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Diffraction of a plane wave by a right-angled penetrable wedge

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[1] The problem of electromagnetic diffraction (E-polarization) by two screens is analyzed. The screens are half-planes which form a right-angled wedge. The first half-plane is an electrically resistive sheet, and the second one is a perfectly magnetically conductive surface (ideal ferrite). The problem is formulated as a boundary-value problem for the Helmholtz equation with respect to the E_z-component of the electric field. On the conductive screen, the normal derivative of the function E_z vanishes. On the resistive half-plane, the function E_z is continuous and it is proportional to the jump of its normal derivative. The Sommerfeld integral representation is used to convert the problem to a difference equation of the second order. For a special value of the impedance parameter the problem reduces to two scalar Riemann-Hilbert (RH) problems on a segment with coefficients having a pole and a zero on the segment. The general solution to the RH problems is derived by quadratures. The RH problems are equivalent to the governing boundary-value problem when certain conditions are satisfied. These conditions are used to determine unknown meromorphic functions in the solution of the RH problems. Exact formulas for the reflected, transmitted, and diffracted waves are derived, and numerical results for the diffraction coefficient are reported.

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1. Introduction

[2] The geometrical theory of diffraction (GTD) is the most widely used theory for analysis of the edge diffraction. The GTD in the context of the time harmonic theory [*Keller*, 1962] deals with high-frequency asymptotic expansions of electromagnetic fields scattered from structures. It requires the solution of canonical problems on scattering of radio waves in order to recover transmitted, surface, and diffracted waves or, equivalently, to evaluate transmission, surface and diffraction coefficients. High fidelity analytical methods are essential for solving the canonical problems. Solution of these problems by pure numerical schemes is not always simple, and it is often costly in computational time. Considerable insight to the problem can be obtained if the solution is carried out analytically.

[3] Reliable detection of targets, specification of antenna ground plates and radar scattering evaluation requires a further investigation into electromagnetic scattering by plates and wedges. The analysis of half-plane diffraction problems is generally carried out using the Wiener-Hopf technique or the Clemmow dual integral equations method [*Senior and Volakis*, 1995, chapter 3]. Model problems for the exterior of wedges with impedance faces are effectively treated by the *Maliuzhinets* [1958] technique. This method is based on the use of the classical Sommerfeld integral representation of the solution to the Helmholtz equation and requires the solution of certain difference equations. When the electric and magnetic fields are not coupled by the boundary conditions and the corresponding difference equations are scalar, the solution can be derived in terms of special functions (known as the Maliuzhinets functions). For penetrative wedges, when the electromagnetic field has to be recovered in the interior and the exterior of the domain, in general, the difference equations are of the second order. For an arbitrary wedge angle, an analytical technique is not available in the literature. For a special case of the second-order difference equation, by using bilinear Riemann relations for abelian differentials, a partial solution was analyzed by *Senior and Legault* [2000] and

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Legault and Senior [2002]. A multi-valued non-physical solution for some canonical diffraction problems on right-angled scatterers was proposed by *Demetrescu et al.* [1998a, 1998b].

[4] Recently, *Antipov and Silvestrov* [2004a, 2004b] developed a novel method for second-order difference equations with meromorphic periodic coefficients and applied it to a diffraction problem for a right-angled wedge. One of the sheets of the scatterer was a conductive surface, and the second one was perfectly conductive. This method is a two-step-procedure which requires first to reduce the model problem to a scalar RH problem on a Riemann surface [*Antipov and Silvestrov*, 2004c] (in particular cases, this results in two scalar RH problems on a complex plane). The general single-valued meromorphic solution to the RH problems is derived in terms of 2π-periodic meromorphic functions with specified zeros and poles. These functions are arbitrary for the solution of the RH problem but not free for the main difference equation. The second step of the method is to find these functions from certain extra conditions. In the case considered by *Antipov and Silvestrov* [2004b], the coefficients $l_1(t)$ and $l_2(t)$ of the RH problems are continuous on the contour and do not vanish. It turns out that for some boundary conditions the coefficients of the associated RH problems may have poles and zeros on the contour and therefore the functions log $l_j(t)$ are multi-valued. The procedure by *Antipov and Silvestrov* [2004b] if remains unchanged does not work for this multi-valued case. The main aims of the current paper are as follows: (1) to analyze the diffraction problem for a right-angled scatterer formed by an electrically resistive half-plane and a perfectly magnetically conductive half-plane; (2) to develop further the theory of second-order difference equations of diffraction theory focusing on the case when the coefficients of the associated RH problems have zeros and poles on the contour; (3) to determine the far field asymptotic expansion of the electric field.

[5] It is known [*Senior and Volakis*, 1995, p. 53] that for an electrically resistive half-plane, the E_z-component of the electric field is continuous, and the H_x-component of the magnetic field is discontinuous:

$$[E_z]_-^+ = 0, \quad E_z = -R_e[H_x]_-^+,$$
$$x > 0,\ y = 0,\ -\infty < z < \infty, \qquad (1)$$

where R_e denotes the surface resistivity, and $[f]_-^+ = f|_{y=+0} - f|_{y=-0}$. The electromagnetic dual of an electrically resistive screen is a magnetically conductive one. The magnetically conductive sheet boundary conditions stipulate the continuity of the H_x-component and the discontinuity of the E_z-component [*Senior and Volakis*, 1995, p. 74]:

$$[H_x]_-^+ = 0, \quad H_x = -R_m[E_z]_-^+,$$
$$x > 0,\ y = 0,\ -\infty < z < \infty, \qquad (2)$$

where R_m is the conductivity. In the case of normal incidence (the incident wave E_z^i is orthogonal to the z-axis), the E_z- and H_x-components are linked by

$$H_x = -\frac{1}{ik_0 Z_0}\frac{\partial E_z}{\partial y}, \qquad (3)$$

where k_0 is the wave number, and Z_0 is the intrinsic impedance of the medium. Therefore, if $R_e = 0$ (a perfectly electrically conductive sheet), then the E_z-component is continuous and vanishes on the faces of the screen while its normal derivative is discontinuous. In the case $R_m = 0$ (a perfectly magnetically conductive sheet), the function $\partial E_z/\partial y$ is continuous and equal to zero, while the E_z-component is discontinuous. If $R_e = \infty$ in the case (1) or $R_m = \infty$ in the case (2), then the sheet ceases to exist.

[6] The paper is organized as follows. In section 2, the canonical problem of diffraction by a right-angled wedge with electrically resistive and perfectly magnetically conductive surfaces is formulated. It is reduced to a certain difference equation in section 3. In the next section, to solve this difference equation we analyze an auxiliary second-order difference equation and convert it into two RH problems. For the particular case $Z_0/R_e = 4/\sqrt{3}$, the general solution to the auxiliary difference equation is derived in terms of some unknown meromorphic periodic functions. In section 5, we establish under which conditions the auxiliary equation is equivalent to the main difference equation. These conditions are used to find the unknown meromorphic functions from section 4. An asymptotic expansion of the E_z-component of the electric field far away from the junction of the screens is derived in section 6. Numerical results for the diffraction coefficient are also reported.

2. Formulation

[7] Consider diffraction of a time-harmonic plane wave of unit amplitude,

$$E_z^i = e^{ik_0 r\cos(\varphi-\varphi_0)+i\omega t}, \qquad (4)$$

by two half-planes, $W_1 = \{0 < r < \infty,\ \varphi = 3\pi/4\}$ and $W_2 = \{0 < r < \infty,\ \varphi = 5\pi/4\}$ in a medium $\{0 < r < \infty, -3\pi/4 < \varphi < 5\pi/4\}$ with permittivity ε, permeability μ, and intrinsic impedance Z_0. The incident wave is normal to

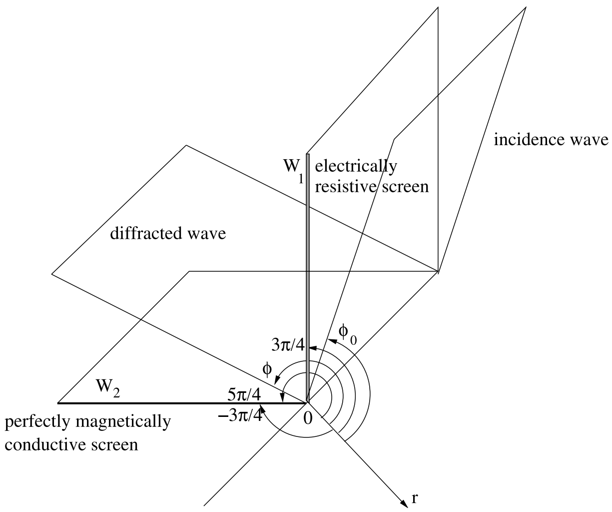

Figure 1. Geometry of the problem.

the z-axis and is traveling at an angle φ_0 ($|\varphi_0| < 3\pi/4$) to the plane $\varphi = 0$. Here k_0 is the wave number and $\omega = k_0/\sqrt{\varepsilon\mu}$. The time factor $e^{i\omega t}$ will be suppressed henceforth. It is assumed that the first screen W_1 is an electrically resistive surface, and the second screen W_2 is a perfectly magnetically conductive sheet (Figure 1).

