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Complex Analysis and Operator Theory

Riemann–Hilbert Problem for Automorphic Functions and the Schottky–Klein Prime Function

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Abstract. The construction of analogues of the Cauchy kernel is crucial for the solution of Riemann–Hilbert problems on compact Riemann surfaces. A formula for the Cauchy kernel can be given as an infinite sum over the elements of a Schottky group, and this sum is often used for the explicit evaluation of the kernel. In this paper a new formula for a quasi-automorphic analogue of the Cauchy kernel in terms of the Schottky–Klein prime function of the associated Schottky double is derived. This formula opens the door to finding new ways to evaluate the analogue of the Cauchy kernel in cases where the infinite sum over a Schottky group is not absolutely convergent. Application of this result to the solution of the Riemann–Hilbert problem with a discontinuous coefficient for symmetric automorphic functions is discussed.

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1. Introduction

The Schottky–Klein prime function is an important transcendental function with a primary role to play in solving problems involving multiply connected domains. It has, however, received almost no attention in the applied mathematics literature until relatively recently. The prime function is documented in Chapter 12 of H. Baker's now classic 19th century monograph [4] and is discussed again in the memoir by Hejhal [15]. In terms of applications, it has recently been demonstrated that the prime function arises naturally in a variety of important applied mathematical problems (e.g., [8, 11, 12]).

One way to define the Schottky–Klein prime function is by means of a classical infinite product formula recorded, for example, in Chapter 12 of Baker's monograph on abelian functions [4]. However, this infinite product does not always converge; there are usually restrictions on the multiply connected domain (or the Schottky group) required in order to ensure convergence. It should be emphasized, however, that the prime function is a well-defined (indeed a uniquely defined) function for *any* multiply connected circular domain [15]. An alternative numerical scheme for the determination of the Schottky–Klein prime function associated with the Schottky double of multiply connected planar domains has recently been presented in [13]. This algorithm can be used to compute the prime function when the infinite product is either divergent, or so slowly convergent as to make its use in applications impracticable.

In this paper we demonstrate how to represent the Chibrikova–Silvestrov [6,7] quasi-automorphic and automorphic analogues of the Cauchy kernel – functions important in the solution of Riemann–Hilbert (RH) problems in the theory of automorphic functions – in terms of the Schottky–Klein prime function. By virtue of this formula, if the prime function can be calculated (by some method, not necessarily by means of a sum or a product over a Schottky group), then the analogues of the Cauchy kernel can also be readily computed. The solution of the Riemann–Hilbert problem for automorphic functions is required for model problems in fluid mechanics [1], electrostatics, and elasticity.

The new representation is used to solve the homogeneous RH problem for Θ-automorphic functions with a discontinuous coefficient. Here Θ is a symmetric Schottky group generated by compositions of Möbius transformations and their inverses. It is known [17] that this problem can equivalently be reduced to the Hilbert problem for a multiply connected circular domain (for a survey, see [18]). The solution we present makes use of the new formula for the Cauchy kernel (in terms of the Schottky–Klein prime function) and it avoids a reduction of the problem to the construction of a finite number of the Schwarz operators (as done in [18]). Instead, the method proceeds by converting the RH problem for automorphic functions to a RH problem on a compact Riemann surface (the Schottky double). It is shown that the terms $\eta_j(\tau)d\tau$ (the functions $\{\eta_j(\tau)\}$ are the cyclic terms in the representation [7] of the quasi-automorphic kernel) form a basis of abelian differentials of the first kind on the Schottky double. The associated factorization problem requires the solution of the Jacobi inversion problem. The final formulas for the solution are thus expressed through the Schottky–Klein prime function and the solution of this inversion problem.

2. The Schottky–Klein prime function

Let D_ζ be the multiply connected circular domain consisting of the unit disk in the ζ-plane with M smaller circular disks excised. Let the unit circle be denoted C_0 and the boundaries of the M enclosed circular disks be denoted $\{C_j | j = 1, \dots, M\}$. Let the radius and center of C_j be denoted ρ_j and δ_j respectively.

