(2\kappa_0^2 + \rho^2)\chi_0^+(\zeta)r^+(\zeta)s^+(\zeta)(\zeta^2 + \kappa_0^2)}$$
$$\times \begin{pmatrix} \frac{i\zeta s^+(\zeta)}{\chi_1^+(\zeta)} & -\left(\kappa_0^2 + \frac{\rho^2}{2}\right)r^+(\zeta) \\ -\frac{\kappa_0^2 s^+(\zeta)}{\chi_1^+(\zeta)} & \left(\kappa_0^2 + \frac{\rho^2}{2}\right)i\zeta r^+(\zeta) \end{pmatrix}[\mathbf{D} + \mathbf{\Psi}^+(\zeta)], \quad \zeta \in \mathbb{C}^+,$$
$$\boldsymbol{\phi}^-(\zeta) = \frac{\gamma\chi_1^-(\zeta)r^-(\zeta)s^-(\zeta)}{2\chi_0^-(\zeta)(\zeta^2 + \kappa_0^2)}\begin{pmatrix} \frac{i\zeta}{\chi_1^-(\zeta)s^-(\zeta)} & -\frac{\kappa_0^2}{r^-(\zeta)} \\ -\frac{1}{s^-(\zeta)\chi_1^-(\zeta)} & \frac{i\zeta}{r^-(\zeta)} \end{pmatrix}[\mathbf{D} + \mathbf{\Psi}^-(\zeta)], \quad \zeta \in \mathbb{C}^-. \tag{2.36}$$

Due to the polynomial $\zeta^2 + \kappa_0^2$ in the denominator, $\boldsymbol{\phi}^+(\zeta)$ and $\boldsymbol{\phi}^-(\zeta)$ in (2.36) have inadmissible poles at the points $\zeta = i\kappa_0 \in \mathbb{C}^+$ and $\zeta = -i\kappa_0 \in \mathbb{C}^-$, respectively. These points become removable singularities if and only if the constants D_1 and D_2 are chosen to be

$$D_1 = \frac{1}{a^+ - a^-}\left\{a^-\Psi_1^+(i\kappa_0) - a^+\Psi_1^-(-i\kappa_0) + a^+a^-[\Psi_2^+(i\kappa_0) - \Psi_2^-(-i\kappa_0)]\right\},$$
$$D_2 = -\frac{1}{a^+ - a^-}[\Psi_1^+(i\kappa_0) - \Psi_1^-(-i\kappa_0) + a^+\Psi_2^+(i\kappa_0) - a^-\Psi_2^-(-i\kappa_0)], \tag{2.37}$$

where

$$a^+ = \frac{(2\kappa_0^2 + \rho^2)\chi_1^+(i\kappa_0)r^+(i\kappa_0)}{2\kappa_0 s^+(i\kappa_0)}, \quad a^- = -\frac{\kappa_0 s^-(-i\kappa_0)\chi_1^-(-i\kappa_0)}{r^-(-i\kappa_0)}. \tag{2.38}$$

2.5 *Weight functions*

Analysis of the solution found indicates that the stresses σ_y and τ_{yx} and the couple–stress m_{yz} have the square root singularities at the tip of the crack

$$\sigma_y \sim \frac{K_{\rm I}}{\sqrt{2\pi}}(-x)^{-1/2}, \quad \tau_{yx} \sim \frac{K_{\rm II}}{\sqrt{2\pi}}(-x)^{-1/2}, \quad m_{yz} \sim \frac{K_{\rm VI}}{\sqrt{2\pi}}(-x)^{-1/2}, \quad y=0, \quad x \to 0^-,$$

where $K_{\rm I}$, $K_{\rm II}$ and $K_{\rm VI}$ are the stress intensity factors. By using the abelian theorem for the Laplace transform, we obtain

$$\begin{aligned}
\Phi_1^-(\zeta) &= \int_{-\infty}^0 \tau_{yx}(x,0)e^{i\zeta x}dx \sim \frac{K_{\rm II}}{\sqrt{2}}e^{-i\pi/4}\zeta^{-1/2},\\
\Phi_2^-(\zeta) &= \int_{-\infty}^0 \sigma_y(x,0)e^{i\zeta x}dx \sim \frac{K_{\rm I}}{\sqrt{2}}e^{-i\pi/4}\zeta^{-1/2},\\
\Phi_3^-(\zeta) &= \int_{-\infty}^0 m_{yz}(x,0)e^{i\zeta x}dx \sim \frac{K_{\rm VI}}{\sqrt{2}}e^{-i\pi/4}\zeta^{-1/2}, \quad \zeta \to \infty, \quad \zeta \in \mathbb{C}^-.
\end{aligned} \tag{2.39}$$

On the other hand, from (2.23) and (2.36), analysing the asymptotics as $\zeta \to \infty$, $\zeta \in \mathbb{C}^-$, of the solution of the RHP, we derive

$$\Phi_1^-(\zeta) \sim -\Psi_0^\circ \zeta^{-1/2}, \quad \Phi_2^-(\zeta) \sim \frac{i\gamma D_1}{2}\zeta^{-1/2}, \quad \Phi_3^-(\zeta) \sim \frac{i\gamma D_2}{2}\zeta^{-1/2}, \quad \zeta \to \infty, \quad \zeta \in \mathbb{C}^-, \tag{2.40}$$

where

$$\Psi_0^\circ = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\chi_0^-(\zeta)\hat{p}_1^+(\zeta)}{s^-(\zeta)}\,d\zeta. \tag{2.41}$$

By comparing formulas (2.39) and (2.40), we determine the stress intensity factors

$$K_{\rm I} = \frac{\gamma}{\sqrt{2}}e^{3i\pi/4}D_1, \quad K_{\rm II} = -\sqrt{2}e^{i\pi/4}\Psi_0^\circ, \quad K_{\rm VI} = \frac{\gamma}{\sqrt{2}}e^{3i\pi/4}D_2. \tag{2.42}$$

These formulas can be rewritten in terms of the weight functions. Remembering (2.11), we substitute the expression (2.41) into (2.42). This gives

$$K_{\rm II} = \int_0^\infty W_{\rm II}(\xi)p_1(\xi)d\xi, \tag{2.43}$$

where $W_{\rm II}(\xi)$ is the mode-II weight function,

$$W_{\rm II}(\xi) = -\frac{e^{i\pi/4}}{\sqrt{2}\pi i}\int_{-\infty}^{\infty}\frac{\chi_0^-(\zeta)e^{i\zeta\xi}}{s^-(\zeta)}\,d\zeta. \tag{2.44}$$

In order to determine the weight functions associated with the modes I and VI, we transform the expressions (2.42) for the stress intensity factors $K_{\rm I}$ and $K_{\rm VI}$ and write them in the form

$$\begin{pmatrix} K_{\rm I} \\ K_{\rm VI} \end{pmatrix} = \frac{\gamma e^{3\pi i/4}}{\sqrt{2}(a^+ - a^-)} \begin{pmatrix} a^-\Psi_1^+(i\kappa_0) - a^+\Psi_1^-(-i\kappa_0) + a^+a^-[\Psi_2^+(i\kappa_0) - \Psi_2^-(-i\kappa_0)] \\ -\Psi_1^+(i\kappa_0) + \Psi_1^-(-i\kappa_0) - a^+\Psi_2^+(i\kappa_0) + a^-\Psi_2^-(-i\kappa_0) \end{pmatrix}, \tag{2.45}$$

where

$$\begin{pmatrix} \Psi_1(\zeta) \\ \Psi_2(\zeta) \end{pmatrix} = -\frac{1}{\pi i \gamma} \int_0^\infty \begin{pmatrix} \upsilon_{\rm I,I}(\zeta,\xi) & \upsilon_{\rm I,VI}(\zeta,\xi) \\ \upsilon_{\rm VI,I}(\zeta,\xi) & \upsilon_{\rm VI,VI}(\zeta,\xi) \end{pmatrix} \begin{pmatrix} p_2(\xi) \\ p_3(\xi) \end{pmatrix} d\xi \tag{2.46}$$

and

$$\upsilon_{\rm I,I}(\zeta,\xi) = i\int_{-\infty}^\infty \frac{\zeta_0\chi_0^-(\zeta_0)e^{i\zeta_0\xi}d\zeta_0}{r^-(\zeta_0)(\zeta_0-\zeta)}, \quad \upsilon_{\rm I,VI}(\zeta,\xi) = \kappa_0^2\int_{-\infty}^\infty \frac{\chi_0^-(\zeta_0)e^{i\zeta_0\xi}d\zeta_0}{r^-(\zeta_0)(\zeta_0-\zeta)},$$

$$\upsilon_{\rm VI,I}(\zeta,\xi) = \int_{-\infty}^\infty \frac{\chi_0^-(\zeta_0)e^{i\zeta_0\xi}d\zeta_0}{s^-(\zeta_0)\chi_1^-(\zeta_0)(\zeta_0-\zeta)}, \quad \upsilon_{\rm VI,VI}(\zeta,\xi) = i\int_{-\infty}^\infty \frac{\zeta_0\chi_0^-(\zeta_0)e^{i\zeta_0\xi}d\zeta_0}{s^-(\zeta_0)\chi_1^-(\zeta_0)(\zeta_0-\zeta)}. \tag{2.47}$$

Denote by $\{W_{\rm I,I}(\xi), W_{\rm I,VI}(\xi)\}$ and $\{W_{\rm VI,I}(\xi), W_{\rm VI,VI}(\xi)\}$ the weight functions for the modes I and VI, respectively. Then,

$$K_{\rm I} = \int_0^\infty W_{\rm I,I}(\xi)p_2(\xi)d\xi + \int_0^\infty W_{\rm I,VI}(\xi)p_3(\xi)d\xi,$$
$$K_{\rm VI} = \int_0^\infty W_{\rm VI,I}(\xi)p_2(\xi)d\xi + \int_0^\infty W_{\rm VI,VI}(\xi)p_3(\xi)d\xi. \tag{2.48}$$

On substituting (2.46) into (2.45) and comparing the result with (2.48), we obtain

$$W_{{\rm I},j}(\xi) = \frac{e^{i\pi/4}}{\sqrt{2}\pi(a^- - a^+)}\{a^-\upsilon_{{\rm I},j}(i\kappa_0,\xi) - a^+\upsilon_{{\rm I},j}(-i\kappa_0,\xi) + a^+a^-[\upsilon_{{\rm VI},j}(i\kappa_0,\xi) - \upsilon_{{\rm VI},j}(-i\kappa_0,\xi)]\},$$

$$W_{{\rm VI},j}(\xi) = \frac{e^{i\pi/4}}{\sqrt{2}\pi(a^- - a^+)}[\upsilon_{{\rm I},j}(-i\kappa_0,\xi) - \upsilon_{{\rm I},j}(i\kappa_0,\xi) + a^-\upsilon_{{\rm VI},j}(-i\kappa_0,\xi) - a^+\upsilon_{{\rm VI},j}(i\kappa_0,\xi)], \tag{2.49}$$

where $j = {\rm I, VI}$.