[8] By expressing the component H_r of the magnetic field through the component E_z of the electric field,

$$Z_0 H_r = -\frac{1}{ik_0 r}\frac{\partial E_z}{\partial \varphi}, \tag{5}$$

we can transform the electrically resistive boundary conditions (1) on the sheet W_1 to the form

$$E_z|_{\varphi=3\pi/4-0} - E_z|_{\varphi=3\pi/4+0} = 0, \quad 0 < r < \infty,$$
$$\frac{1}{r}\frac{\partial}{\partial\varphi}\left(E_z|_{\varphi=3\pi/4-0} - E_z|_{\varphi=3\pi/4+0}\right) + 2ik_0\gamma E_z|_{\varphi=3\pi/4-0} = 0, \quad 0 < r < \infty. \tag{6}$$

Here $\gamma = Z_0/(2R_e) \neq 0$ and R_e is the surface resistivity. It will be convenient to represent the complex parameter γ as $\gamma = \sin\theta$, $0 < \mathrm{Re}\,\theta < \pi/2$.

[9] On the second sheet W_2, a perfectly magnetically conductive surface, the parameter R_m vanishes, and the boundary conditions (2) become

$$\frac{\partial}{\partial\varphi}E_z|_{\varphi=5\pi/4-0} = \frac{\partial}{\partial\varphi}E_z|_{\varphi=-3\pi/4+0} = 0, \quad 0 < r < \infty. \tag{7}$$

Thus, the canonical diffraction problem reduces to the Helmholtz equation in the free space for the E_z-component of the electric field

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\varphi^2} + k_0^2\right)E_z = 0,$$
$$0 < r < \infty, \ \varphi \in \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \tag{8}$$

which has to satisfy the boundary conditions (6) and (7) on the lines $\varphi = 3\pi/4$ and $\varphi = 5\pi/4$, respectively.

3. Main Difference Equation

[10] To solve the problem, we use the Sommerfeld integral representations

$$E_z(r,\varphi) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{ik_0 r\cos s}\mathcal{S}_1(s+\varphi)ds, \quad |\varphi| < \frac{3\pi}{4},$$
$$E_z(r,\varphi) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{ik_0 r\cos s}\mathcal{S}_2(s+\varphi-\pi)ds,$$
$$\frac{3\pi}{4} < \varphi < \frac{5\pi}{4}. \tag{9}$$

which satisfy the Helmholtz equation. Here $\mathcal{L}$ is the Sommerfeld double loop contour (Figure 2). It is symmetric with respect to the origin $s = 0$. For large positive Im s, the upper contour asymptotically approaches the lines $s = 3\pi/2$ (from the left) and $s = -\pi/2$ (from the right), whilst the lower contour approaches the lines $s = \pi/2$ and $s = -3\pi/2$. The function $\mathcal{S}_1(s)$ is analytic in the strip $|\mathrm{Re}\, s| < 3\pi/4$ and continuous in the strip up to the boundary $|\mathrm{Re}\, s| = 3\pi/4$ apart from the geometrical optics pole at the point $s = \varphi_0$ with the prescribed residue

$$\mathrm{res}_{s=\varphi_0}\mathcal{S}_1(s) = 1. \tag{10}$$

The second spectral function $\mathcal{S}_2(s)$ is analytic in the strip $|\mathrm{Re}\, s| < \pi/4$ and continuous everywhere in the strip $|\mathrm{Re}\, s| \leq \pi/4$. At infinity, as Im $s \to \pm\infty$ and Re s is finite, both functions are assumed to be bounded: $|\mathcal{S}_j(s)| \leq$ const, $j = 1, 2$.

[11] To determine the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ and to satisfy the boundary conditions, we substitute the integral representations (9) into the boundary conditions (6) and (7). Then, taking into account the symmetry of the contour $\mathcal{L}$ with respect to the origin,

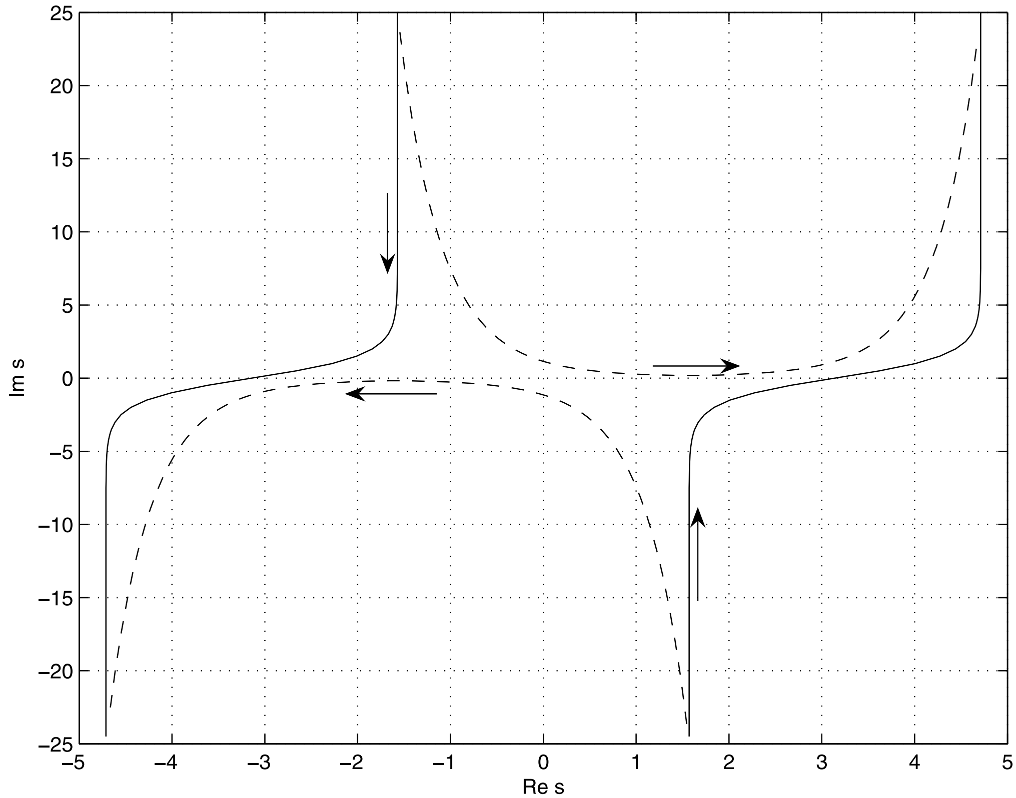

Figure 2. Sommerfeld double loop contour $\mathcal{L}$ (the broken lines) and the two steepest descent paths (the solid lines).

we arrive at the following system of difference equations

$$\begin{aligned}
&\mathcal{S}_1\left(s-\frac{3\pi}{4}\right) = -\mathcal{S}_1\left(-s-\frac{3\pi}{4}\right), \quad \mathcal{S}_2\left(s+\frac{\pi}{4}\right) \\
&\qquad = -\mathcal{S}_2\left(-s+\frac{\pi}{4}\right), \\
&(\sin s+\gamma)\left[\mathcal{S}_1\left(s+\frac{3\pi}{4}\right) - \mathcal{S}_2\left(-s-\frac{\pi}{4}\right)\right] \\
&\qquad = (\sin s-\gamma)\left[-\mathcal{S}_1\left(-s+\frac{3\pi}{4}\right) + \mathcal{S}_2\left(s-\frac{\pi}{4}\right)\right], \\
&\mathcal{S}_1\left(s+\frac{3\pi}{4}\right) + \mathcal{S}_2\left(-s-\frac{\pi}{4}\right) = \mathcal{S}_1\left(-s+\frac{3\pi}{4}\right) \\
&\quad + \mathcal{S}_2\left(s-\frac{\pi}{4}\right).
\end{aligned} \tag{11}$$

By using the first two equations in (11), we find

$$\begin{aligned}
\mathcal{S}_1\left(-s+\frac{3\pi}{4}\right) &= -\mathcal{S}_1\left(s-\frac{9\pi}{4}\right), \\
\mathcal{S}_2\left(-s-\frac{\pi}{4}\right) &= -\mathcal{S}_2\left(s+\frac{3\pi}{4}\right),
\end{aligned} \tag{12}$$

Then we transform the third and the forth equations in (11) to new ones with respect to the terms $\mathcal{S}_1(s + 3\pi/4)$, $\mathcal{S}_1(s - 9\pi/4)$, $\mathcal{S}_2(s + 3\pi/4)$, and $\mathcal{S}_2(s - \pi/4)$. It will be helpful to express the function $\mathcal{S}_2(s - \pi/4)$ through $\mathcal{S}_1(s + 3\pi/4)$ and $\mathcal{S}_1(s - 9\pi/4)$. This can be done by eliminating the function $\mathcal{S}_2(s + 3\pi/4)$ from those two new equations. After the substitution $s - \pi/4 = s'$, we have

$$\begin{aligned}
\mathcal{S}_2(s) &= \left(1+\frac{\gamma}{\sin(s+\pi/4)}\right)\mathcal{S}_1(s+\pi) \\
&\quad + \frac{\gamma}{\sin(s+\pi/4)}\mathcal{S}_1(s-2\pi).
\end{aligned} \tag{13}$$