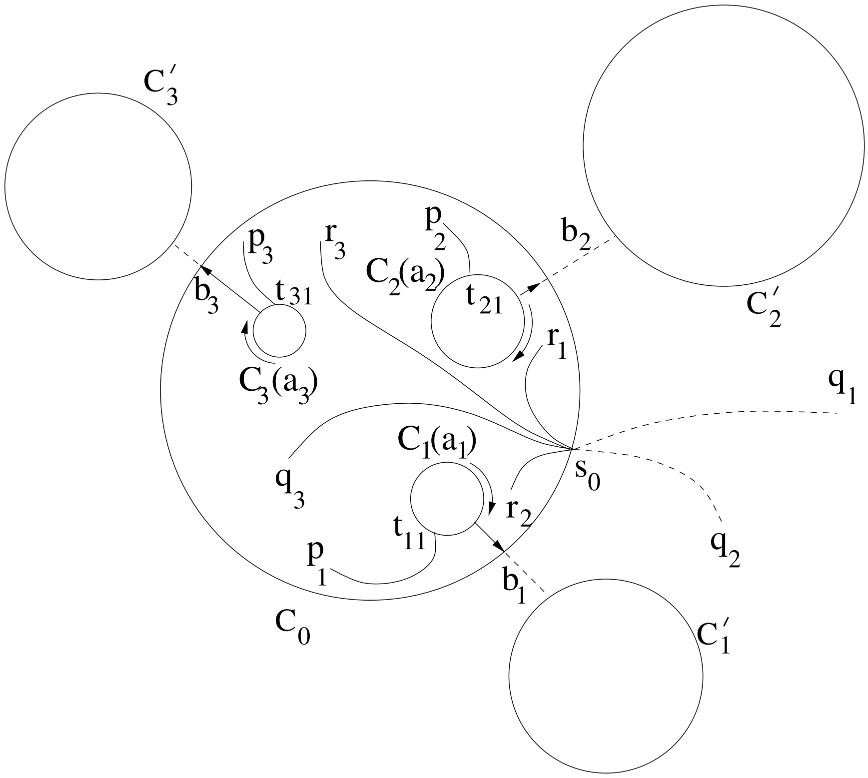

FIGURE 1. Schematic illustrating the circles C_j and C'_j in a quadruply connected case ($M = 3$).

First define M Möbius maps $\{\phi_j | j = 1, \ldots, M\}$ corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = \rho_j^2 \,, \tag{2.1}$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{\rho_j^2}{\zeta - \delta_j} \,, \tag{2.2}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{\rho_j^2}{\zeta - \delta_j} \,. \tag{2.3}$$

If ζ is a point on C_j then its complex conjugate is given by $\bar{\zeta} = \phi_j(\zeta)$.

Next, introduce the M Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{\rho_j^2 \zeta}{1 - \bar{\delta}_j \zeta} \,. \tag{2.4}$$

Let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$, that is, the circle obtained by the anti-holomorphic transformation $T(\zeta)$ given by

$$\zeta \mapsto T(\zeta) = \bar{\zeta}^{-1} . \tag{2.5}$$

Figure 1 shows a schematic in a quadruply connected case. It is easily verified that the image of the circle C_j' under the transformation θ_j is the circle C_j. Thus, θ_j identifies circle C_j' with circle C_j. Since the M circles $\{C_j | j = 1, \ldots, M\}$ are non-overlapping, so are the M circles $\{C_j' | j = 1, \ldots, M\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of mappings $\theta(\zeta)$ generated by compositions of the M basic Möbius maps $\{\theta_j | j = 1, \ldots, M\}$, their M inverses $\{\theta_j^{-1} | j = 1, \ldots, M\}$ and including the identity map $\theta_0(\zeta) = \zeta$. All maps of the group Θ admit the representation

$$\theta(\zeta) = \frac{a_\theta \zeta + b_\theta}{c_\theta \zeta + d_\theta} . \tag{2.6}$$