2.6 *Evaluation of the weight functions*

The problem of finding the numerical values of the weight functions involves the evaluation of the integrals (2.44), (2.47) and the limiting value of the singular integrals (2.20) and (2.33) defined by the Sokhotski–Plemelj formulas as

$$\chi_j^-(\zeta) = \exp\left\{-\frac{1}{2}\ln\delta_j(\zeta) + \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\ln\delta_j(\zeta_0)d\zeta_0}{\zeta_0-\zeta}\right\}, \quad \zeta\in L, \quad j=0,1. \tag{2.50}$$

Since the functions $\delta_j(\zeta)$ are even, the principal value of the integrals can be evaluated by converting them into the integrals over the interval $(-1, 1)$ as

$$\chi_j^-(\zeta) = \exp\left\{-\frac{1}{2}\ln\delta_j(\zeta) + \frac{2\zeta}{\pi i}\int_{-1}^{1}\ln\delta_j\left(\frac{1+\eta}{1-\eta}\right)\frac{d\eta}{(1+\eta)^2-(1-\eta)^2\zeta^2}\right\}, \quad \zeta \in L.$$

In order to compute the weight functions $W_{\mathrm{II}}(\xi)$ and $W_{m,j}(\xi)$ ($m, j =$ I, VI), we represent the integrals (2.44) and (2.49) in the form

$$W(\xi) = a_*\left\{\int_0^\infty \frac{\cos\zeta\xi}{\sqrt{\zeta}}[\Lambda(\zeta)+i\Lambda(-\zeta)]d\zeta + i\int_0^\infty \frac{\sin\zeta\xi}{\sqrt{\zeta}}[\Lambda(\zeta)-i\Lambda(-\zeta)]d\zeta\right\}. \tag{2.51}$$

Here, for the mode-II weight function, $W(\xi) = W_{\mathrm{II}}(\xi)$,

$$a_* = \frac{i-1}{2\pi}, \quad \Lambda(\zeta) = \chi_0^-(\zeta), \tag{2.52}$$

and for the other modes, $W(\xi) = W_{m,j}(\xi)$,

$$a_* = \frac{1+i}{2\pi(a^- - a^+)}, \quad \Lambda(\zeta) = \Lambda_{mj}(\zeta), \quad m, j = I, VI, \tag{2.53}$$

where

$$\Lambda_{\mathrm{I,I}}(\zeta) = \frac{i\chi_0^-(\zeta)}{\zeta^2+\kappa_0^2}\left\{\frac{\zeta s^-(\zeta)}{r^-(\zeta)}\left[(\zeta+i\kappa_0)a^- - (\zeta-i\kappa_0)a^+\right] + \frac{2\kappa_0 a^+ a^-}{\chi_1^-(\zeta)}\right\},$$

$$\Lambda_{\mathrm{I,VI}}(\zeta) = \frac{\chi_0^-(\zeta)}{\zeta^2+\kappa_0^2}\left\{\frac{\kappa_0^2 s^-(\zeta)}{r^-(\zeta)}\left[(\zeta+i\kappa_0)a^- - (\zeta-i\kappa_0)a^+\right] - \frac{2\kappa_0\zeta a^+ a^-}{\chi_1^-(\zeta)}\right\},$$

$$\Lambda_{\mathrm{VI,I}}(\zeta) = \frac{\chi_0^-(\zeta)}{\zeta^2+\kappa_0^2}\left\{\frac{2\kappa_0\zeta s^-(\zeta)}{r^-(\zeta)} + \frac{(\zeta-i\kappa_0)a^- - (\zeta+i\kappa_0)a^+}{\chi_1^-(\zeta)}\right\},$$

$$\Lambda_{\mathrm{VI,VI}}(\zeta) = \frac{i\chi_0^-(\zeta)}{\zeta^2+\kappa_0^2}\left\{-\frac{2\kappa_0^3 s^-(\zeta)}{r^-(\zeta)} + \frac{\zeta}{\chi_1^-(\zeta)}\left[(\zeta-i\kappa_0)a^- - (\zeta+i\kappa_0)a^+\right]\right\}.$$

For computational purposes, it is desirable to improve the rate of convergence of the integral (2.51). Notice that $\Lambda(\zeta) = \Lambda^\circ + O(\zeta)$ as $\zeta \to 0$ and $\Lambda(\zeta) \sim \Lambda^* + \Lambda^\infty/\zeta$ as $\zeta \to \infty$, where for mode II,

$$\Lambda^\circ = \sqrt{1+\rho^2/(2\kappa_0^2)}, \quad \Lambda^* = 1, \quad \Lambda^\infty = \chi_0^\infty, \tag{2.54}$$

and for the other modes,

$$\Lambda^\circ = \Lambda^\circ_{mj}, \quad \Lambda^* = \Lambda^*_{mj}, \quad \Lambda^\infty = \Lambda^\infty_{mj},$$

$$\Lambda^\circ_{\mathrm{I,I}} = 2ia^+a^-/\kappa_0, \quad \Lambda^\circ_{\mathrm{I,VI}} = \Lambda^\circ_{\mathrm{VI,VI}} = 0, \quad \Lambda^\circ_{\mathrm{VI,I}} = -i(a^+ + a^-)/\kappa_0,$$

$$\Lambda^*_{\mathrm{I,I}} = \Lambda^*_{\mathrm{VI,VI}} = i(a^- - a^+), \quad \Lambda^*_{\mathrm{I,VI}} = \Lambda^*_{\mathrm{VI,I}} = 0,$$

$$\Lambda^\infty_{\mathrm{I,I}} = i(\chi_0^\infty + i\rho/2)(a^- - a^+) - \kappa_0(a^- + a^+), \quad \Lambda^\infty_{\mathrm{I,VI}} = \kappa_0^2(a^- - a^+) - 2\kappa_0 a^+ a^-,$$

$$\Lambda^\infty_{\mathrm{VI,I}} = 2\kappa_0 + a^- - a^+, \quad \Lambda^\infty_{\mathrm{VI,VI}} = i(\chi_0^\infty - \chi_1^\infty)(a^- - a^+) + \kappa_0(a^- + a^+).$$

Here,

$$\chi_j^\infty = -\frac{1}{\pi i}\int_0^\infty \ln\delta_j(\zeta)d\zeta = -\frac{2}{\pi i}\int_{-1}^1 \ln\delta_j\left(\frac{1+\eta}{1-\eta}\right)\frac{d\eta}{(1-\eta)^2}, \tag{2.55}$$

and $\ln\delta_j((1+\eta)/(1-\eta)) = O((1-\eta)^2)$, $\eta\to 1$, $j=0,1$. Next, we use the following integrals:

$$\int_0^1 x^{a-1}e^{i\xi x}dx = S(a;\xi), \quad \mathrm{Re}\, a > 0,$$

$$\int_1^\infty x^{a-1}e^{i\xi x}dx = \frac{\Gamma(a)}{\xi^a}e^{i\pi a/2} - S(a;\xi), \quad \mathrm{Re}\, a < 1, \tag{2.56}$$

where

$$S(a;\xi) = \sum_{j=0}^\infty \frac{(i\xi)^j}{j!(a+j)}. \tag{2.57}$$

These formulas furnish the resulting representation for the weight functions used for the computations

$$\frac{W(\xi)}{a^*} = \int_0^\infty F(\zeta,\xi)d\zeta + (1+i)\Lambda^*\sqrt{\frac{2\pi}{\xi}} + (1+i)(\Lambda^\circ-\Lambda^*)\left[\mathrm{Re}\, S\left(\frac{1}{2};\xi\right) + \mathrm{Im}\, S\left(\frac{1}{2};\xi\right)\right]$$
$$-(1-i)\Lambda^\infty\left[2\sqrt{2\pi\xi} + \mathrm{Re}\, S\left(-\frac{1}{2};\xi\right) - \mathrm{Im}\, S\left(-\frac{1}{2};\xi\right)\right]. \tag{2.58}$$

Here,

$$F(\zeta,\xi) = \frac{\cos\zeta\xi}{\sqrt{\zeta}}\left\{\Lambda(\zeta) + i\Lambda(-\zeta) - (1+i)\Lambda^\circ\psi_-(\zeta) - \left[(1+i)\Lambda^* + (1-i)\frac{\Lambda^\infty}{\zeta}\right]\psi_+(\zeta)\right\}$$
$$+ i\frac{\sin\zeta\xi}{\sqrt{\zeta}}\left\{\Lambda(\zeta) - i\Lambda(-\zeta) - (1-i)\Lambda^\circ\psi_-(\zeta) - \left[(1-i)\Lambda^* + (1+i)\frac{\Lambda^\infty}{\zeta}\right]\psi_+(\zeta)\right\},$$

$$\psi_-(\zeta) = \begin{cases}1, & 0<\zeta<1,\\ 0, & 1<\zeta<\infty,\end{cases} \qquad \psi_+(\zeta) = \begin{cases}0, & 0<\zeta<1,\\ 1, & 1<\zeta<\infty.\end{cases}$$

The function $F(\zeta,\xi)$ is bounded as $\zeta\to 0$, and $F(\zeta,\xi) = O(\zeta^{-5/2})$ as $\zeta\to\infty$.

Figure 2 shows the weight functions $W_{\mathrm{I,I}}(\xi)$, $W_{\mathrm{I,VI}}(\xi)$, $W_{\mathrm{VI,I}}(\xi)$, $W_{\mathrm{VI,VI}}(\xi)$ and $W_{\mathrm{II}}(\xi)$ for the parameters $\kappa/\mu = 1$, $\lambda/\mu = 1$ and $\gamma/\mu = 1$. It turns out that the weight functions $W_{\mathrm{I,I}}(\xi)$ and $W_{\mathrm{VI,VI}}(\xi)$ tend to $-\infty$ as $\xi\to 0^+$, while the off-diagonal functions $W_{\mathrm{I,VI}}(\xi)$ and $W_{\mathrm{VI,I}}(\xi)$ are bounded. The function $W_{\mathrm{I,I}}$ is very close to the classical elastic weight function for modes I and II, $W_0 = -\sqrt{2/(\pi\xi)}$. The differences between W_{II} and W_0 are substantial. If the parameters ξ, κ/μ and λ/μ are fixed, and the parameter γ/μ approaches zero, then the weight functions $W_{\mathrm{I,VI}}$, $W_{\mathrm{VI,VI}}$ and W_{II} change their values drastically, while the changes in the other weight functions, $W_{\mathrm{I,I}}$ and $W_{\mathrm{VI,I}}$, are insignificant (Fig. 3).