Thus, the function $\mathcal{S}_2(s)$ can be excluded from (11). By doing this, we derive a single difference equation for the function $\mathcal{S}_1$

$$(\gamma - \sin s)\left[\mathcal{S}_1\left(s + \frac{7\pi}{4}\right) - \mathcal{S}_1\left(s - \frac{9\pi}{4}\right)\right] = \gamma\left[\mathcal{S}_1\left(s + \frac{3\pi}{4}\right) - \mathcal{S}_1\left(s - \frac{5\pi}{4}\right)\right]. \quad (14)$$

By following *Maliuzhinets* [1958], in order to eliminate the pole at the point $s = \varphi_0$, we split the function $\mathcal{S}_1(s)$ into two parts

$$\mathcal{S}_1(s) = \Sigma(s)\psi\left(s + \frac{3\pi}{4}\right), \quad \Sigma(s) = \frac{2\cos 2\varphi_0}{\sin 2s - \sin 2\varphi_0}, \quad (15)$$

where the function $\psi(s)$ is free of the geometrical optics pole of the function $\mathcal{S}_1(s)$. Equation (14) is a difference equation of order 3. To reduce it to a new one of order 2, we introduce a function

$$f(s) = \psi(s + \pi) - \psi(s - \pi) \quad (16)$$

and transform equation (14) to the form

$$(\sin\theta + \cos s)[f(s + \pi) + f(s - \pi)] = \sin\theta f(s), \quad (17)$$

where $\sin\theta = \gamma$. It follows from (15), (12), and (16) that

$$\psi(s) = -\psi(-s), \quad f(s) = f(-s). \quad (18)$$

At infinity, the function $f(s)$ may grow:

$$|f(s)| \le Be^{2|\mathrm{Im}\, s|}, \quad \mathrm{Im}\, s \to \infty, \quad \mathrm{Re}\, s \text{ is finite}, \quad B = \mathrm{const}. \quad (19)$$

[12] It is clear that in order to recover the function $\mathcal{S}_1(s)$, it is required to express the function $\psi(s)$ through the solution to equation (17). From (16), we find the most general form of the function $\psi(s)$

$$\psi(s) = C_1 \sin s + C_2 \sin 2s + \mathcal{F}(s),$$
$$\mathcal{F}(s) = \frac{\sin s}{4\pi i}\int_\Omega \left(\cot\frac{\tau - s}{2} - \cot\frac{\tau - s_0}{2}\right) \cdot \frac{f(\tau - \pi)d\tau}{\sin\tau}, \quad -\pi < \mathrm{Re}\, s < \pi. \quad (20)$$

Here C_1 and C_2 are constants to be fixed, $\Omega = \{s \in \mathbf{C}_s\colon \mathrm{Re}\ s = \pi\}$, $\mathbf{C}_s$ is a complex plane, and s_0 is an arbitrary fixed internal point in the strip $\Pi = \{s \in \mathbf{C}_s\colon -\pi < \mathrm{Re}\ s < \pi\}$ such that $\mathrm{Re}\ s_0 \neq 0$. The choice of the kernel guarantees the convergence of the integral (20) with the density growing at infinity as in (19). The values of the function $\psi(s)$ as $\mathrm{Re}\ s = \pm\pi$ are recovered by the Sokhotski-Plemelj formulas for the Hilbert kernel (a periodic analogue of the Cauchy kernel)

$$\psi(\sigma \pm \pi) = -C_1 \sin\sigma + C_2 \sin 2\sigma \pm \frac{f(\sigma)}{2} + \frac{\sin\sigma}{4\pi i}\int_\Omega \left(\tan\frac{\tau - \sigma}{2} + \cot\frac{\tau - s_0}{2}\right) \cdot \frac{f(\tau - \pi)d\tau}{\sin\tau} \quad \mathrm{Re}\,\sigma = 0, \quad (21)$$

where the integral in (21) is understood in the sense of the principal value.

4. Auxiliary Second-Order Difference Equation

[13] The main step of the method is to determine the function $f(s)$. If this function is known, then the E_z-component of the electric field is found by formulas (9), (13), (15), and (20). The function $f(s)$ satisfies the difference equation (17). A second-order difference equation

$$a(s)f(s + h) + b(s)f(s) + c(s)f(s - h) = d(s) \quad (22)$$

with meromorphic T-periodic coefficients $a(s)$, $b(s)$, and $c(s)$ can be reduced to a scalar Riemann-Hilbert problem on a Riemann surface [*Antipov and Silvestrov*, 2004a] if $h = mT$, where m is an integer. In certain cases, the genus of the surface equals zero, and the equation is equivalent to two scalar Riemann-Hilbert problems on a complex plane. Clearly, in the case of equation (17), $h = \pi$, $T = 2\pi$, and the condition $h = mT$ is not valid (m is not an integer in this case). To transform the equation to another one which meets the condition $h = mT$ with m being an integer, replace first s by $s + \pi$ and then s by $s - \pi$, sum the new equations and finally obtain

$$\left(\cos^2 s - \sin^2\theta\right)[f(s + 2\pi) + f(s - 2\pi)] + \left(2\cos^2 s - \sin^2\theta\right)f(s) = 0. \quad (23)$$

Now we have $T = \pi$, $h = 2\pi$, $m = 2$, and the technique by *Antipov and Silvestrov* [2004a] may be applied to equation (23). Obviously, the main equation (17) and the new one (23) are not equivalent. Not each solution to equation (23) satisfies equation (17). However, each solution to equation (17) is a solution to equation (23). That is why it is crucial to separate the functions which do not satisfy equation (17) from the general solution to the auxiliary equation (23).

[14] First we need to construct the general solution to the auxiliary equation (23). By following *Antipov and Silvestrov* [2004a] we introduce two new functions

$$\Phi_1(s) = f(s), \quad \Phi_2(s) = f(s + 2\pi),$$
$$s \in \bar{\Pi} = \{s \in \mathbf{C}_s : -\pi \leq \operatorname{Re} s \leq \pi\}, \tag{24}$$

and rewrite equation (23) as a system of two difference equations of the first order

$$\mathbf{\Phi}(\sigma) = \mathbf{G}(\sigma)\mathbf{\Phi}(\sigma - 2\pi), \quad \sigma \in \Omega, \tag{25}$$

where

$$\mathbf{\Phi}(s) = \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix}, \quad \mathbf{G}(s) = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{2\cos^2 s - \sin^2\theta}{\cos^2 s - \sin^2\theta} \end{pmatrix}. \tag{26}$$

In general, the eigenvalues of the matrix $\mathbf{G}(s)$ are two-valued functions

$$\lambda_j(s) = \frac{\sin^2\theta - 2\cos^2 s + (-1)^{j-1}\Delta^{1/2}(s)}{2(\cos^2 s - \sin^2\theta)}, \quad j = 1, 2, \tag{27}$$

where

$$\Delta(s) = \sin^2\theta(4\cos^2 s - 3\sin^2\theta). \tag{28}$$

In what follows we confine ourselves to considering the case when the function $\Delta^{1/2}(s)$ does not have branch points, namely when $\sin\theta = 2/\sqrt{3}$ ($\theta = \pi/2 \pm i\log\sqrt{3}$). In this case, the eigenvalues become single-valued

$$\lambda_1(s) = \frac{i + \sqrt{3}\sin s}{i - \sqrt{3}\sin s}, \quad \lambda_2(s) = \frac{1}{\lambda_1(s)}, \tag{29}$$

and the problem (25) can be decoupled

$$\mathbf{G}(s) = \mathbf{T}(s)\mathbf{\Lambda}(s)[\mathbf{T}(s - 2\pi)]^{-1},$$
$$\mathbf{\Lambda}(s) = \begin{pmatrix} \lambda_1(s) & 0 \\ 0 & \lambda_2(s) \end{pmatrix}, \tag{30}$$

by using the following single-valued transformation matrix

$$\mathbf{T}(s) = \begin{pmatrix} 1 & 1 \\ \lambda_1(s) & \lambda_2(s) \end{pmatrix}. \tag{31}$$

The next step of the solution procedure is to introduce a new vector $\boldsymbol{\phi}(s) = [\mathbf{T}(s)]^{-1}\mathbf{\Phi}(s)$, $s \in \Pi$. The components of the vector, the functions $\phi_1(s)$ and $\phi_2(s)$, form the general solution of the two scalar first-order difference equations

$$\phi_j(\sigma) = \lambda_j(\sigma)\phi_j(\sigma - 2\pi), \quad \sigma \in \Omega, \quad j = 1, 2. \tag{32}$$

These two equations can be transformed into two scalar Riemann-Hilbert problems

$$F_j^+(t) = l_j(t)F_j^-(t), \quad t \in [-1, 1], \quad j = 1, 2, \tag{33}$$

by the map

$$z = i\cot\frac{s}{2}, \qquad s = \pi + i\log\frac{1+z}{1-z},$$
$$z \in \mathbf{C}_z \backslash [-1, 1], \quad s \in \Pi. \tag{34}$$

Here $\mathbf{C}_z$ is a complex plane,

$$F_j^+(t) = \phi_j(\sigma), \quad F^-(t) = \phi_j(\sigma - 2\pi), \quad \sigma \in \Omega,$$
$$l_1(t) = \lambda_1(\sigma) = \frac{(t-a)(t+b)}{(t+a)(t-b)}, \quad l_2(t) = \frac{1}{l_1(t)}, \tag{35}$$

$a = 2 - \sqrt{3}$, $b = 1/a = 2 + \sqrt{3}$, and $t = z(\sigma)$. The function $z = i\cot\frac{1}{2}s$ maps the strip Π into the z-plane cut along the segment $[-1, 1]$. The upper side of the cut corresponds to the contour Ω, and the lower bank $z = x - i0$ ($-1 \leq x \leq 1$) corresponds to the left boundary $\operatorname{Re} s = -\pi$ of the strip.