It is easy to verify that, with the normalization $a_\theta d_\theta - b_\theta c_\theta = 1$,

$$\theta'(\zeta) = \frac{1}{(c_\theta \zeta + d_\theta)^2} . \tag{2.7}$$

Let the radius and center of C_j' be denoted ρ_j' and δ_j' respectively. Then it follows that

$$\rho_j' = \frac{\rho_j}{||\delta_j|^2 - \rho_j^2|} , \quad \delta_j' = \frac{\delta_j}{|\delta_j|^2 - \rho_j^2} . \tag{2.8}$$

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j, C_j' | j = 1, \ldots, M\}$. Let this region be called F. F is known as the *fundamental region* associated with the Schottky group generated by the Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses. This is because the entire plane is tessellated with copies of this fundamental region obtained by mapping F by the elements of the Schottky group. This fundamental region can be understood as having two "halves" – the half that is inside the unit circle but exterior to the circles $\{C_j | j = 1, \ldots, M\}$ is the domain D_ζ, the other half is the region outside the unit circle and exterior to the circles $\{C_j' | j = 1, \ldots, M\}$.

These two halves of F, one just a reflection through the unit circle of the other, can be viewed as a model of the two "sides" of a compact (symmetric) Riemann surface associated with D_ζ known as its *Schottky double*. The genus of this compact Riemann surface is M. Indeed, in [4] it is discussed how the circles $\{C_j | j = 1, \ldots, M\}$ (or, equivalently, the identified circles $\{C_j' | j = 1, \ldots, M\}$) can be understood, in the language of Riemann surface theory, as M a-cycles on a genus-M Riemann surface; further, any line joining a pair of identified points on C_j and C_j' can be viewed as a b-cycle (there are also M of these). The schematic in Figure 1 illustrates the a and b-cycles for the case $M = 3$.

It is also well-known that any compact Riemann surface of genus-M also possesses exactly M holomorphic differentials which will here be called $\{dv_j(\zeta) | j = 1, \ldots, M\}$. The functions $\{v_j(\zeta) | j = 1, \ldots, M\}$ are the *integrals of the first kind* and each is uniquely determined, up to an additive constant, by their periods around the a- and b-cycles. These functions are analytic, but not single-valued, everywhere in F. Let a_k denote the k-th a-cycle (which can be taken to be the circle C_k) and let b_k denote the k-th b-cycle (which can be taken to be any line joining identified points on C_k and C_k'). Here we normalize the holomorphic differentials so that

$$\oint_{a_k} dv_j = \delta_{kj}\,, \quad \oint_{b_k} dv_j = B_{kj} \tag{2.9}$$

for some set of constants B_{kj}.

Armed with a normalized basis of a and b-cycles, the M integrals of the first kind and the Schottky group Θ, we have now set up all the necessary machinery to be able to define the Schottky–Klein prime function. The following theorem is established in [15]; it holds for any compact Riemann surface, not just the Schottky double of a planar domain considered here:

Theorem 2.1. *There is a unique function $X(\zeta,\tau)$ defined by the properties:*

(i) *$X(\zeta,\tau)$ is analytic everywhere in F.*

(ii) *For any $\tau \in F$, $X(\zeta,\tau)$ has a second-order zero at each of the points $\{\theta(\tau)|\theta \in \Theta\}$.*

(iii) *For any $\tau \in F$,*

$$\lim_{\zeta\to\tau} \frac{X(\zeta,\tau)}{(\zeta-\tau)^2} = 1\,. \tag{2.10}$$

(iv) *For $j = 1, 2, \ldots, M$,*

$$X\big(\theta_j(\zeta),\tau\big) = \exp\bigg(-2\pi i\Big(2\big(v_j(\zeta)-v_j(\tau)\big)+B_{jj}\Big)\bigg)\frac{d\theta_j(\zeta)}{d\zeta}X(\zeta,\tau)\,. \tag{2.11}$$