2.7 *Limiting case* $\kappa\to 0^+$

We now analyse the limit case $\kappa\to 0^+$ assuming that $\gamma > 0$. The third equation in (2.5) is separated from the other two, and the microrotation ϕ_z is a harmonic function. The vector RHP (2.14) is uncoupled and has the form

$$\mathbf{\Phi}^+(\zeta) = -|\zeta|^{-1}\,\mathrm{diag}\{e_0, e_0, 2/\gamma\}[\mathbf{\Phi}^-(\zeta) + \hat{\mathbf{p}}^+(\zeta)], \quad \zeta\in L, \tag{2.59}$$

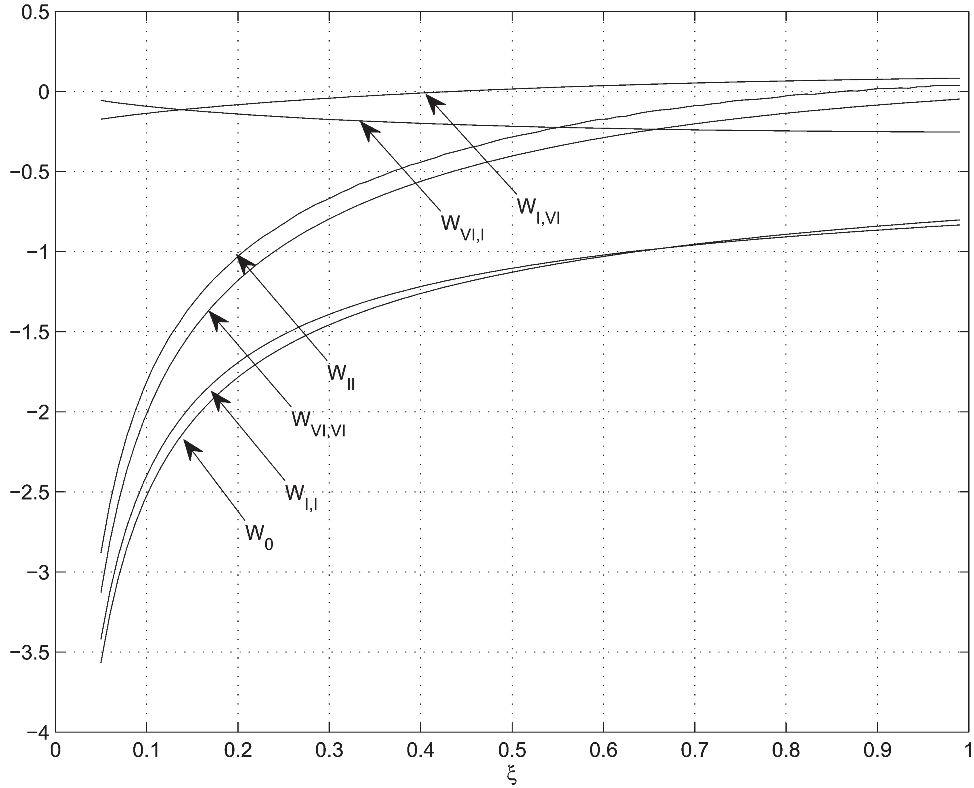

Fig. 2 The weight functions $W_{\mathrm{I,I}}(\xi)$, $W_{\mathrm{I,VI}}(\xi)$, $W_{\mathrm{VI,I}}(\xi)$, $W_{\mathrm{VI,VI}}(\xi)$ and $W_{\mathrm{II}}(\xi)$ for the parameters $\kappa/\mu = 1$, $\lambda/\mu = 1$ and $\gamma/\mu = 1$. $W_0(\xi)$ is the classical elastic mode-I weight function

where $e_0 = (\lambda + 2\mu)/\{\mu(\lambda + \mu)\}$. The solution is immediately derived as

$$\Phi_j^+(\zeta) = -\frac{e_j \psi_j^+(\zeta)}{s^+(\zeta)}, \quad \Phi_j^-(\zeta) = s^-(\zeta)\psi_j^-(\zeta), \quad j = 1, 2, 3. \tag{2.60}$$

Here, $e_1 = e_2 = e_0$, $e_3 = 2/\gamma$, and

$$\psi_j(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{p}_j^+(\zeta_0)d\zeta_0}{s^-(\zeta_0)(\zeta_0 - \zeta)}. \tag{2.61}$$

By evaluating the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\zeta\xi}d\zeta}{s^-(\zeta)} = (1+i)\sqrt{\frac{2\pi}{\xi}}, \quad \xi > 0, \tag{2.62}$$

we obtain that the modes I, II and VI share the same weight function

$$W_0(\xi) = -\sqrt{2/(\pi\xi)}. \tag{2.63}$$

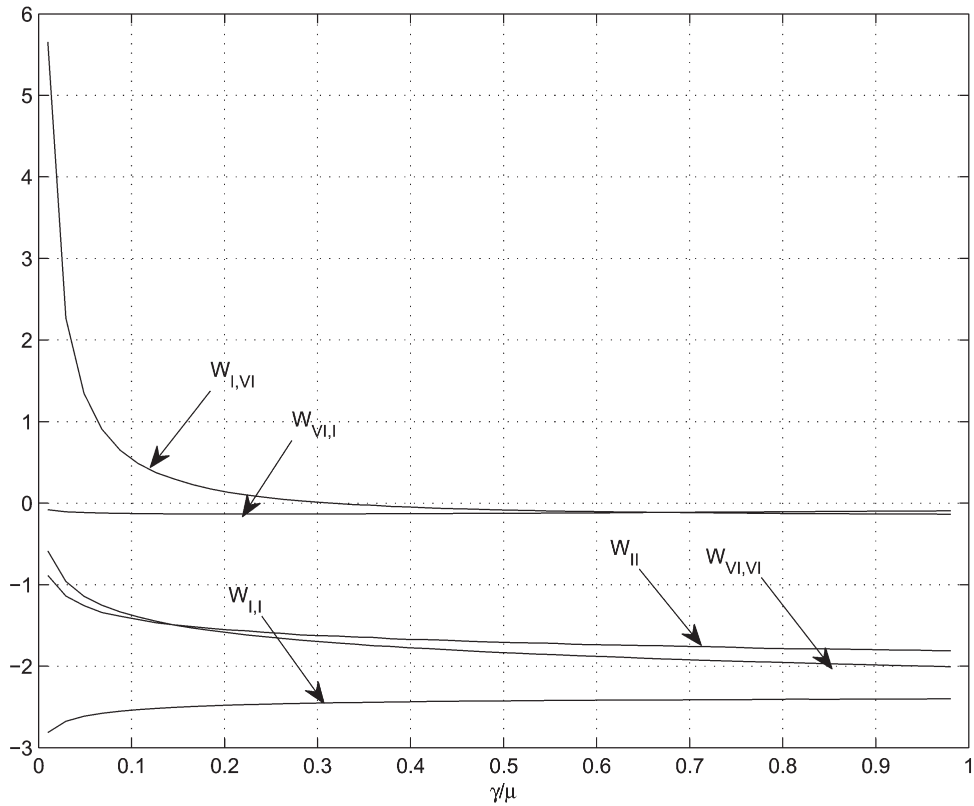

Fig. 3 The weight functions $W_{\mathrm{I,I}}$, $W_{\mathrm{I,VI}}$, $W_{\mathrm{VI,I}}$, $W_{\mathrm{VI,VI}}$ and W_{II} versus the parameter γ/μ when $\xi = 0.1$, $\kappa/\mu = 1$ and $\lambda/\mu = 1$

It is also possible to pass to the limit $\kappa \to 0^+$ in the final formulas (2.44) and (2.49) for the weight functions obtained for the case $\kappa > 0$. As $\kappa \to 0^+$, the functions $\delta_j(\zeta) \to 1$ $(j = 0, 1)$ and $r(\zeta) \to |\zeta|$. Therefore, $\chi(\zeta)^{\pm} \to 1$ and $r^{\pm}(\zeta) \to s^{\pm}(\zeta)$. Formula (2.44) becomes identical to (2.63). Set now $\kappa = 0$ in (2.38) and (2.47). This gives $a^{\pm} = \pm\kappa_0$ and

$$\begin{pmatrix} v_{\mathrm{I,I}} & v_{\mathrm{I,VI}} \\ v_{\mathrm{VI,I}} & v_{\mathrm{VI,VI}} \end{pmatrix}(\zeta, \xi) = \int_{-\infty}^{\infty} \begin{pmatrix} i\zeta_0 & \kappa_0^2 \\ 1 & i\zeta_0 \end{pmatrix} \frac{e^{i\zeta_0\xi}\, d\zeta_0}{s^-(\zeta_0)(\zeta_0 - \zeta)}. \tag{2.64}$$

By substituting these formulas into (2.49), we derive ultimately

$$W_{\mathrm{I,I}} = W_{\mathrm{VI,VI}} = -\sqrt{2/(\pi\xi)}, \quad W_{\mathrm{I,VI}} = W_{\mathrm{VI,I}} = 0. \tag{2.65}$$

Figure 4 shows that if $\kappa/\mu \to 0^+$, then the weight functions $W_{\mathrm{I,VI}}$ and $W_{\mathrm{VI,I}}$ vanish, while the other three functions tend to $-\sqrt{2/(\pi\xi)}$ that is approximately equal to -2.5231 at $\xi = 0.1$.

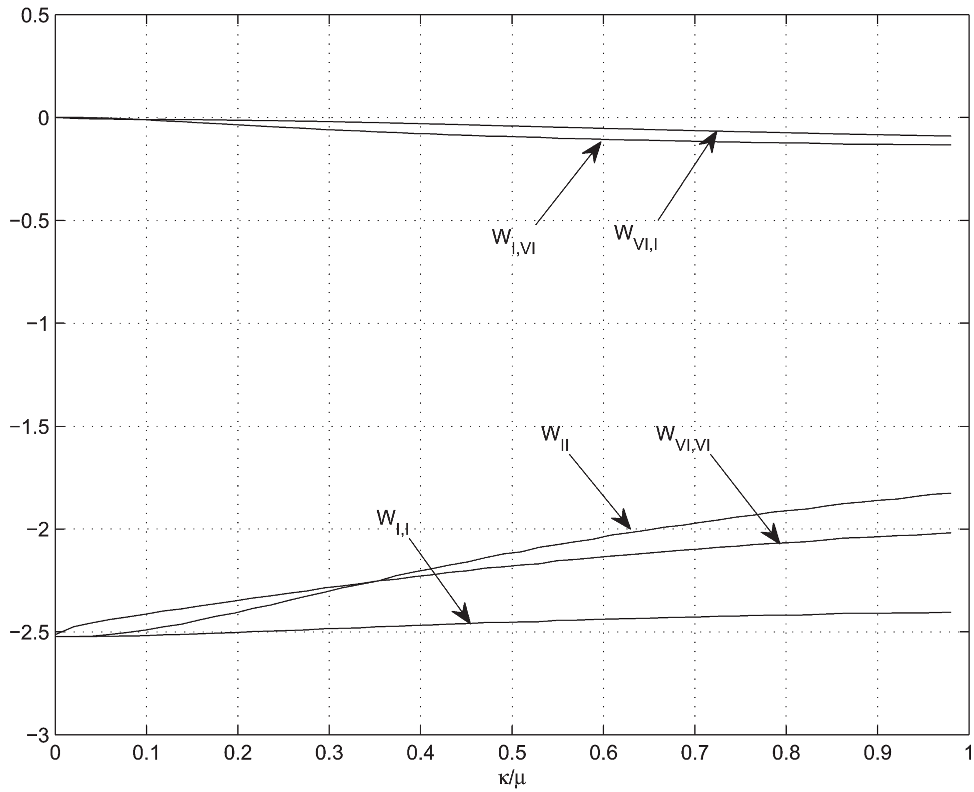

Fig. 4 The weight functions $W_{\mathrm{I,I}}$, $W_{\mathrm{I,VI}}$, $W_{\mathrm{VI,I}}$, $W_{\mathrm{VI,VI}}$ and W_{II} versus the parameter κ/μ when $\xi = 0.1$, $\gamma/\mu = 1$ and $\lambda/\mu = 1$

3. Antiplane-strain problem of micropolar elasticity

3.1 *Derivation of the governing vector RHP*

The balance laws of micropolar elasticity in the antiplane-strain case reduce to

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \quad \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} + \tau_{yz} - \tau_{zy} = 0, \quad \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y} - \tau_{xz} + \tau_{zx} = 0,$$

where τ_{xz}, τ_{zx}, τ_{yz} and τ_{zy} are the stress tensor components, and m_{xx}, m_{yy}, m_{xy} and m_{yx} are the couple–stress tensor components. The constitutive and kinematic equations when combined lead to the relations

$$\tau_{xz} = (\mu + \kappa)\frac{\partial u_z}{\partial x} + \kappa\phi_y, \quad \tau_{zx} = \mu\frac{\partial u_z}{\partial x} - \kappa\phi_y,$$

$$\tau_{yz} = (\mu + \kappa)\frac{\partial u_z}{\partial y} - \kappa\phi_x, \quad \tau_{zy} = \mu\frac{\partial u_z}{\partial y} + \kappa\phi_x,$$

$$m_{xx}=\delta\frac{\partial\phi_x}{\partial x}+\alpha\frac{\partial\phi_y}{\partial y},\quad m_{yy}=\delta\frac{\partial\phi_y}{\partial y}+\alpha\frac{\partial\phi_x}{\partial x},$$

$$m_{xy}=\beta\frac{\partial\phi_x}{\partial y}+\gamma\frac{\partial\phi_y}{\partial x},\quad m_{yx}=\beta\frac{\partial\phi_y}{\partial x}+\gamma\frac{\partial\phi_x}{\partial y},\tag{3.1}$$

where $\delta=\alpha+\beta+\gamma$, and $\alpha,\beta,\gamma,\kappa$ and μ are material constants such that (**28**, **29**)

$$\gamma>|\beta|,\quad \kappa>0,\quad \kappa+2\mu>0,\quad 3\alpha+\beta+\gamma>0.\tag{3.2}$$