[15] To factorize the function $l_1(t)$, fix its argument by the condition $\arg l_1(t) = 0$ as $t > b$. Because the points $t = -a$ and $t = b$ are poles and $t = a$ and $t = -b$ are zeros of the function $l_1(t)$ we have

$$\arg l_1(t \pm i0) = \begin{cases} 0, & t \in (-\infty, -b) \cup (-a, a) \cup (b, \infty) \\ \mp\pi, & t \in (-b, -a) \cup (a, b) \end{cases}. \tag{36}$$

Consequently,

$$\log l_1(\pm i0) = \log l_1(\infty) = 0,$$
$$\log l_1(\pm 1 + i0) = -\log l_1(\pm 1 - i0) = -\pi i. \tag{37}$$

Observe now that $l_1(t)l_1(-t) = 1$, $t \in [-1, 1]$ and establish the following properties

$$\log l_1(t \pm i0) + \log l_1(-t \pm i0) = 0, \quad t \in (-a, a),$$
$$\log l_1(t \pm i0) + \log l_1(-t \pm i0) = \mp 2\pi i,$$
$$t \in (-b, -a) \cup (a, b). \tag{38}$$

In view of these relations it can be shown that

$$\exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\log l_1(t)dt}{t - z}\right\} = \frac{z+1}{z+a}$$
$$\cdot \exp\left\{\frac{1}{\pi i}\int_{\Gamma_0} \frac{t\log l_1(t)dt}{t^2 - z^2}\right\}, \tag{39}$$

where Γ is the upper side of the cut $[-1, 1]$, and $\Gamma_0 = \{z \in \mathbf{C}_z: z = x + i0,\ 0 \leq x \leq 1\}$. Therefore, the functions

$$F_{10}(z) = \frac{z+a}{z+1}\exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\log l_1(t)dt}{t-z}\right\} = \exp\left\{\frac{1}{\pi i}\int_{\Gamma_0}\frac{t\log l_1(t)dt}{t^2-z^2}\right\},$$
$$F_{20}(z) = \frac{1}{F_{10}(z)} \tag{40}$$

are the canonical functions to the Riemann-Hilbert problems (33). They are even and factorize the functions $l_j(t)$:

$$l_j(t) = \frac{F_{j0}^+(t)}{F_{j0}^-(t)}, \quad t \in [1,1], \quad j = 1,2. \tag{41}$$

The general solution to the auxiliary difference equation (23) can be written in terms of the canonical functions in the form

$$f(s) = \Phi_{10}(s)P_1(s) + \Phi_{20}(s)P_2(s), \tag{42}$$

where $\Phi_{j0}(s) = F_{j0}(z)$. The functions $P_j(s)$ ($j = 1, 2$) are even 2π-periodic meromorphic functions.

5. Exact Solution to the Diffraction Problem

[16] In what follows we find the most general form of the functions $P_1(s)$ and $P_2(s)$ for which the function (42) is the general solution to the main difference equation (17). Afterwards we determine the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ and therefore complete the solution procedure for the diffraction problem (8), (6), and (7).

5.1. Connection Between the Functions $P_1(s)$ and $P_2(s)$

[17] Substitute the expression (42) into (17)

$$\left(\cos s + \frac{2}{\sqrt{3}}\right)\{[\Phi_{10}(s+\pi) + \Phi_{10}(s-\pi)]P_1(s+\pi) + [\Phi_{20}(s+\pi) + \Phi_{20}(s-\pi)]P_2(s+\pi)\} = \frac{2}{\sqrt{3}}[\Phi_{10}(s)P_1(s) + \Phi_{20}(s)P_2(s)]. \tag{43}$$

Let Re $s = 0$. The function $z = i\cot\frac{1}{2}s$ maps the imaginary axis into the two segments $(-\infty, -1) \cup (1, \infty)$, while the points $s \pm \pi$ fall into the points $(1/x)^\pm = 1/x \pm i0$, $1/x \in [-1, 1]$. Represent next the functions $\Phi_{10}(s)$ and $\Phi_{10}(s \pm \pi)$ (Re $s = 0$) in the form

$$\Phi_{10}(s) = F_{10}(x) = \frac{x+a}{x+1}e^{\chi_1(x)},$$
$$\Phi_{10}(s\pm\pi) = F_{10}^\pm\left(\frac{1}{x}\right) = \frac{a(x+b)}{x+1}\left[l_1\left(\frac{1}{x}\right)\right]^{\pm 1/2} e^{\chi_*(x)},$$
$$\chi_1(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\log l_1(t)dt}{t-x}, \quad \chi_*(x) = \frac{1}{2\pi i}\int_\Gamma\frac{\log l_1(t)dt}{t-1/x}. \tag{44}$$

Here we used the Sokhotski-Plemelj formulas. The integral $\chi_*(x)$ is understood in the sense of the principal value. Establish now a connection between $\chi_1(x)$ and $\chi_*(x)$. It is directly verified that

$$l_1(1/x)l_1(x) = 1, \tag{45}$$

and since $\log\ l_1(\pm 1) = -\pi i$ we have

$$\log l_1(1/x + i0) + \log l_1(x+i0) = \begin{cases} 0, & |x| < a,\ |x| > b \\ -2\pi i, & a < |x| < b \end{cases}. \tag{46}$$

By making the substitution $t \to 1/t$ we find

$$\chi_*(x) = \frac{1}{2\pi i}\left(\int_{-\infty}^{-1} + \int_1^\infty\right)\frac{x\log l_1(1/t)dt}{t(x-t)} = \frac{1}{2\pi i}\left(\int_{-\infty}^{-1} + \int_1^\infty\right)\frac{\log l_1(t)dt}{t-x} + \log l_0(x) - C, \tag{47}$$

where

$$l_0(x) = \frac{(x+1)(x-b)}{(x-1)(x+b)},$$
$$C = -\frac{1}{2\pi i}\int_\Gamma \frac{\log l_1(t)dt}{t}. \tag{48}$$

Notice that $\log\ l_1(t)/t$ is bounded at the point $t = 0$, and Re $C = 0$. By using now the Cauchy theorem

$$\int_{-\infty}^{\infty}\frac{\log l_1(t)dt}{t-x} = \frac{1}{2}\log l_1(x) \tag{49}$$

and comparing (44), (47), and (49) we express the function $\chi_*(x)$ through the function $\chi_1(x)$

$$\chi_*(x) = -\chi_1(x) + \frac{1}{2}\log l_1(x) + \log l_0(x) - C. \quad (50)$$

On using (44) and (50), we observe that the boundary values $\Phi_{j0}(s \pm \pi)$, $j = 1, 2$, have the following representations

$$\Phi_{10}(s \pm \pi) = \frac{ae^{-C}(x \pm a)(x \mp b)}{x^2 - 1}\Phi_{20}(s),$$
$$\Phi_{20}(s \pm \pi) = \frac{1}{\Phi_{10}(s \pm \pi)}, \quad (51)$$

which are required in the sequel. Therefore,

$$\Phi_{10}(s + \pi) + \Phi_{10}(s - \pi) = 2ae^{-C}\Phi_{20}(s),$$
$$\Phi_{20}(s + \pi) + \Phi_{20}(s - \pi) = \frac{2be^C(x^2 - 1)^2\Phi_{10}(s)}{(x^2 + 1)^2 - 16x^2}. \quad (52)$$

Substitute now these expressions into equation (43)

$$\left(\cos s + \frac{2}{\sqrt{3}}\right)\left[2ae^{-C}\Phi_{20}(s)P_1(s + \pi) + \frac{2be^C(x^2 - 1)^2}{(x^2 + 1)^2 - 16x^2}\Phi_{10}(s)P_2(s + \pi)\right] = \frac{2}{\sqrt{3}}[\Phi_{10}(s)P_1(s) + \Phi_{20}(s)P_2(s)]. \quad (53)$$

Because of the linear independence of the functions $\Phi_{10}(s)$ and $\Phi_{20}(s)$, on one hand,

$$P_2(s) = A(s)P_1(s + \pi), \quad A(s) = ae^{-C}\left(2 + \sqrt{3}\cos s\right). \quad (54)$$

On the other hand,

$$P_1(s) = be^C\left(\sqrt{3}\cos s + 2\right)\frac{(x^2 - 1)^2 P_2(s + \pi)}{(x^2 + 1)^2 - 16x^2}. \quad (55)$$

Replace s by $s + \pi$. Then x will be replaced by $1/x$, and we obtain

$$P_2(s) = B(s)P_1(s + \pi),$$
$$B(s) = ae^{-C}\frac{(x^2 + 1)^2 - 16x^2}{(x^2 - 1)^2\left(2 - \sqrt{3}\cos s\right)}. \quad (56)$$