In [15], the *Klein prime function* $\omega(\zeta,\tau)$ (or what we will call, following [4], the *Schottky–Klein prime function*) is defined as the square root of this function, i.e.,

$$\omega(\zeta,\tau) = \big(X(\zeta,\tau)\big)^{\frac{1}{2}}\,, \tag{2.12}$$

where the branch of the square root is chosen so that $\omega(\zeta,\gamma)$ behaves like $(\zeta-\tau)$ as $\zeta\to\tau$. A further property of $X(\zeta,\tau)$ established in [15] is that

$$X(\zeta,\tau) = X(\tau,\zeta)\,. \tag{2.13}$$

Finally it should be noted that, apart from the zero at τ, $X(\zeta,\tau)$ has no other zeros in the fundamental region F.

3. Quasi-automorphic analogue of the Cauchy kernel

The solution of the RH problem in the theory of automorphic functions and singular integral equations with automorphic kernels can be expressed through automorphic or quasi-automorphic analogues of the Cauchy integral. For the remainder of this paper, let $K(\zeta,\tau)$ and $A(\zeta,\tau)$ respectively denote the quasi-automorphic and automorphic analogues of the Cauchy kernel. In this section, we show how to derive a formula for $K(\zeta,\tau)$ in terms of the function $X(\zeta,\tau)$ defined in the previous section. As mentioned earlier, the function $X(\zeta,\tau)$, with the properties defined above, is a well-defined (indeed, uniquely defined) function on *any* compact Riemann surface.

Alternative representations for the kernel $K(\zeta,\tau)$ (not expressed in terms of the Schottky–Klein prime function) are already known. For finite groups of Möbius transformations the automorphic kernel of the integral is the finite sum [14]

$$K(\zeta,\tau)=\sum_{j=0}^{n}\frac{\theta_j'(\tau)}{\theta_j(\tau)-\zeta}\,,\quad \zeta\in F,\quad \tau\in C\,, \tag{3.1}$$

where C is a smooth contour in the fundamental domain F. For an infinite group Θ, a series representation for $K(\zeta,\tau)$ was proposed in [6, 7] (see also [1]):

$$K(\zeta,\tau)=\sum_{\theta\in\Theta}\left(\frac{1}{\theta(\tau)-\zeta}-\frac{1}{\theta(\tau)-\zeta_*}\right)\theta'(\tau)\,,\quad \zeta\in F,\quad \tau\in C\,, \tag{3.2}$$

where ζ_* is an arbitrary fixed point of the domain F. The convergence of the series (3.2) is guaranteed if the series

$$\sum_{\theta\in\Theta}\theta'(\tau)\,,\quad \tau\in C\,, \tag{3.3}$$

is convergent. If, in addition, the above series converges absolutely, then the series (3.2) converges absolutely and uniformly with respect to ζ in the fundamental domain F and with respect to τ on the line C. Since $-d_\theta/c_\theta\notin F$, then there exists a positive constant M_0 such that $|\tau+d_\theta/c_\theta|>M_0$. Thus, the absolute convergence of the series (3.2) is guaranteed by the convergence of the series (3.3), or, equivalently, by the fact that the group Θ is of the first class (the Burnside classification [5]). For such groups, it is possible to represent a Θ-automorphic function as a series whose elements are simple fractions (the Poincaré series of dimension -2).

Let $H(\zeta,\tau)$ be the logarithmic derivative, with respect to the second argument τ, of the Schottky–Klein prime function, i.e.,

$$H(\zeta,\tau)\equiv\frac{d}{d\tau}\log\omega(\zeta,\tau)=\frac{1}{2}\frac{d}{d\tau}\log X(\zeta,\tau)\,. \tag{3.4}$$

Also define

$$K(\zeta,\tau)\equiv H(\zeta,\tau)-H(\zeta_*,\tau)\,. \tag{3.5}$$

Here ζ_* is an arbitrary fixed point of the domain F. Note also that, by virtue of (2.10), then

$$X(\zeta,\tau)=(\zeta-\tau)^2+O(\zeta-\tau)^3\,, \tag{3.6}$$

so that, as $\zeta\to\tau$,

$$K(\zeta,\tau)=\frac{1}{\tau-\zeta}+R(\zeta,\tau)\,, \tag{3.7}$$

where $R(\zeta,\tau)$ is an analytic function with respect to ζ everywhere in the domain F.