The problem to be solved here consists in finding a solution $\{w,\phi_x,\phi_y\}$ of the system of differential equations

$$(\mu+\kappa)\Delta u_z+\kappa\frac{\partial\phi_y}{\partial x}-\kappa\frac{\partial\phi_x}{\partial y}=0,$$

$$\delta\frac{\partial^2\phi_x}{\partial x^2}+\gamma\frac{\partial^2\phi_x}{\partial y^2}+(\alpha+\beta)\frac{\partial^2\phi_y}{\partial x\partial y}-2\kappa\phi_x+\kappa\frac{\partial u_z}{\partial y}=0,$$

$$\gamma\frac{\partial^2\phi_y}{\partial x^2}+\delta\frac{\partial^2\phi_y}{\partial y^2}+(\alpha+\beta)\frac{\partial^2\phi_x}{\partial x\partial y}-2\kappa\phi_y-\kappa\frac{\partial u_z}{\partial x}=0,\tag{3.3}$$

satisfying the boundary conditions

$$\tau_{yz}=p_1(x),\quad m_{yx}=p_2(x),\quad m_{yy}=p_3(x),\quad 0<x<\infty,\quad y=0^{\pm},\tag{3.4}$$

and the continuity conditions

$$u_z(x,0^+)-u_z(x,0^-)=0,\quad \phi_x(x,0^+)-\phi_x(x,0^-)=0,$$

$$\phi_y(x,0^+)-\phi_y(x,0^-)=0,\quad -\infty<x\leqslant 0.\tag{3.5}$$

The solution can be represented in terms of the Fourier integrals

$$\begin{pmatrix}u_z\\ \phi_x\\ \phi_y\end{pmatrix}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}\hat{u}_z\\ \hat{\phi}_x\\ \hat{\phi}_y\end{pmatrix}(\zeta;y)e^{-i\zeta x}dx,\tag{3.6}$$

where the functions $\hat{u}_z$, $\hat{\phi}_x$ and $\hat{\phi}_y$ yield the solution to the system of ordinary differential equations

$$(\mu+\kappa)(\hat{u}_z''-\zeta^2\hat{u}_z)-\kappa(i\zeta\hat{\phi}_y+\phi_x')=0,$$

$$-\delta\zeta^2\hat{\phi}_x+\gamma\hat{\phi}_x''-i\zeta(\alpha+\beta)\hat{\phi}_y'-2\kappa\hat{\phi}_x+\kappa\hat{u}_z'=0,$$

$$\delta\hat{\phi}_y''-\gamma\zeta^2\phi_y-i\zeta(\alpha+\beta)\hat{\phi}_x'-2\kappa\hat{\phi}_y+i\kappa\zeta\hat{u}_z=0,\quad y\in\mathbb{R}^1\setminus\{0\},\tag{3.7}$$

subject to the boundary conditions

$$(\mu+\kappa)\hat{u}_z' - \kappa\hat{\phi}_x = \hat{p}_1, \quad -i\zeta\beta\hat{\phi}_y + \gamma\hat{\phi}_x' = \hat{p}_2, \quad \delta\hat{\phi}_y' - i\alpha\zeta\hat{\phi}_x = \hat{p}_3, \quad y = 0^{\pm}, \quad (3.8)$$

where

$$\hat{p}_j(\zeta) = \Phi_j^-(\zeta) + \hat{p}_j^+(\zeta), \quad \hat{p}_j^+(\zeta) = \int_0^\infty p_j(x)e^{i\zeta x}dx, \quad j = 1, 2, 3,$$

$$\Phi_1^-(\zeta) = \int_{-\infty}^0 \tau_{yz}(x,0)e^{i\zeta x}dx, \quad \Phi_2^-(\zeta) = \int_{-\infty}^0 m_{yx}(x,0)e^{i\zeta x}dx,$$

$$\Phi_3^-(\zeta) = \int_{-\infty}^0 m_{yy}(x,0)e^{i\zeta x}dx. \quad (3.9)$$

This boundary value problem, similarly to the plane-strain case, can be reduced to a vector RHP. The general solution of the system (3.7) in the case $0 < y < \infty$ is given by

$$\hat{u}_z = -\frac{2iC_1}{\zeta}e^{-|\zeta|y} - \frac{i\kappa C_2}{(\kappa+\mu)\zeta}e^{-r_0 y},$$

$$\hat{\phi}_x = \frac{i|\zeta|C_1}{\zeta}e^{-|\zeta|y} + \frac{ir_0C_2}{\zeta}e^{-r_0y} + \frac{i\delta\zeta r_1C_3}{2\kappa+\delta\zeta^2}e^{-r_1y},$$

$$\hat{\phi}_y = C_1e^{-|\zeta|y} + C_2e^{-r_0y} + C_3e^{-r_1y}, \quad (3.10)$$

where C_1, C_2 and C_3 are arbitrary constants, and

$$r_j = \sqrt{\zeta^2+\rho_j^2}, \quad j = 0, 1, \quad \rho_0 = \sqrt{\frac{\kappa(\kappa+2\mu)}{\gamma(\kappa+\mu)}} > 0, \quad \rho_1 = \sqrt{\frac{2\kappa}{\delta}} > 0.$$

By employing the boundary conditions (3.8), we express the constants C_j through $\hat{p}_j$ ($j = 1, 2, 3$). Then, we substitute them into (3.10) and fix $y = 0$. This gives us expressions for $\hat{u}_z(\zeta;0^+)$, $\hat{\phi}_x(\zeta;0^+)$ and $\hat{\phi}_y(\zeta;0^+)$. After that we obtain similar expressions for the functions $\hat{u}_z(\zeta;0^-)$, $\hat{\phi}_x(\zeta;0^-)$ and $\hat{\phi}_y(\zeta;0^-)$. This can be done by replacing $|\zeta|$, r_0 and r_1 in the solution on the line $y = 0^+$ by $-|\zeta|$, $-r_0$ and $-r_1$, respectively. We then introduce the vector function

$$\mathbf{\Phi}^+(\zeta) = \int_0^\infty \begin{pmatrix} u_z(x,0^+) - u_z(x,0^-) \\ \phi_x(x,0^+) - \phi_x(x,0^-) \\ \phi_y(x,0^+) - \phi_y(x,0^-) \end{pmatrix} e^{i\zeta x}dx, \quad (3.11)$$

and, as in the plane-strain case, derive the following vector RHP:

$$\mathbf{\Phi}^+(\zeta) = [h(\zeta)]^{-1}G(\zeta)[\mathbf{\Phi}^-(\zeta) + \hat{\mathbf{p}}^+(\zeta)], \quad \zeta \in L, \quad (3.12)$$

where L is the real axis,

$$h(\zeta) = (\beta+\gamma)^2 r_0\left[(\kappa+2\mu)|\zeta| + \frac{\kappa\zeta^2}{r_1}\right] - \frac{2[\kappa(\kappa+2\mu)+(\beta+\gamma)(\kappa+\mu)\zeta^2]^2}{(\kappa+\mu)|\zeta|r_1}, \quad (3.13)$$

$$G(\zeta)=\begin{pmatrix} G_{11}(\zeta) & 0 & iG_{13}(\zeta) \\ 0 & G_{22}(\zeta) & 0 \\ -iG_{13}(\zeta) & 0 & G_{33}(\zeta) \end{pmatrix},$$

$$G_{11}(\zeta)=2(\beta+\gamma)^2\left(\frac{\kappa|\zeta|}{\kappa+\mu}-2r_0\right)+\frac{2(\kappa+2\mu)[2\kappa+(\beta+\gamma)\zeta^2]^2}{(\kappa+\mu)\zeta^2 r_1},$$

$$G_{13}(\zeta)=\frac{2\kappa(\kappa+2\mu)}{(\kappa+\mu)\zeta}\left[(\beta+\gamma)|\zeta|-\frac{2\kappa+(\beta+\gamma)\zeta^2}{r_1}\right],$$

$$G_{22}(\zeta)=\frac{4\kappa(\kappa+2\mu)r_0}{|\zeta|r_1},\quad G_{33}(\zeta)=\frac{2\kappa(\kappa+2\mu)}{\kappa+\mu}\left(\frac{\kappa+2\mu}{|\zeta|}+\frac{\kappa}{r_1}\right).$$

As in the plane-strain problem, one of the equations (the second one) is separated from the others. Therefore, the m_{yx}-mode (mode-V) is uncoupled, while the τ_{yz}- and m_{yy}-modes, modes III and IV, respectively, are coupled.

3.2 *Mode-V*

The scalar RHP associated with mode-V has the form

$$\Phi_2^+(\zeta)=\frac{G_{22}(\zeta)}{h(\zeta)}[\Phi_2^-(\zeta)+\hat{p}_2^+(\zeta)],\quad \zeta\in L. \tag{3.14}$$

The analysis of the behaviour of the function $h(\zeta)$ at infinity shows that

$$h(\zeta)=-\kappa(\kappa+2\mu)h_1+O(\zeta^{-2}),\quad \zeta\to\infty, \tag{3.15}$$

where

$$h_1=(\beta+\gamma)\left(\frac{3\alpha+2\beta+2\gamma}{\alpha+\beta+\gamma}-\frac{\beta}{\gamma}\right). \tag{3.16}$$

Because of the conditions (3.2), the parameter h_1 is always positive. It will be convenient to introduce a new function

$$G_{22}^\circ(\zeta)=-\frac{h_1 r_0(\zeta)G_{22}(\zeta)}{4h(\zeta)}. \tag{3.17}$$

This function is even, positive at $\zeta=0$ and at infinity,

$$G_{22}^\circ(\zeta)\sim\frac{h_1}{2\gamma}>0,\quad \zeta\to 0,\quad G_{22}^\circ(\zeta)=1+O(\zeta^{-2}),\quad \zeta\to\infty, \tag{3.18}$$

and does not have poles or zeros in the real axis. Therefore, it can be factorized in terms of the Cauchy integral as follows:

$$G_{22}^\circ(\zeta)=\frac{\chi^+(\zeta)}{\chi^-(\zeta)},\quad \zeta\in L, \tag{3.19}$$

where

$$\chi(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_{22}^{\circ}(\zeta_0)d\zeta_0}{\zeta_0-\zeta}\right\}, \quad \zeta \in \mathbb{C}^{\pm}. \tag{3.20}$$

Hence, the solution to the problem exists and is unique in the class of functions whose asymptotics at infinity are

$$\Phi_2^+(\zeta) = O(\zeta^{-3/2}), \quad \Phi_2^-(\zeta) = O(\zeta^{-1/2}), \quad \zeta \to \infty. \tag{3.21}$$

The solution to the RHP is given by

$$\Phi_2^+(\zeta) = \frac{\chi^+(\zeta)}{r_0^+(\zeta)}\Psi^+(\zeta), \quad \Phi_2^-(\zeta) = -\frac{h_1}{4}r_0^-(\zeta)\chi^-(\zeta)\Psi^-(\zeta), \tag{3.22}$$

where

$$\Psi(\zeta) = -\frac{2}{\pi i h_1}\int_L \frac{\hat{p}_2^+(\zeta_0)d\zeta_0}{r_0^-(\zeta_0)\chi^-(\zeta_0)(\zeta_0-\zeta)}, \tag{3.23}$$

and $r_0^{\pm}(\zeta) = (\zeta \pm i\rho_0)^{1/2}$ are analytic functions in the half-planes $\mathbb{C}^{\pm}$, and $\arg(\zeta + i\rho_0) \in [0, \pi]$, $\arg(\zeta - i\rho_0) \in [-\pi, 0]$. As in the mode-II case, we derive expressions for the stress intensity factor K_{V} and the weight function. They are

$$K_{\mathrm{V}} = \sqrt{2\pi}\lim_{x\to 0^-}\sqrt{-x}\,m_{yx}(x,0) = \int_0^{\infty} W_{\mathrm{V}}(\xi)p_2(\xi)d\xi,$$

$$W_{\mathrm{V}}(\xi) = \frac{i-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\zeta\xi}d\zeta}{r_0^-(\zeta)\chi^-(\zeta)}, \tag{3.24}$$

where $p_2(\xi) = m_{yx}(\xi, 0^{\pm})$, $0 < \xi < \infty$.