Since $x = i\cot\frac{1}{2}s$, we have the relations

$$\cos s = \frac{x^2 + 1}{x^2 - 1}, \quad 4 - 3\cos^2 s = \frac{x^4 - 14x^2 + 1}{(x^2 - 1)^2}, \quad (57)$$

and it follows that $B(s) = A(s)$ as Re $s = 0$. By principle of analytical continuation, this is valid everywhere in the strip Π, and therefore, the general solution (42) becomes

$$f(s) = \Phi_{10}(s)P_1(s) + ae^{-C}\left(2 + \sqrt{3}\cos s\right)\frac{P_1(s + \pi)}{\Phi_{10}(s)}. \quad (58)$$

5.2. Function $\mathbf{P_1(s)}$

[18] The determination of the function $P_1(s)$ requires a further study into the properties of the functions $\Phi_{10}(s)$ and $f(s)$. It will be helpful to rewrite formulas (51) for $\Phi_{10}(s \pm \pi)$ as follows

$$\Phi_{10}(s + \pi) = \frac{ae^{-C}(x - b)}{x - 1}e^{-\chi_1(x)},$$
$$\Phi_{10}(s - \pi) = \frac{ae^{-C}(x - a)(x + b)}{(x - 1)(x + a)}e^{-\chi_1(x)}, \quad (59)$$

where $x = i\cot\frac{1}{2}s$, x is real, $x > 1$, and Re $s = 0$. Clearly, if $s = \pm i\log\sqrt{3}$, then $x = \pm b$. Therefore, the function $\Phi(\sigma)$ has simple zeros at the points $\sigma = \pi + i\log\sqrt{3}$ and $\sigma = -\pi - i\log\sqrt{3}$. At all other finite points of the strip Π including the points $\sigma = \pi - i\log\sqrt{3}$ and $\sigma = -\pi + i\log\sqrt{3}$, it is bounded and nonzero. Now, because of the factor $2 + \sqrt{3}\cos s$, the second term in (58) is bounded and nonzero everywhere in the strip Π apart from the points $s = \pi - i\log\sqrt{3}$ and $s = -\pi + i\log\sqrt{3}$ which are simple zeros of the function $(2 + \sqrt{3}\cos s)/\Phi_{10}(s)$.

[19] Show next that the 2π-periodic meromorphic function $P_1(s)$ may grow as $\exp\{|\text{Im } s|\}$ when Im $s \to \pm\infty$. Indeed, since $\log l_1(\pm 1 + i0) = -\log l_1(\pm 1 - i0) = -\pi$, by analyzing the Cauchy integral (39) at the points $z = \pm 1$, we find $F_{10}(z) = O((1 \mp z)^{-1/2})$ as $z \to \pm 1$. Correspondingly, at the infinite points of the strip,

$$\Phi_{10}(s) \sim A_0 e^{|\text{Im}\, s|/2}, \quad \text{Im}\, s \to \pm\infty, \quad A_0 \neq 0. \quad (60)$$

Check now the asymptotics of the function $\mathcal{F}(s)$ at infinity. By using the connection between the Cauchy and Hilbert kernels

$$\cot\frac{\tau - s}{2} = -i\frac{1 - tz}{t - z}, \quad (61)$$

we have

$$\mathcal{F}(s) = \frac{z(z - z_0)}{2\pi i(1 - z^2)}\int_{-1}^{1}\frac{(1 - t^2)\hat{f}(t)dt}{t(t - z)(t - z_0)}, \quad (62)$$

where $\hat{f}(t) = f(i\log\{(1 + t)/(1 - t)\})$. Since $\hat{f}(t) = O((t \pm 1)^{-3/2})$, $t \to \mp 1$, the function $\mathcal{F}(s)$ grows as $e^{3/2|\text{Im } s|}$ as $|\text{Im } s| \to \infty$. This means that the function $\psi(s)$ given by (20) grows at infinity as $e^{2|\text{Im } s|}$ as it must be. Therefore, the most general form of the even 2π-periodic meromorphic function $P_1(s)$ which grows at infinity as $e^{|\mathrm{Im}\, s|}$ and which has simple poles at the zeros of the function $\Phi_{10}(s)$ is given by

$$P_1(s) = \frac{E_0 + E_1\cos s + E_2\cos 2s}{2+\sqrt{3}\cos s}, \tag{63}$$

and from (58) we have

$$f(s) = \frac{E_0 + E_1\cos s + E_2\cos 2s}{2+\sqrt{3}\cos s}\Phi_{10}(s) + \frac{2+\sqrt{3}\cos s}{2-\sqrt{3}\cos s}\,\frac{E_0 - E_1\cos s + E_2\cos 2s}{e^C b\Phi_{10}(s)}. \tag{64}$$

Clearly, for arbitrary constants E_0, E_1, and E_2, the function $f(s)$ has inadmissible poles at the points $s = \pm(\pi - i\log\sqrt{3})$ (the first term) and $s = \pm i\log\sqrt{3}$ (the second term). To remove them, we require

$$E_0 = \frac{2}{\sqrt{3}}E_1 - \frac{5}{3}E_2. \tag{65}$$

This condition reduces formula (63) to the form $P_1(s) = D_0 + D_1\cos s$, where $D_0 = E_1/\sqrt{3} - 4E_2/3$ and $D_1 = 2E_2/\sqrt{3}$ are free constants. This simplifies the expression for the function $f(s)$

$$f(s) = D_0 f_0(s) + D_1 f_1(s), \tag{66}$$

where

$$f_j(s) = \left[\Phi_{10}(s) + (-1)^j \frac{e^{-C}\left(2+\sqrt{3}\cos s\right)}{b\Phi_{10}(s)}\right]\cos^j s, \quad j = 0, 1. \tag{67}$$

Formula (66) represents the general solution to the main difference equation (17). It is expressed through one quadrature $\chi_1(z)$ given by (44) ($C = -\chi_1(0)$). The function $\psi(s)$ and therefore the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ have 4 unknown constants C_1, C_2, D_0, and D_1. They will be fixed next.

5.3. Definition of the Constants

[20] In general, the spectral functions may have some inadmissible poles which should be eliminated by fixing the free constants. The function $\mathcal{S}_1(s)$ must be analytic everywhere in the strip $-3\pi/4 \le \mathrm{Re}\, s \le 3\pi/4$ apart from the point $s = \varphi_0$, where it has to have a simple pole to reproduce the incident field (4). The function $\mathcal{S}_2(s)$ must be free of poles everywhere in the strip $-\pi/4 \le \mathrm{Re}\, s \le \pi/4$. To analyze the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$, express them through the functions $f(s)$ and $\psi(s)$. To do this we continue analytically the function $\psi(s)$ into the strips $-2\pi \le \mathrm{Re}\, s \le -\pi$ and $\pi \le \mathrm{Re}\, s \le 2\pi$

$$\psi(s) = \psi(s+2\pi) - f(s+\pi), \quad -2\pi \le \mathrm{Re}\, s \le -\pi,$$
$$\psi(s) = \psi(s-2\pi) + f(s-\pi), \quad \pi \le \mathrm{Re}\, s \le 2\pi, \tag{68}$$

and find then

$$\mathcal{S}_1(s) = \Sigma(s)\psi\left(s+\frac{3\pi}{4}\right), \quad -\frac{7\pi}{4} \le \mathrm{Re}\, s \le \frac{\pi}{4},$$
$$\mathcal{S}_1(s) = \Sigma(s)\left[\psi\left(s-\frac{5\pi}{4}\right) + f\left(s-\frac{\pi}{4}\right)\right], \quad \frac{\pi}{4} \le \mathrm{Re}\, s \le \frac{5\pi}{4},$$
$$\mathcal{S}_2(s) = \Sigma(s)\left\{\left(1+\frac{\sin\theta}{\sin(s+\pi/4)}\right)\cdot\left[\psi\left(s-\frac{\pi}{4}\right) + f\left(s+\frac{3\pi}{4}\right)\right] + \frac{\sin\theta}{\sin(s+\pi/4)}\left[\psi\left(s+\frac{3\pi}{4}\right) - f\left(s-\frac{\pi}{4}\right)\right]\right\}, \quad -\frac{3\pi}{4} \le \mathrm{Re}\, s \le \frac{\pi}{4}. \tag{69}$$

Notice that the point $s = -\pi/4$ is a removable singularity of the function $\mathcal{S}_2(s)$. This follows from (18). Analysis of formulas (69) [see also *Antipov and Silvestrov*, 2004b] shows that the function $\mathcal{S}_1(s)$ has two simple poles at the points $s = \alpha_1$ and $s = \alpha_2$, and the function $\mathcal{S}_2(s)$ has one pole at the point $s = \alpha_3$:

$$\alpha_1 = \begin{cases} \varphi_0 - \pi, & \pi/4 \le \varphi_0 \le 3\pi/4 \\ -\varphi_0 - \pi/2, & -3\pi/4 \le \varphi_0 \le \pi/4 \end{cases},$$
$$\alpha_2 = \begin{cases} -\varphi_0 + \pi/2, & -\pi/4 \le \varphi_0 \le 3\pi/4 \\ \varphi_0 + \pi, & -3\pi/4 \le \varphi_0 \le -\pi/4 \end{cases},$$
$$\alpha_3 = \begin{cases} -\varphi_0 + \pi/2, & \pi/4 \le \varphi_0 \le 3\pi/4 \\ \varphi_0, & -\pi/4 \le \varphi_0 \le \pi/4 \\ -\varphi_0 - \pi/2, & -3\pi/4 \le \varphi_0 \le -\pi/4 \end{cases}. \tag{70}$$