Now, the transformation property (2.11) can be written in the form

$$X\big(\theta_j(\zeta),\tau\big)=F_j(\zeta)G_j(\tau)X(\zeta,\tau)\,, \tag{3.8}$$

where

$$\begin{aligned} F_j(\zeta) &= \exp\big(-4\pi i v_j(\zeta) - 2\pi i B_{jj}\big)\frac{d\theta_j(\zeta)}{d\zeta}\,, \\ G_j(\tau) &= \exp\big(4\pi i v_j(\tau)\big)\,. \end{aligned} \tag{3.9}$$

It also follows from (2.13) that

$$X\big(\zeta,\theta_j(\tau)\big) = X\big(\theta_j(\tau),\zeta\big) = F_j(\tau)G_j(\zeta)X(\tau,\zeta) = F_j(\tau)G_j(\zeta)X(\zeta,\tau)\,, \tag{3.10}$$

where we have used (3.8).

Now consider

$$\begin{aligned} H(\theta_j(\zeta),\tau) &= \frac{1}{2}\frac{d}{d\tau}\log X\big(\theta_j(\zeta),\tau\big) \\ &= \frac{1}{2}\frac{d}{d\tau}\big(\log F_j(\zeta) + \log G_j(\tau) + \log X(\zeta,\tau)\big) \\ &= \frac{1}{2}\frac{d}{d\tau}\log X(\zeta,\tau) + \frac{1}{2}\frac{d}{d\tau}\log G_j(\tau) \\ &= H(\zeta,\tau) + \eta_j(\tau)\,, \end{aligned} \tag{3.11}$$

where

$$\eta_j(\tau) \equiv \frac{1}{2}\frac{d}{d\tau}\log G_j(\tau) = 2\pi i v_j'(\tau)\,. \tag{3.12}$$

Similarly, for any map $\theta(\zeta)$ of the group Θ,

$$H\big(\theta(\zeta),\tau\big) = H(\zeta,\tau) + \eta_\theta(\tau)\,. \tag{3.13}$$

It follows that

$$\begin{aligned} K\big(\theta(\zeta),\tau\big) &= H\big(\theta(\zeta),\tau\big) - H(\zeta_*,\tau) \\ &= H(\zeta,\tau) - H(\zeta_*,\tau) + \eta_\theta(\tau) \\ &= K(\zeta,\tau) + \eta_\theta(\tau)\,, \quad \theta(\zeta)\in\Theta\,. \end{aligned} \tag{3.14}$$

Therefore, as a function of ζ, the kernel $K(\zeta,\tau)$ is a quasi-automorphic function with the cyclic terms $\eta_\theta(\tau)$ [7].

It will now be shown that, with respect to the argument τ, the kernel $K(\zeta,\tau)$ is an automorphic form of dimension -2. Consider

$$\begin{aligned} H\big(\zeta,\theta(\tau)\big) &= \frac{1}{2}\frac{d}{d\theta(\tau)}\log X\big(\zeta,\theta(\tau)\big) \\ &= \frac{1}{2}\frac{d\tau}{d\theta(\tau)}\frac{d}{d\tau}\log X\big(\zeta,\theta(\tau)\big) \\ &= \frac{1}{2}\frac{1}{\theta'(\tau)}\frac{d}{d\tau}\log X\big(\zeta,\theta(\tau)\big)\,, \quad \theta(\zeta)\in\Theta\,. \end{aligned} \tag{3.15}$$