3.3 *System of singular integral equations for modes III and IV: a finite crack*

To find the functions $\Phi_1^{\pm}(\zeta)$ and $\Phi_3^{\pm}(\zeta)$, we need to solve the following RHP:

$$\boldsymbol{\phi}^+(\zeta) = [h(\zeta)]^{-1}A(\zeta)[\boldsymbol{\phi}^-(\zeta) + \tilde{\mathbf{p}}^+(\zeta)], \quad \zeta \in L, \tag{3.25}$$

where $A(\zeta)$ is the 2×2 matrix

$$A(\zeta) = \begin{pmatrix} G_{11}(\zeta) & iG_{13}(\zeta) \\ -iG_{13}(\zeta) & G_{33}(\zeta)\end{pmatrix} = \frac{2\kappa}{(\kappa+\mu)|\zeta|}Q_0(\zeta) + \frac{2(\kappa+2\mu)}{(\kappa+\mu)\zeta^2 r_1}Q_1(\zeta) + 4(\beta+\gamma)^2 r_0 Q_2,$$

Q_j are the polynomial matrices

$$Q_0(\zeta) = \begin{pmatrix} (\beta+\gamma)^2\zeta^2 & i(\kappa+2\mu)(\beta+\gamma)\zeta \\ -i(\kappa+2\mu)(\beta+\gamma)\zeta & (\kappa+2\mu)^2 \end{pmatrix},$$

$$Q_1(\zeta) = \begin{pmatrix} [2\kappa+(\beta+\gamma)\zeta^2]^2 & -i\kappa\zeta[2\kappa+(\beta+\gamma)\zeta^2] \\ i\kappa\zeta[2\kappa+(\beta+\gamma)\zeta^2] & \kappa^2\zeta^2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.26}$$

and

$$\boldsymbol{\phi}^{\pm}(\zeta) = \begin{pmatrix} \Phi_1^{\pm}(\zeta) \\ \Phi_3^{\pm}(\zeta) \end{pmatrix}, \quad \tilde{\mathbf{p}}^{+}(\zeta) = \begin{pmatrix} \hat{p}_1^{+}(\zeta) \\ \hat{p}_3^{+}(\zeta) \end{pmatrix}. \tag{3.27}$$

Unfortunately, the structure of the matrix $A(\zeta)$ does not allow for exact factorization by methods available in the literature, and therefore, modes III and IV cannot be decoupled. However, it is possible to determine the partial indices of the RHP. The eigenvalues of the matrix $A(\zeta)$ have the form

$$\lambda_1 = \frac{G_{11}+G_{33}}{2} + \frac{1}{2}\sqrt{\Delta_0}, \quad \lambda_2 = \frac{G_{11}+G_{33}}{2} - \frac{1}{2}\sqrt{\Delta_0}, \tag{3.28}$$

where $\Delta_0 = (G_{11}-G_{33})^2 + 4G_{13}^2$. It can be directly verified that both eigenvalues are positive on the real axis, and therefore, the Hermitian matrix $A(\zeta)$ is positive definite. The partial indices, ν_1 and ν_2, are equal to zero (**25**). This implies that the difference between the number of arbitrary constants in the solution of the problem (3.25) and the number of additional conditions equals zero, and the solution of the RHP is stable (**27**). This guarantees that if $|A(\zeta) - \hat{A}(\zeta)| < \varepsilon$, $-\infty < \zeta < +\infty$, and $\hat{A}(\zeta)$ admits a factorization $\hat{A}(\zeta) = \hat{X}^{+}(\zeta)[\hat{X}^{-}(\zeta)]^{-1}$, then the factors $\hat{X}^{\pm}(\zeta)$ approach the exact factors $X^{\pm}(\zeta)$ of the matrix $A(\zeta)$ as $\varepsilon \to 0$.

By focusing on the case of a finite crack $\{0 < x < a, y = \pm 0\}$, we transform the associated vector RHP into a system of singular integral equations. This system admits an efficient approximate solution. On using the convolution theorem and integration by parts, we have

$$\begin{pmatrix} p_1(x) \\ p_3(x) \end{pmatrix} = \frac{1}{\pi}\int_0^a K(x-\tau)\begin{pmatrix} \chi_1(\tau) \\ \chi_3(\tau) \end{pmatrix} d\tau, \quad 0 < x < a, \tag{3.29}$$

where

$$K(x) = -\frac{1}{2i}\int_{-\infty}^{\infty} h(\zeta)[A(\zeta)]^{-1} e^{-i\zeta x}\frac{d\zeta}{\zeta},$$

$$[A(\zeta)]^{-1} = \frac{1}{\Delta(\zeta)}\begin{pmatrix} G_{33}(\zeta) & -iG_{13}(\zeta) \\ iG_{13}(Gz) & G_{11}(\zeta) \end{pmatrix}, \quad \Delta(\zeta) = G_{11}(\zeta)G_{33}(\zeta) - G_{13}^2(\zeta),$$

$$\chi_1(x) = \frac{d}{dx}\big[u_z(x,0^+) - u_z(x,0^-)\big], \quad \chi_3(x) = \frac{d}{dx}\big[\phi_y(x,0^+) - \phi_y(x,0^-)\big]. \tag{3.30}$$

To understand the structure of the kernel, it is crucial to determine the asymptotics of the matrix $h(\zeta)[A(\zeta)]^{-1}$ as $\zeta \to 0$ and $\zeta \to \infty$. We have

$$h(\zeta)[A(\zeta)]^{-1} \sim -\frac{1}{4}\begin{pmatrix} (\kappa+2\mu)|\zeta| & -i(\beta+\gamma)|\zeta|^2 \operatorname{sgn}\zeta \\ i(\beta+\gamma)|\zeta|^2 \operatorname{sgn}\zeta & \frac{4\kappa}{\rho_1} \end{pmatrix}, \ \zeta \to 0. \tag{3.31}$$

Analysis of the entries of the matrix $A(\zeta)$ yields

$$G_{mm}(\zeta) = \frac{G_{mm}^0}{|\zeta|} + \frac{G_{mm}^1}{|\zeta|^3} + O(|\zeta|^{-5}), \ m = 1, 3, \quad G_{13}(\zeta) = \frac{G_{13}^0 \operatorname{sgn}\zeta}{|\zeta|^2} + O(|\zeta|^{-4}), \ \zeta \to \infty,$$

where

$$G_{11}^0 = \frac{2\kappa(\kappa+2\mu)h_1}{\kappa+\mu}, \quad G_{11}^1 = \frac{\kappa+2\mu}{\kappa+\mu}\left[8\kappa^2 - 4\kappa(\beta+\gamma)\rho_1^2 + \frac{3}{4}\rho_1^4(\beta+\gamma)^2\right] + \frac{\rho_0^4}{2}(\beta+\gamma)^2,$$
$$G_{33}^0 = 4\kappa(\kappa+2\mu), \quad G_{33}^1 = -\frac{\kappa^2(\kappa+2\mu)\rho_1^2}{\kappa+\mu}, \quad G_{13}^0 = \frac{\kappa(\kappa+2\mu)}{\kappa+\mu}[(\beta+\gamma)\rho_1^2 - 4\kappa],$$

and therefore, as $\zeta \to \infty$,

$$h(\zeta)[A(\zeta)]^{-1} = \frac{1}{4}\begin{pmatrix} -2(\kappa+\mu)|\zeta| + c_0|\zeta|^{-1} & i\frac{(\beta+\gamma)\rho_1^2-4\kappa}{2}\operatorname{sgn}\zeta + ic_1(\zeta|\zeta|)^{-1} \\ -i\frac{(\beta+\gamma)\rho_1^2-4\kappa}{2}\operatorname{sgn}\zeta - ic_1(\zeta|\zeta|)^{-1} & -h_1|\zeta| + c_2|\zeta|^{-1}\end{pmatrix},$$

where c_j, $(j = 0, 1, 2)$ are real constants. Ultimately, we arrive at the following system of singular integral equations:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^a \left[\frac{\nu_0}{\tau - x} + k_{11}(x-\tau)\right]\chi_1(\tau)d\tau \\
&\quad + \frac{1}{\pi}\int_0^a \left[\nu_1 \ln|\tau - x| + k_{13}(x-\tau)\right]\chi_3(\tau)d\tau = p_1(x), \quad 0 < x < a, \\
&-\frac{1}{\pi}\int_0^a \left[\nu_1 \ln|\tau - x| + k_{13}(x-\tau)\right]\chi_1(\tau)d\tau \\
&\quad + \frac{1}{\pi}\int_0^a \left[\frac{\nu_2}{\tau - x} + k_{33}(x-\tau)\right]\chi_3(\tau)d\tau = p_3(x), \quad 0 < x < a,
\end{aligned} \tag{3.32}$$

where $\nu_0 = \frac{1}{2}(\kappa+\mu)$, $\nu_1 = \frac{1}{8}\{(\beta+\gamma)\rho_1^2 - 4\kappa\}$, $\nu_2 = \frac{1}{4}h_1$, k_{mj} $(m, j = 1, 3)$ are regular kernels given by

$$k_{mm}(t) = \int_0^\infty L_{mm}(\zeta)\sin\zeta t d\zeta + \nu_{m-1}^*\Lambda(t), \quad m = 1, 3,$$
$$k_{13}(t) = \int_0^\infty L_{13}(\zeta)\cos\zeta t d\zeta + \nu_1 C + \nu_1\sum_{k=1}^\infty \frac{(-1)^k t^{2k}}{2k(2k)!},$$
$$L_{11}(\zeta) = \frac{h(\zeta)G_{33}(\zeta)}{\zeta\Delta(\zeta)} + \nu_0 + \frac{\nu_0^*\psi_+(\zeta)}{\zeta^2}, \quad L_{33}(\zeta) = \frac{h(\zeta)G_{11}(\zeta)}{\zeta\Delta(\zeta)} + \nu_2 + \frac{\nu_2^*\psi_+(\zeta)}{\zeta^2},$$
$$L_{13}(\zeta) = \frac{h(\zeta)G_{13}(\zeta)}{\zeta\Delta(\zeta)} + \frac{\nu_1\psi_+(\zeta)}{\zeta}, \quad \psi_+(\zeta) = \begin{cases} 0, & 0 < \zeta < 1, \\ 1, & 1 < \zeta < \infty.\end{cases}$$
$$\Lambda(t) = t(C + \ln|t| - 1) + \sum_{k=1}^\infty \frac{(-1)^k t^{2k+1}}{2k(2k+1)!}, \quad \nu_0^* = \frac{\nu_2 G_{33}^1 - \delta_*}{G_{11}^0}, \quad \nu_2^* = \frac{\nu_0 G_{11}^1 - \delta_*}{G_{33}^0},$$
$$\delta_* = \delta_2 - \delta_1\left(\frac{G_{11}^1}{G_{11}^0} + \frac{G_{33}^1}{G_{33}^0} - \frac{(G_{13}^0)^2}{G_{11}^0 G_{33}^0}\right), \quad \delta_1 = -\kappa(\kappa+2\mu)h_1,$$
$$\begin{aligned}\delta_2 = {}& \frac{(\beta+\gamma)^2}{4}\left[\frac{3}{2}\rho_1^4 - \kappa\rho_0^2\rho_1^2 - \rho_0^4(\kappa+\mu)\right] \\ & - \frac{2}{\kappa+\mu}\left[\kappa^2(\kappa+2\mu)^2 - \rho_1^2\kappa(\kappa+\mu)(\kappa+2\mu)(\beta+\gamma) + \frac{3}{8}\rho_1^4(\kappa+\mu)^2(\beta+\gamma)^2\right].\end{aligned}$$