All these poles are inadmissible singularities and should be removed. Thus, we have four additional conditions

$$\mathrm{res}_{s=\varphi_0}\mathcal{S}_1(s) = 1, \quad \mathrm{res}_{s=\alpha_1}\mathcal{S}_1(s) = 0,$$
$$\mathrm{res}_{s=\alpha_2}\mathcal{S}_1(s) = 0, \quad \mathrm{res}_{s=\alpha_3}\mathcal{S}_2(s) = 0. \tag{71}$$

By evaluating these residues we rewrite the conditions (71) in terms of the functions $f(s)$ and $\psi(s)$

$$f(\beta_0) = 1, \quad f(\beta_1) = \delta_0, \quad \psi(\beta_0) = 0, \quad \psi(\beta_1) = \delta_1, \tag{72}$$

where

$$\beta_0 = \varphi_0 - \pi/4,$$
$$\beta_1 = \begin{cases} \varphi_0 + 3\pi/4, & -3\pi/4 \le \varphi_0 \le \pi/4 \\ -\varphi_0 + 5\pi/4, & \pi/4 \le \varphi_0 \le 3\pi/4 \end{cases},$$
$$\delta_0 = \begin{cases} 0, & -3\pi/4 \le \varphi_0 \le \pi/4 \\ \mu_0, & \pi/4 \le \varphi_0 \le 3\pi/4 \end{cases},$$
$$\delta_1 = \begin{cases} 1, & -3\pi/4 \le \varphi_0 \le \pi/4 \\ 0, & \pi/4 \le \varphi_0 \le 3\pi/4 \end{cases},$$
$$\mu_0 = \frac{2/\sqrt{3}}{2/\sqrt{3} + \sin(\pi/4 + \varphi_0)}. \quad (73)$$

The requirements (72) when satisfied give the following values for the free constants:

$$D_0 = \frac{f_1(\beta_1) - \delta_0 f_1(\beta_0)}{\Delta_0},$$
$$D_1 = \frac{-f_0(\beta_1) + \delta_0 f_0(\beta_0)}{\Delta_0},$$
$$C_1 = -\frac{\mathcal{F}(\beta_0)\sin 2\beta_1 + [\delta_1 - \mathcal{F}(\beta_1)]\sin 2\beta_0}{\Delta_1},$$
$$C_2 = \frac{\mathcal{F}(\beta_0)\sin \beta_1 + [\delta_1 - \mathcal{F}(\beta_1)]\sin \beta_0}{\Delta_1}, \quad (74)$$

where

$$\Delta_0 = f_0(\beta_0)f_1(\beta_1) - f_1(\beta_0)f_0(\beta_1),$$
$$\Delta_1 = 2(-1)^{\nu}\sin(\varphi_0 - \pi/4)\cos 2\varphi_0, \quad (75)$$

$\nu = 1$ for $-3\pi/4 \le \varphi_0 \le \pi/4$, and $\nu = 2$ for $\pi/4 \le \varphi_0 \le 3\pi/4$.

6. Reflected, Transmitted, Surface, and Diffracted Waves

[21] In this section we will derive the asymptotic expansion for the far field. For large $k_0 r$, the E_z-component of the electric field can be represented as follows

$$E_z \sim E_z^i + E_z^r + E_z^t + E_z^s + E_z^d, \quad (76)$$

where E_z^i, E_z^r, E_z^t, E_z^s, and E_z^d are the incident, reflected, transmitted, surface, and diffracted waves, respectively. By using the method of steepest descent we deform the contour $\mathcal{L}$ into another one consisting of two steepest descent paths (Figure 2). The right-hand path is given by $\mathrm{Re}\, s = \pi + \mathrm{gd}(\mathrm{Im}\, s)\mathrm{sgn}\, \mathrm{Im}\, s$, where $\mathrm{gd}\, x = \arccos(1/\cosh x)$ is the Gudermann function. This curve goes from the infinite point $s = \pi/2 - i\infty$, crosses the real axis at the point $s = \pi$ and then travels to the upper infinite point $s = 3\pi/2 + i\infty$. The lines $\mathrm{Re}\, s = \pi/2$ and $\mathrm{Re}\, s = 3\pi/2$ are the asymptotes for the lower and the upper part of the path, respectively. The second path is symmetric to the first one with respect to the origin. To derive the expansion (76), we need to continue the function $\mathcal{S}_1(s)$ into the strip $5\pi/4 \le \mathrm{Re}\, s \le 9\pi/4$ and the function $\mathcal{S}_2(s)$ into the strips $-7\pi/4 \le \mathrm{Re}\, s \le -3\pi/4$ and $\pi/4 \le \mathrm{Re}\, s \le 5\pi/4$

$$\mathcal{S}_1(s) = \Sigma(s)\left[\psi\left(s - \frac{5\pi}{4}\right) - f\left(s - \frac{9\pi}{4}\right) + \frac{\gamma f(s - 5\pi/4)}{\gamma - \sin(s + \pi/4)}\right], \quad \frac{5\pi}{4} \le \mathrm{Re}\, s \le \frac{9\pi}{4},$$
$$\mathcal{S}_2(s) = \Sigma(s)\left\{\left(1 + \frac{\gamma}{\sin(s+\pi/4)}\right)\psi\left(s + \frac{7\pi}{4}\right) + \frac{\gamma}{\sin(s+\pi/4)} \times\left[\psi\left(s + \frac{3\pi}{4}\right) + f\left(s + \frac{7\pi}{4}\right) - \frac{\gamma f(s + 3\pi/4)}{\gamma - \sin(s+\pi/4)}\right]\right\}, -\frac{7\pi}{4} \le \mathrm{Re}\, s \le -\frac{3\pi}{4},$$
$$\mathcal{S}_2(s) = \Sigma(s)\left\{\left(1 + \frac{\gamma}{\sin(s+\pi/4)}\right) \cdot \left[\psi\left(s - \frac{\pi}{4}\right) - f\left(s - \frac{5\pi}{4}\right)\right] + \frac{\gamma}{\sin(s+\pi/4)}\left[\psi\left(s - \frac{5\pi}{4}\right) + f\left(s - \frac{\pi}{4}\right)\right]\right\},$$
$$\frac{\pi}{4} \le \mathrm{Re}\, s \le \frac{5\pi}{4}. \quad (77)$$

Let

$$\omega_\varphi(a, b) = \begin{cases} 1, & \varphi \in [a, b] \\ 0, & \varphi \notin [a, b] \end{cases}. \quad (78)$$

In view of formulas (69) and (77) we deduce the following result. Let first $|\varphi| < 3\pi/4$. If $-3\pi/4 < \varphi_0 < -\pi/4$, then

$$E_z^i = e^{ik_0 r\cos(\varphi - \varphi_0)}\omega_\varphi\left(-\frac{3\pi}{4}, \pi + \varphi_0\right),$$
$$E_z^t = 0, E_z^r = e^{ik_0 r\sin(\varphi + \varphi_0)}\omega_\varphi\left(-\frac{3\pi}{4}, -\frac{\pi}{2} - \varphi_0\right). \quad (79)$$

In the case $-\pi/4 < \varphi_0 < \pi/4$ the incident, transmitted and reflected waves are

$$E_z^i = e^{ik_0 r\cos(\varphi - \varphi_0)}, \quad E_z^t = 0,$$
$$E_z^r = \mu_0 e^{-ik_0 r\sin(\varphi + \varphi_0)}\omega_\varphi(\pi/2 - \varphi_0, 3\pi/4) + e^{-ik_0 r\sin(\varphi + \varphi_0)}\omega_\varphi(-3\pi/4, -\pi/2 - \varphi_0). \quad (80)$$