Here, the series representation of the cosine integral (**30**)

$$\mathrm{ci}(t) = C + \ln|t| + \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{2k(2k)!} \tag{3.33}$$

(C is the Euler constant) was used.

In order to solve the system of integral equations (3.32), we employ the method of orthogonal polynomials. As a matter of utility, it will be desirable to have the system (3.32) in the interval $(-1, 1)$ with respect to the functions $\chi_m(x) = \tilde{\chi}_m(2x/a - 1)$. We expand the unknown functions $\tilde{\chi}_m(t)$ in terms of the Chebyshev polynomials of the first kind $T_n(x)$ as

$$\tilde{\chi}_m(t) = \frac{1}{\sqrt{1-t^2}} \sum_{j=0}^{\infty} a_j^{(m)} T_j(t), \quad m = 1, 3, \tag{3.34}$$

where the coefficients $a_j^{(m)}$ are to be determined. From the natural condition (it follows from the definition (3.30))

$$\int_{-1}^{1} \tilde{\chi}_m(t)dt = 0, \tag{3.35}$$

we immediately obtain that $a_0^{(m)} = 0$, $m = 1, 3$. The next step of the method is the use of the spectral relations

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_j(t)dt}{\sqrt{1-t^2}(t-\eta)} = \begin{cases} 0, & j = 0, \\ U_{j-1}(\eta), & j = 1, 2, \dots, \end{cases} \quad -1 < \eta < 1,$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_j(t)}{\sqrt{1-t^2}} \ln|t-\eta|dt = \begin{cases} -\ln 2, & j = 0, \\ j^{-1} T_j(\eta), & j = 1, 2, \dots, \end{cases} \quad -1 < \eta < 1,$$

the relation

$$\int_{-1}^{1} T_j(t) U_n(t) \sqrt{1-t^2} dt = \begin{cases} \pi/2, & j = n = 0, \\ \pi/4, & j = n \neq 0, \\ -\pi/4, & j = n+2, \end{cases} \tag{3.36}$$

and the orthogonality condition for the Chebyshev polynomials of the second kind $U_n(t)$. Ultimately, the system of integral equations reduces to the following infinite system of linear algebraic equations:

$$\frac{\pi \nu_0}{2} a_{n+1}^{(1)} + \sum_{j=1}^{\infty} c_{nj}^{(1,1)} a_j^{(1)} - \nu_1[\sigma_n a_n^{(3)} - \sigma_{n+2} a_{n+2}^{(3)}] + \sum_{j=1}^{\infty} c_{nj}^{(1,3)} a_j^{(3)} = b_n^{(1)}, \quad n = 0, 1, \dots,$$

$$\frac{\pi \nu_2}{2} a_{n+1}^{(3)} + \sum_{j=1}^{\infty} c_{nj}^{(3,3)} a_j^{(3)} + \nu_1[\sigma_n a_n^{(1)} - \sigma_{n+2} a_{n+2}^{(1)}] - \sum_{j=1}^{\infty} c_{nj}^{(1,3)} a_j^{(1)} = b_n^{(3)}, \quad n = 0, 1, \dots. \tag{3.37}$$

Here,

$$\sigma_0 a_0^{(1)} = \sigma_0 a_0^{(3)} = 0, \quad \sigma_n = \frac{\pi}{4n} \quad (n = 1, 2 \ldots),$$

$$c_{nj}^{(m,l)} = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{T_j(t)}{\sqrt{1-t^2}} \sqrt{1-\eta^2} U_n(\eta) \tilde{k}_{ml}(\eta - t) d\eta dt,$$

$$b_n^{(m)} = \int_{-1}^{1} \sqrt{1-\eta^2} U_n(\eta) \tilde{p}_m(\eta) d\eta, \quad \tilde{p}_m(t) = p_m\left(\frac{a}{2}(t+1)\right),$$

$$\tilde{k}_{mm}(t) = \frac{a}{2} k_{mm}\left(\frac{at}{2}\right), \quad \tilde{k}_{13}(t) = \frac{a}{2}\left[\nu_1 \ln \frac{a}{2} + k_{13}\left(\frac{at}{2}\right)\right].$$

By using the properties of the kernels and the Chebyshev polynomials, we find that

$$c_{nj}^{(11)} = c_{nj}^{(33)} = 0 \quad \text{if} \ \ n + j = 2, 4, 6, \ldots \quad \text{and} \quad c_{nj}^{(13)} = 0 \quad \text{if} \ \ n + j = 1, 3, 5, \ldots.$$

It can be verified (**31**) that the non-zero coefficients decay as n or j grow ($c_{nj}^{(ml)} = o(n^{-1})$ as $n \to \infty$ when j is fixed, and $c_{nj}^{(ml)} = o(j^{-1})$ as $j \to \infty$ when n is fixed, $m, l = 1, 3$), and the infinite system (3.37) can be solved approximately by the reduction method.

3.4 *Weight functions for modes III and IV*

Determine now the stress intensity factors and the weight functions pertaining to them. At the tips of the crack, the stress τ_{yz} and the couple–stress m_{yy} have the square root singularity:

$$\tau_{yz}(x, 0) \sim \frac{K_{\rm III}^{(0)}}{\sqrt{-2\pi x}}, \quad m_{yy}(x, 0) \sim \frac{K_{\rm IV}^{(0)}}{\sqrt{-2\pi x}}, \quad x \to 0^-,$$

$$\tau_{yz}(x, 0) \sim \frac{K_{\rm III}^{(a)}}{\sqrt{2\pi(x-a)}}, \quad m_{yy}(x, 0) \sim \frac{K_{\rm IV}^{(a)}}{\sqrt{2\pi(x-a)}}, \quad x \to a^+. \tag{3.38}$$

On the other hand, as $x \to 0^-$, we have

$$\tau_{yz}(x, 0) \sim \frac{\nu_0}{\pi} \int_0^a \frac{\chi_1(\tau) d\tau}{\tau - x}, \quad m_{yy}(x, 0) \sim \frac{\nu_2}{\pi} \int_0^a \frac{\chi_3(\tau) d\tau}{\tau - x}, \quad x \to 0^-. \tag{3.39}$$

On replacing the functions $\chi_m(\tau)$ by $\tilde{\chi}_m(t) = \chi(a(t+1)/2)$, using (3.34) and the asymptotic formula for the singular integral

$$I(x') = \frac{1}{\pi} \int_0^1 \frac{T_j(2\tau' - 1) d\tau'}{\sqrt{\tau'(1-\tau')}(\tau' - x')} \sim (-1)^j (-x')^{-1/2}, \quad x' \to 0^-, \tag{3.40}$$

we obtain

$$\tau_{yz} \sim \frac{\nu_0}{2}\left(-\frac{x}{a}\right)^{-1/2} \sum_{j=1}^{\infty} (-1)^j a_j^{(1)}, \quad m_{yy} \sim \frac{\nu_2}{2}\left(-\frac{x}{a}\right)^{-1/2} \sum_{j=1}^{\infty} (-1)^j a_j^{(3)}, \quad x \to 0^-. \tag{3.41}$$

By comparing formulas (3.39) and (3.41), we find that

$$K_{\rm III}^{(0)} = \frac{\nu_0}{2}\sqrt{2\pi a} \sum_{j=1}^{\infty} (-1)^j a_j^{(1)}, \quad K_{\rm IV}^{(0)} = \frac{\nu_2}{2}\sqrt{2\pi a} \sum_{j=1}^{\infty} (-1)^j a_j^{(3)}. \tag{3.42}$$

Similarly,

$$K_{\mathrm{III}}^{(a)} = -\frac{\nu_0}{2}\sqrt{2\pi a}\sum_{j=1}^{\infty} a_j^{(1)}, \quad K_{\mathrm{IV}}^{(a)} = -\frac{\nu_2}{2}\sqrt{2\pi a}\sum_{j=1}^{\infty} a_j^{(3)}. \tag{3.43}$$

Having determined formulas for the stress intensity factors $K_{\mathrm{III}}^{(c)}$ and $K_{\mathrm{IV}}^{(c)}$ ($c = 0, a$), we now introduce the associated weight functions $W_{j,m}^{(0)}$ and $W_{j,m}^{(a)}$, $j, m = \mathrm{III}, \mathrm{IV}$, such that

$$\begin{pmatrix} K_{\mathrm{III}}^{(c)} \\ K_{\mathrm{IV}}^{(c)} \end{pmatrix} = \int_0^a \begin{pmatrix} W_{\mathrm{III,III}}^{(c)}(\xi) & W_{\mathrm{III,IV}}^{(c)}(\xi) \\ W_{\mathrm{IV,III}}^{(c)}(\xi) & W_{\mathrm{IV,IV}}^{(c)}(\xi) \end{pmatrix} \begin{pmatrix} p_1(\xi) \\ p_3(\xi) \end{pmatrix} d\xi, \quad c = 0, a. \tag{3.44}$$

By choosing the functions $p_1(x)$ and $p_3(x)$ as $p_1(x) = \delta(x - \xi)$ and $p_3(x) = 0$ ($\delta(x)$ is the generalized δ-function), we find from (3.44)

$$W_{\mathrm{III,III}}^{(c)}(\xi) = K_{\mathrm{III}}^{(c,1)}, \quad W_{\mathrm{IV,III}}^{(c)}(\xi) = K_{\mathrm{IV}}^{(c,1)}, \quad c = 0, a, \tag{3.45}$$

where $K_{\mathrm{III}}^{(c,1)} = K_{\mathrm{III}}^{(c)}$ and $K_{\mathrm{IV}}^{(c,1)} = K_{\mathrm{IV}}^{(c)}$ are expressed through the solution $a_n^{(m)}$ ($n = 1, 2, \ldots; m = 1, 3$) of the infinite system (3.37) by (3.42) and (3.43). This system has to be solved with the special right-hand side

$$b_n^{(1)} = q_n, \quad b_n^{(3)} = 0, \quad q_n = (4/a^2)\sqrt{\xi(a-\xi)}\, U_n(2[\xi/a] - 1). \tag{3.46}$$

By solving next the infinite system (3.37) with the right-hand side

$$b_n^{(1)} = 0, \quad b_n^{(3)} = q_n, \tag{3.47}$$

we determine the associated coefficients $a_n^{(m)}$ ($n = 1, 2, \ldots, m = 1, 3$). By means of (3.42) and (3.43), the coefficients $K_{\mathrm{III}}^{(c,3)} = K_{\mathrm{III}}^{(c)}$ and $K_{\mathrm{IV}}^{(c,3)} = K_{\mathrm{IV}}^{(c)}$ may be computed and identified as the other weight functions

$$W_{\mathrm{III,IV}}^{(c)}(\xi) = K_{\mathrm{III}}^{(c,3)}, \quad W_{\mathrm{IV,IV}}^{(c)}(\xi) = K_{\mathrm{IV}}^{(c,3)}, \quad c = 0, a. \tag{3.48}$$

Analyse now the series representations of the the weight functions. Consider first $W_{\mathrm{III,III}}^{(c)}$ and $W_{\mathrm{IV,III}}^{(c)}$, $c = 0, a$. In this case, $b_n^{(1)} = q_n$ and $b_n^{(3)} = 0$. From the infinite system (3.37),

$$\begin{aligned} a_n^{(1)} &= \frac{2}{\pi\nu_0} q_{n-1} + \tilde{a}_n^{(1)}, \quad a_n^{(3)} = \hat{a}_n^{(3)} + \tilde{a}_n^{(3)}, \\ \hat{a}_n^{(3)} &= -\frac{4\nu_1}{\pi^2\nu_0\nu_2}(\sigma_{n-1}q_{n-2} - \sigma_{n+1}q_n), \quad \tilde{a}_n^{(m)} = o(n^{-1}), \quad m = 1, 3, \quad n \to \infty. \end{aligned} \tag{3.49}$$

Here, $\sigma_0 q_{-1} = 0$, $\tilde{a}_0^{(m)} = \hat{a}_0^{(m)} = 0$, $m = 1, 3$. Introduce next the power series

$$F(z) = \nu_0\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}(-1)^j a_j^{(1)} z^j, \tag{3.50}$$

which is an analytic function of z in the open disc $|z| < 1$. On substituting $a_n^{(1)} = 2(\pi\nu_0)^{-1}q_{n-1} + \tilde{a}_n^{(1)}$ along with (3.46) into (3.50) and employing the generating function for the Chebyshev polynomials $U_n(x)$

$$\sum_{j=0}^{\infty} U_j(x)z^j = \frac{1}{1-2xz+z^2}, \quad |z|<1, \tag{3.51}$$

we obtain

$$F(z) = \nu_0\sqrt{\frac{\pi a}{2}}\left[-\frac{8z}{\pi a\nu_0}\frac{\sqrt{\xi(a-\xi)}}{a+2(2\xi-a)z+az^2} + \sum_{j=1}^{\infty}(-1)^j\tilde{a}_j^{(1)}z^j\right]. \tag{3.52}$$

By evaluating the limit of $F(z)$ as $z \to 1^-$, we determine the factor $K_{\mathrm{III}}^{(0)}$ for the loading functions $p_1(x) = \delta(x-\xi)$, $p_3(x) = 0$. That factor is the weight function $W_{\mathrm{III,III}}^{(0)}$ given by

$$W_{\mathrm{III,III}}^{(0)} = -\sqrt{\frac{2(a-\xi)}{\pi a\xi}} + \nu_0\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}(-1)^j\tilde{a}_j^{(1)}, \tag{3.53}$$

and $\tilde{a}_j^{(1)} = o(j^{-1})$ as $j \to \infty$. Similarly,

$$W_{\mathrm{III,III}}^{(a)} = -\sqrt{\frac{2\xi}{\pi a(a-\xi)}} - \nu_0\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}\tilde{a}_j^{(1)}. \tag{3.54}$$

We wish to evaluate next the weight functions $W_{\mathrm{IV,III}}^{(c)}$ or, equivalently, the factors $K_{\mathrm{IV}}^{(c)}$, $c = 0, a$, as $p_1(x) = \delta(x-\xi)$, $p_3(x) = 0$. Since

$$\sum_{j=1}^{\infty}(\sigma_{j-1}q_{j-2} - \sigma_{j+1}q_j) = \sigma_1 q_0, \tag{3.55}$$

from (3.42) and (3.49),

$$W_{\mathrm{IV,III}}^{(0)} = -\frac{2\nu_1}{\nu_0 a}\sqrt{\frac{2\xi(a-\xi)}{\pi a}} + \nu_2\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}(-1)^j\tilde{a}_j^{(3)},$$

$$W_{\mathrm{IV,III}}^{(a)} = \frac{2\nu_1}{\nu_0 a}\sqrt{\frac{2\xi(a-\xi)}{\pi a}} - \nu_2\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}\tilde{a}_j^{(3)}.$$

By solving the system (3.37) with $b_n^{(1)} = 0$ and $b_n^{(3)} = q_n$, we determine a new set of the coefficients $a_n^{(1)}$ and $a_n^{(3)}$. These coefficients admit the representations

$$a_n^{(3)} = \frac{2}{\pi\nu_2}q_{n-1} + \tilde{a}_n^{(3)}, \quad a_n^{(1)} = \hat{a}_n^{(1)} + \tilde{a}_n^{(1)},$$

$$\hat{a}_n^{(1)} = \frac{4\nu_1}{\pi^2\nu_0\nu_2}(\sigma_{n-1}q_{n-2} - \sigma_{n+1}q_n), \quad \tilde{a}_n^{(m)} = o(n^{-1}), \quad m = 1, 3, \quad n \to \infty.$$

Similarly to the previous case,

$$W_{\mathrm{III,IV}}^{(0)} = \frac{2\nu_1}{\nu_2 a}\sqrt{\frac{2\xi(a-\xi)}{\pi a}} + \nu_0\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}(-1)^j \tilde{a}_j^{(1)},$$

$$W_{\mathrm{III,IV}}^{(a)} = -\frac{2\nu_1}{\nu_2 a}\sqrt{\frac{2\xi(a-\xi)}{\pi a}} - \nu_0\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty} \tilde{a}_j^{(1)},$$

$$W_{\mathrm{IV,IV}}^{(0)} = -\sqrt{\frac{2(a-\xi)}{\pi a\xi}} + \nu_2\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty}(-1)^j \tilde{a}_j^{(3)},$$

$$W_{\mathrm{IV,IV}}^{(a)} = -\sqrt{\frac{2\xi}{\pi a(a-\xi)}} - \nu_2\sqrt{\frac{\pi a}{2}}\sum_{j=1}^{\infty} \tilde{a}_j^{(3)}. \tag{3.56}$$

It is seen that the weight functions are expressed in terms of the new coefficients $\tilde{a}_n^{(1)}$ and $\tilde{a}_n^{(3)}$. To determine these coefficients associated with the weight functions $W_{\mathrm{III,III}}^{(c)}$ and $W_{\mathrm{IV,III}}^{(c)}$ $(c = 0, a)$, one needs to solve the infinite system (3.37) with the right-hand side $b_n^{(1)} = \tilde{b}_n^{(1)}$ and $b_n^{(3)} = \tilde{b}_n^{(3)}$, where

$$\tilde{b}_n^{(1)} = -\frac{4\nu_1^2}{\pi^2\nu_0\nu_2}\left[\sigma_n(\sigma_{n-1}q_{n-2} - \sigma_{n+1}q_n) - \sigma_{n+2}(\sigma_{n+1}q_n - \sigma_{n+3}q_{n+2})\right]$$
$$- \frac{2}{\pi\nu_0}\sum_{j=1}^{\infty} c_{nj}^{(1,1)} q_{j-1} + \frac{4\nu_1}{\pi^2\nu_0\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(1,3)}(\sigma_{j-1}q_{j-2} - \sigma_{j+1}q_j),$$

$$\tilde{b}_n^{(3)} = \frac{2}{\pi\nu_0}\sum_{j=1}^{\infty} c_{nj}^{(1,3)} q_{j-1} + \frac{4\nu_1}{\pi^2\nu_0\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(3,3)}(\sigma_{j-1}q_{j-2} - \sigma_{j+1}q_j), \tag{3.57}$$

where $\sigma_0 = \sigma_{-1} = 0$. To make computations of the weight functions $W_{\mathrm{III,IV}}^{(c)}$ and $W_{\mathrm{IV,IV}}^{(c)}$ $(c = 0, a)$ by means of (3.56), we have to replace the right-hand side in (3.37) by $\tilde{b}_n^{(1)}$ and $\tilde{b}_n^{(3)}$ given by

$$\tilde{b}_n^{(1)} = -\frac{4\nu_1}{\pi^2\nu_0\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(1,1)}(\sigma_{j-1}q_{j-2} - \sigma_{j+1}q_j) - \frac{2}{\pi\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(1,3)} q_{j-1},$$

$$\tilde{b}_n^{(3)} = -\frac{4\nu_1^2}{\pi^2\nu_0\nu_2}\left[\sigma_n(\sigma_{n-1}q_{n-2} - \sigma_{n+1}q_n) - \sigma_{n+2}(\sigma_{n+1}q_n - \sigma_{n+3}q_{n+2})\right]$$
$$+ \frac{4\nu_1}{\pi^2\nu_0\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(1,3)}(\sigma_{j-1}q_{j-2} - \sigma_{j+1}q_j) - \frac{2}{\pi\nu_2}\sum_{j=1}^{\infty} c_{nj}^{(3,3)} q_{j-1} \tag{3.58}$$

and solve the infinite system for the coefficients $a_n^{(1)} = \tilde{a}_n^{(1)}$ and $a_n^{(3)} = \tilde{a}_n^{(3)}$. It can be directly verified that in both cases, (3.57) and (3.58), the coefficients $\tilde{b}_n^{(1)}$ and $\tilde{b}_n^{(3)}$ decay as $n \to \infty$, and $\tilde{b}_n^{(m)} = o(n^{-1})$, $n \to \infty$, $m = 1, 3$.