In the last possible case $\pi/4 < \varphi_0 < 3\pi/4$, the waves have

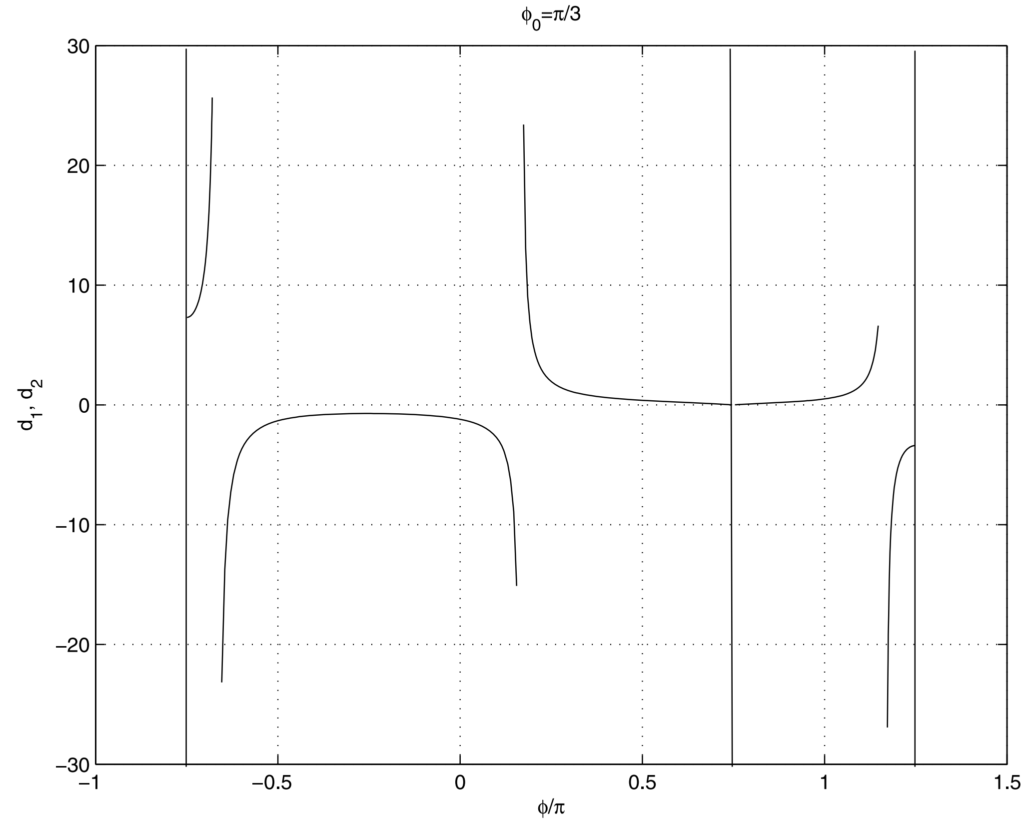

Figure 3. The dependence of the diffraction coefficients d_1 and d_2 on the angle of observation $\varphi \in (-3\pi/4,\ 5\pi/4)$ for $\varphi_0 = \pi/3$.

the form

$$E_z^i = e^{ik_0 r\cos(\varphi-\varphi_0)}\omega_\varphi(\varphi_0-\pi, 3\pi/4), \quad E_z^t = 0,$$
$$E_z^r = -\mu_0 e^{-ik_0 r\sin(\varphi+\varphi_0)}\omega_\varphi(\pi/2-\varphi_0, 3\pi/4). \quad (81)$$

The reflected and transmitted waves in the sector $3\pi/4 < \varphi < 5\pi/4$ are defined by the corresponding poles of the function $\mathcal{S}_2(s)$. The reflected and transmitted waves can be written straightforwardly ($E_z^i = 0$)

$$E_z^r = 0, \quad E_z^t = 0, \quad -3\pi/4 < \varphi_0 < -\pi/4,$$
$$E_z^r = 0, \quad E_z^t = \mu_1\omega_\varphi\left(\frac{3\pi}{4}, \pi+\varphi_0\right)e^{ik_0 r\cos(\varphi-\varphi_0)},$$
$$-\frac{\pi}{4} < \varphi_0 < \frac{\pi}{4},$$
$$E_z^r = \mu_1 e^{ik_0 r\sin(\varphi+\varphi_0)}\omega_\varphi\left(\frac{3\pi}{2}-\varphi_0, \frac{5\pi}{4}\right),$$
$$E_z^t = \mu_1 e^{ik_0 r\cos(\varphi-\varphi_0)}, \pi/4 < \varphi_0 < 3\pi/4, \quad (82)$$

where

$$\mu_1 = \frac{\sin(\varphi_0+\pi/4)}{2/\sqrt{3}+\sin(\varphi_0+\pi/4)}. \quad (83)$$

[22] Check next if there are surface waves for $-\frac{3\pi}{4} < \varphi < \frac{3\pi}{4}$. They may be produced by the zeros s_m of the function $\sin(s+\frac{1}{4}\pi) - \sin\theta$, $\theta = \frac{1}{2}\pi \pm i\log\sqrt{3}$ which satisfy the condition

$$-\pi + g_0 \operatorname{sgn}\operatorname{Im}s_m < \operatorname{Re}s_m - \varphi < \pi + g_0\operatorname{sgn}\operatorname{Im}s_m, \quad (84)$$

where $g_0 = \operatorname{gd}\log\sqrt{3} = \frac{\pi}{6}$. Analysis of formulas (69) and (77) shows that all possible zeros (surface poles) of the function $\sin(s+\frac{1}{4}\pi) - \sin\theta$ in the strip (84) give φ which is outside the range $(-\frac{3\pi}{4}, \frac{3\pi}{4})$ that is impossible. Therefore, no surface waves are observed for $|\varphi| < \frac{3\pi}{4}$. The use of a similar argument yields the same result for the second wedge $\varphi \in (\frac{3\pi}{4}, \frac{5\pi}{4})$, and we conclude that $E_z^s = 0$ everywhere in the medium for the impedance parameter $\gamma = 2/\sqrt{3}$.

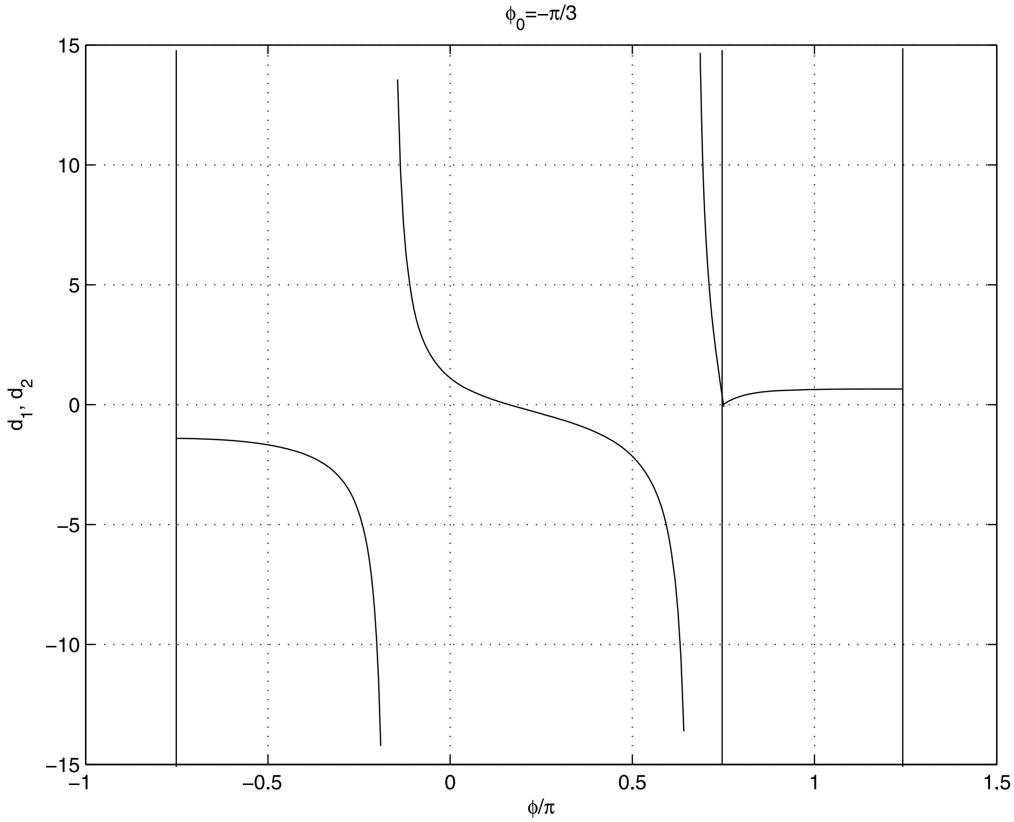

Figure 4. The dependence of the diffraction coefficients d_1 and d_2 on the angle of observation $\varphi \in (-3\pi/4,\ 5\pi/4)$ for $\varphi_0 = -\pi/3$.