The diagonal weight functions $W_{\mathrm{III,III}}^{(c)}(\xi)$ and $W_{\mathrm{IV,IV}}^{(c)}(\xi)$ $(c = 0, a)$ as functions of ξ when $\alpha/\mu = 1$, $\beta/\mu = 1$, $\gamma/\mu = 2$ and $\kappa/\mu = 1$ are plotted in Fig. 5. The off-diagonal functions are presented in Fig. 6. It is seen that when $\xi \to c$, then the absolute values of the diagonal functions $W_{\mathrm{III,III}}^{(c)}(\xi)$ and $W_{\mathrm{IV,IV}}^{(c)}(\xi)$ grow to infinity, while the absolute values of the other two diagonal functions become

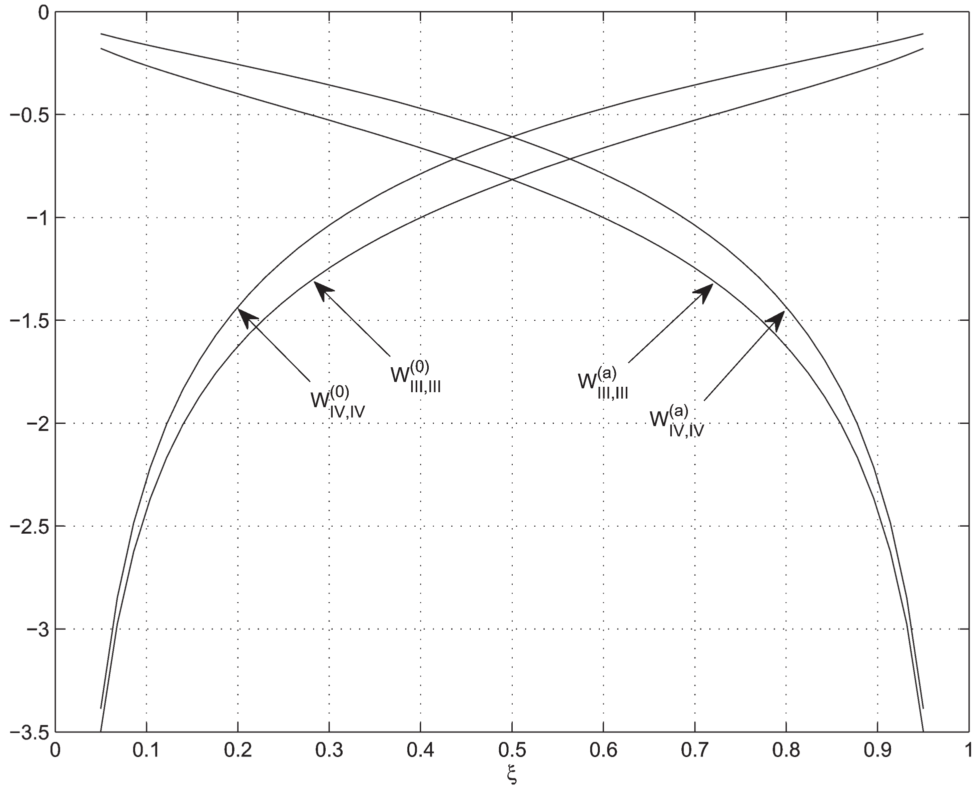

Fig. 5 The weight functions $W^{(0)}_{\text{III,III}}(\xi)$, $W^{(a)}_{\text{III,III}}(\xi)$, $W^{(0)}_{\text{IV,IV}}(\xi)$ and $W^{(a)}_{\text{IV,IV}}(\xi)$, for the parameters $a = 1$, $\alpha/\mu = 1$, $\beta/\mu = 1$, $\gamma/\mu = 2$ and $\kappa/\mu = 1$

small. The absolute values of the four off-diagonal functions are small as ξ approach both tips of the crack. Figure 7 shows the effect on the weight functions of changing the parameter κ when $\xi = 0.5a$ and therefore, $W^{(0)}_{mj} = W^{(a)}_{mj}$, $m, j = \text{III, IV}$. Regardless of ξ, as $\kappa \to 0^+$, the diagonal functions $W^{(c)}_{\text{III,III}}$ and $W^{(c)}_{\text{IV,IV}}$ tend to the classical elastic mode-I weight functions for a finite crack, while the off-diagonal functions $W^{(c)}_{\text{III,IV}}$ and $W^{(c)}_{\text{IV,III}}$ vanish.

4. Conclusions

We have studied two 2D problems of a crack in an unbounded solid modelled in the framework of the unconstrained couple–stress theory (micropolar elasticity). In the first problem, the crack is semi-infinite, and the conditions of plane strain are assumed. That problem has been reduced to a scalar RHP for mode-II (the shear τ_{yx}-mode) and an order-2 vector RHP for modes I and VI, the σ_y- and m_{yz}-modes, respectively. These two problems have been solved by quadratures. We have found the stress intensity factors and the five associated weight functions, one for the uncoupled mode and two for each of the coupled modes. It has been shown that if the micropolar parameter $\kappa \to 0^+$

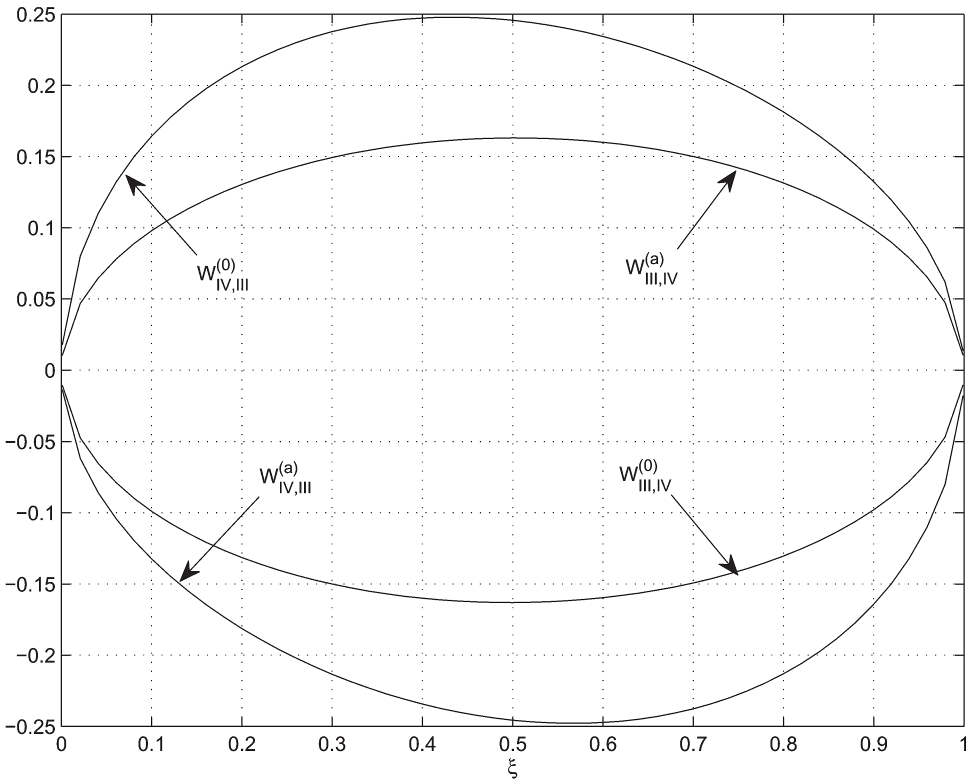

Fig. 6 The weight functions $W^{(0)}_{\mathrm{III,IV}}(\xi)$, $W^{(a)}_{\mathrm{III,IV}}(\xi)$, $W^{(0)}_{\mathrm{IV,III}}(\xi)$ and $W^{(a)}_{\mathrm{IV,III}}(\xi)$, for the parameters $a = 1$, $\alpha/\mu = 1$, $\beta/\mu = 1$, $\gamma/\mu = 2$ and $\kappa/\mu = 1$

and $\gamma \neq 0$, then the stress intensity factors and the diagonal weight functions $W_{\mathrm{I,I}}$, W_{II} and $W_{\mathrm{VI,VI}}$ tend to the function $W_0(\xi) = -\sqrt{2/(\pi\xi)}$, the weight function for the modes I and II in the classical elasticity, while the off-diagonal functions $W_{\mathrm{I,VI}}$ and $W_{\mathrm{VI,I}}$ vanish. If $\gamma \to 0^+$ and $\kappa = o(\gamma)$, then the couple–stress $m_{yz} \to 0$, $W_{\mathrm{I,I}}$ and W_{II} tend to $W_0(\xi)$, and the other weight functions vanish.

The second model problem concerns the antiplane-strain problem of micropolar elasticity for a crack subject to loading $\tau_{yz} = p_1(x)$, $m_{yx} = p_2(x)$, $m_{yy} = p_3(x)$. We have shown that the mode V (the m_{yx}-mode) is uncoupled and found the associated weight function W_{V} by quadratures. The other two modes, III and IV (the τ_{yz}- and m_{yy}-modes), are coupled. It turned out that to find the weight functions for these modes, we needed to solve a vector RHP with the coefficient admitted the representation $G = b_0 Q_0 + b_1 Q_1 + b_2 Q_2$ (b_j are Hölder functions and Q_j are 2×2 polynomial matrices). We were not able to factorize this matrix. However, we have managed to determine that it has zero partial indices. Therefore, they are stable, and the number of arbitrary constants in the solution of the RHP coincides with the number of solvability conditions.

For a finite antiplane-strain crack, we have derived a system of singular integral equations and solved it approximately by the method of orthogonal polynomials. The weight functions have been found in a series form in terms of the solution of an infinite system of linear algebraic equations. As in the plane-strain case, if the micropolar parameter $\kappa \to 0^+$, while the other parameters meet the

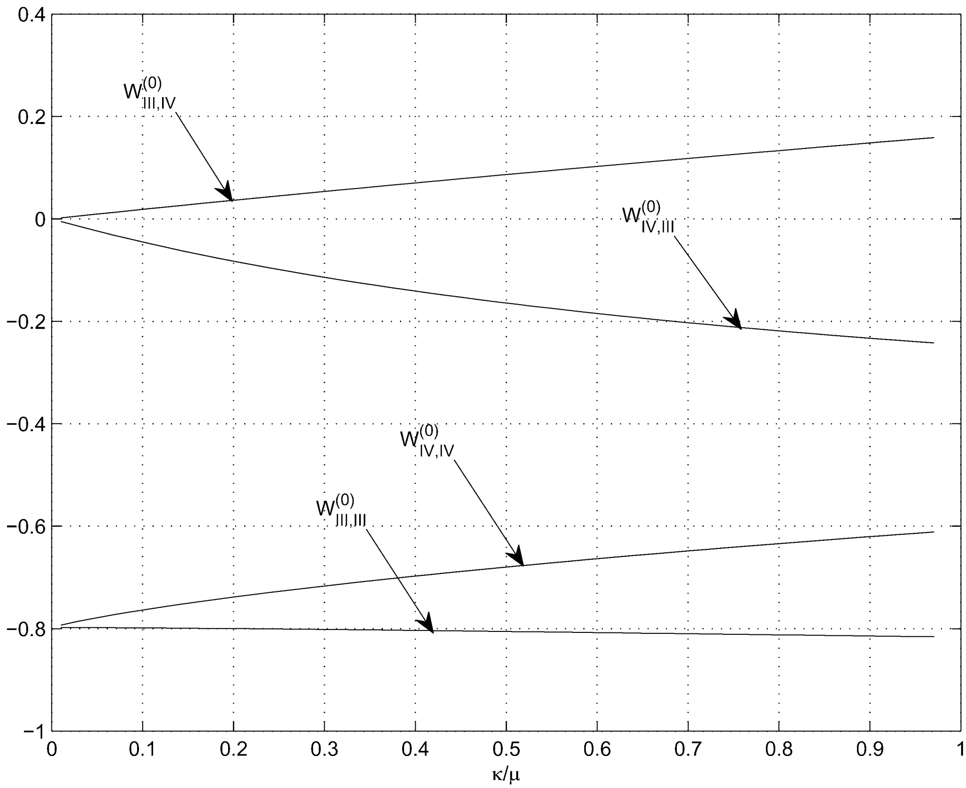

Fig. 7 The weight functions $W^{(0)}_{III,III}(\xi)$, $W^{(0)}_{III,IV}(\xi)$, $W^{(0)}_{IV,III}(\xi)$ and $W^{(0)}_{IV,IV}(\xi)$ versus κ/μ when $\xi = 0.5a$, $a = 1$, $\alpha/\mu = 1$, $\beta/\mu = 1$ and $\gamma/\mu = 2$

physical conditions $\gamma > |\beta|$, $\mu > 0$, $3\alpha + \beta + \gamma > 0$, then the weight functions $W_{III,III}$ and $W_{IV,IV}$ associated with the tips of the crack coincide with the classical elastic weight functions for a finite mode-III crack. The other weight functions, $W_{III,IV}$ and $W_{IV,III}$, tend to zero.

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