[23] Finally, define the diffracted field. By applying the steepest descent method for $k_0 r \gg 1$ we obtain

$$E_z^d = \frac{e^{-ik_0 r}}{\sqrt{k_0 r}} D(\varphi), \quad \varphi \in (-3\pi/4, 5\pi/4) \setminus \{\varphi = 3\pi/4\}, \tag{85}$$

where

$$D(\varphi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \begin{cases} d_1(\varphi), & -3\pi/4 < \varphi < 3\pi/4 \\ d_2(\varphi), & 3\pi/4 < \varphi < 5\pi/4 \end{cases},$$
$$d_1(\varphi) = \mathcal{S}_1(\varphi - \pi) - \mathcal{S}_1(\varphi + \pi),$$
$$d_2(\varphi) = \mathcal{S}_2(\varphi - 2\pi) - \mathcal{S}_2(\varphi). \tag{86}$$

Numerical computations are implemented for the diffraction coefficient in the case $\gamma = 2/\sqrt{3}$. The numerical algorithm is very simple and efficient. To evaluate the diffraction coefficient D, one needs to compute just two integral F_{10} and $\mathcal{F}$ given by (40) and (62). Notice that these two integrals are convergent as improper ones. The density of the former integral has the logarithmic singularity at the point $x = a$, while the density for the integral $\mathcal{F}$ has the square root singularities at the ends $z = \pm 1$. For integration, we used the Simpson formula for F_{10} and the Gauss formula for $\mathcal{F}$. Sample plots of the coefficients d_1 and d_2 against the angle of observation φ for the incident angles $\varphi_0 = \pi/3$ and $\varphi_0 = -\pi/3$ are given in Figures 3 and 4, respectively. We also present two graphs which show how the coefficients Re d_1 and Re d_2 depend on the angle of incidence. For Figure 5, the angle of diffraction is $\pi/8$ (the corresponding diffraction coefficient is d_1) which is in the larger wedge, and for Figure 6, the point of observation is in the wedge $3\pi/4 < \varphi < 5\pi/4$ ($\varphi_0 = 7\pi/8$ and therefore the diffraction coefficient is d_2). Notice

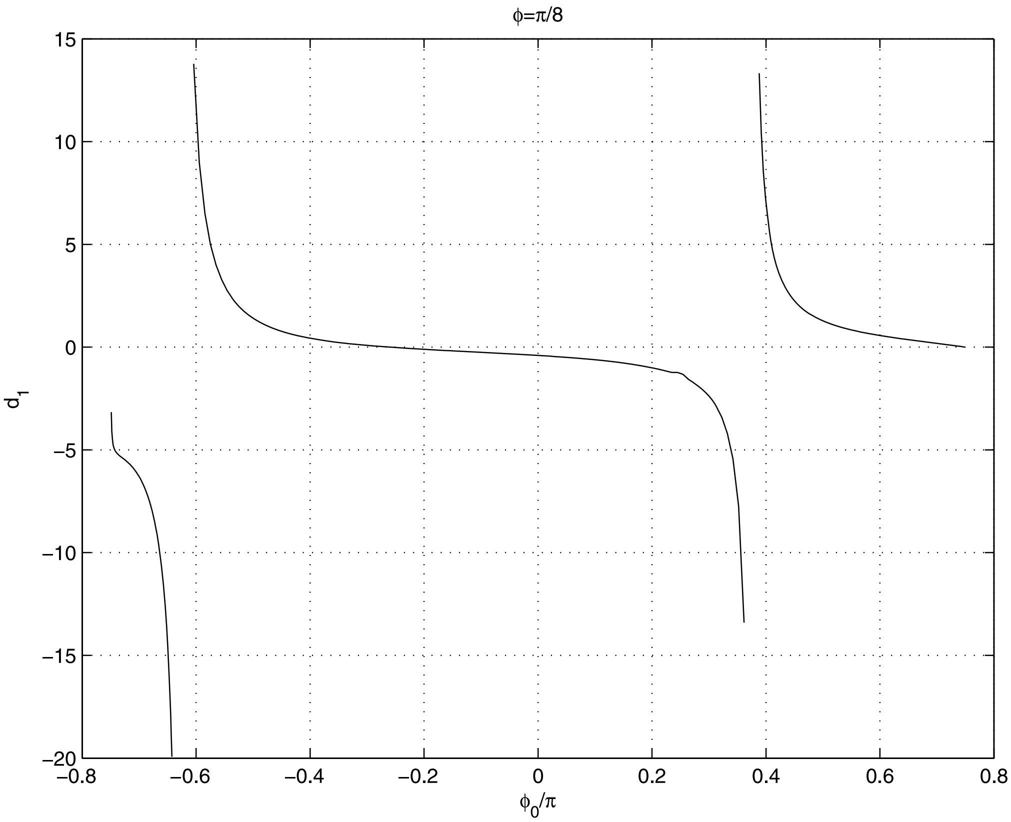

Figure 5. The dependence of the diffraction coefficient d_1 on the angle of incident $\varphi_0 \in (-3\pi/4, 3\pi/4)$ for $\varphi = \pi/8$.

that in the cases considered, the imaginary parts of the coefficients d_1 and d_2 if plotted together with Re d_1 and Re d_2 are negligible to be visible.

7. Conclusion

[24] Extension of the Sommerfeld-Maliuzhinets theory for canonical problems of diffraction for penetrable wedges is an important and not trivial task. If a wedge is of an arbitrary aperture, this is still an open problem. The main contribution of this paper is the development of an exact method for a right-angled penetrable wedge with one of its walls being electrically resistive and the second one being perfectly magnetically conductive. In comparison with the previous work by *Antipov and Silvestrov* [2004b] dealing with another type of penetrable boundary conditions on a right-angled wedge, the main difficulty of the problem in this paper was poles and zeros of the coefficients of the associated Riemann-Hilbert problems. When the penetrable wall is magnetically conductive [*Antipov and Silvestrov*, 2004b], the corresponding coefficients are continuous and do not vanish on the contour of discontinuity of the Riemann-Hilbert problems. The presence of zeros and poles makes the level of mathematical difficulties high even in the particular case of the impedance parameter ($\gamma = 2/\sqrt{3}$) when the main second-order difference equation can be reduced to a pair of first-order difference equations, and the transformation matrix is single-valued.

[25] In this paper, we have derived an exact solution to the diffraction problem for the case $\gamma = 2/\sqrt{3}$. The solution has been presented in a simple form convenient for computation. To evaluate the diffracted waves in the far field expansion of the electric field, it is required to compute only two integrals. To find the numerical values of the E_z-component of the electric field, one needs to evaluate three integrals. It has been shown that no surface wave can be observed in the medium $-3\pi/4 < \varphi < 5\pi/4$. It has also been found that the diffraction coefficient is continuous through the electrically resistive wall but it is discontinuous and bounded through the perfectly magnetically conductive surface.

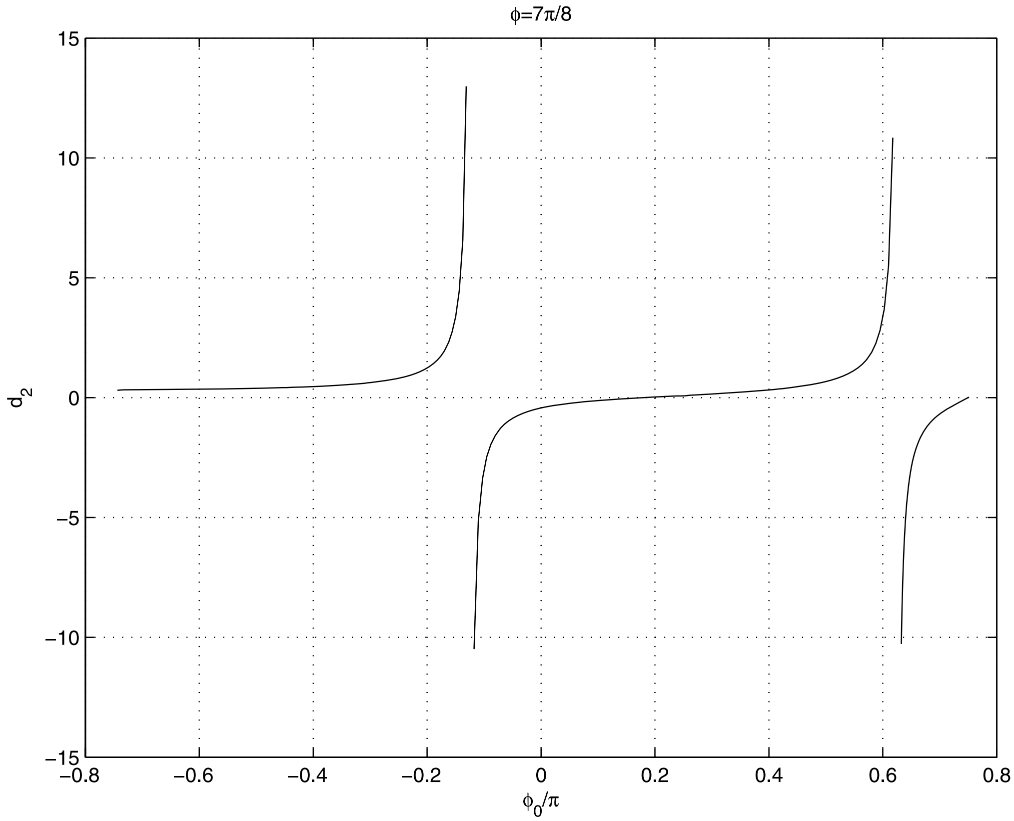

Figure 6. The dependence of the diffraction coefficient d_1 on the angle of incident $\varphi_0 \in (-3\pi/4, 3\pi/4)$ for $\varphi = 7\pi/8$.

[26] In the general case of the impedance parameter, $\gamma \neq 2/\sqrt{3}$, the transformation matrix is two-valued, and the use of Riemann surfaces is unavoidable. Because of the zeros and poles of the coefficient, the technique by *Antipov and Silvestrov* [2004b] has to be modified accordingly. It turns out that the first step of the procedure, the solution of the auxiliary second-order difference equation can be implemented without essential difficulties. What is hard is to satisfy the conditions under which the auxiliary equation is equivalent to the main second-order difference equation and to find unknown periodic meromorphic functions on the Riemann surface. We hope to report a closed-form solution to the general case in the nearest future. The solution presented in this paper for a particular value of the impedance parameter provides a reliable test for the validation of future analytical solutions for the general case and also for pure numerical schemes for more general diffraction problems.

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