But

$$X\big(\zeta,\theta(\tau)\big) = F_\theta(\tau)G_\theta(\zeta)X(\zeta,\tau)\,, \tag{3.16}$$

so that

$$X_2\big(\zeta,\theta(\tau)\big)\theta'(\tau) = F_\theta'(\tau)G_\theta(\zeta)X(\zeta,\tau) + F_\theta(\tau)G_\theta(\zeta)X_2(\zeta,\tau)\,, \tag{3.17}$$

where $X_2(\zeta,\tau)$ denotes the derivative of $X(\zeta,\tau)$ with respect to its second argument. Dividing (3.17) by (3.16) yields

$$\frac{X_2\big(\zeta,\theta(\tau)\big)}{X\big(\zeta,\theta(\tau)\big)}\theta'(\tau)=\frac{F'_\theta(\tau)}{F_\theta(\tau)}+\frac{X_2(\zeta,\tau)}{X(\zeta,\tau)} \tag{3.18}$$

or, equivalently,

$$\frac{d}{d\tau}\log X\big(\zeta,\theta(\tau)\big)=\frac{F'_\theta(\tau)}{F_\theta(\tau)}+\frac{X_2(\zeta,\tau)}{X(\zeta,\tau)}\,. \tag{3.19}$$

Substitution of (3.19) into (3.15) produces

$$H\big(\zeta,\theta(\tau)\big)=\frac{1}{2}\frac{1}{\theta'(\tau)}\left[\frac{F'_\theta(\tau)}{F_\theta(\tau)}+\frac{X_2(\zeta,\tau)}{X(\zeta,\tau)}\right]. \tag{3.20}$$

Similarly,

$$H\big(\zeta_*,\theta(\tau)\big)=\frac{1}{2}\frac{1}{\theta'(\tau)}\left[\frac{F'_\theta(\tau)}{F_\theta(\tau)}+\frac{X_2(\zeta_*,\tau)}{X(\zeta_*,\tau)}\right]. \tag{3.21}$$

Therefore, subtraction of (3.20) and (3.21) implies

$$K\big(\zeta,\theta(\tau)\big)=\frac{1}{\theta'(\tau)}K(\zeta,\tau)\,. \tag{3.22}$$

However, on use of (2.7), it follows that

$$K\big(\zeta,\theta(\tau)\big)=(c_\theta\zeta+d_\theta)^2K(\zeta,\tau)\,. \tag{3.23}$$

This means that the kernel $K(\zeta,\tau)$, as a function of τ, is the Poincaré series of dimension -2 [7].

We can state the following result.

Theorem 3.1. *Let $\omega(\zeta,\tau)$ be the Schottky–Klein prime function associated with the group Θ, ζ_* be an arbitrary point in the fundamental domain F, $\Phi(\zeta)$ be the integral*

$$\Phi(\zeta)=\frac{1}{2\pi i}\int_C\varphi(\tau)K(\zeta,\tau)d\tau \tag{3.24}$$

with the kernel

$$K(\zeta,\tau)=\frac{d}{d\tau}\log\left(\frac{\omega(\zeta,\tau)}{\omega(\zeta_*,\tau)}\right), \tag{3.25}$$

and $\varphi(\tau)\in H(C)$. Then

(i) *For any $\tau\in C$, $K(\zeta_*,\tau)=0$.*

(ii) *As a function of $\zeta\in F$, $K(\zeta,\tau)$ is a quasi-automorphic function with cyclic terms $\eta_j(\tau)=2\pi i v'_j(\tau)$: $K(\theta_j(\zeta),\tau)=K(\zeta,\tau)+\eta_j(\tau)$, $j=1,2,\dots,M$, $\tau\in C$.*

(iii) *As a function of $\tau\in C$, $K(\zeta,\tau)$ is the Poincaré series of dimension -2 $K(\zeta,\theta(\tau))=(c_\theta\zeta+d_\theta)^2K(\zeta,\tau)$.*

(iv) $K(\zeta,\tau)$ *is an analogue of the Cauchy kernel, and the boundary values of the singular integral* (3.24) *satisfy the Sokhotski–Plemelj formulas*

$$\Phi^{\pm}(t) = \pm\frac{\varphi(t)}{2} + \frac{1}{2\pi i} V.p. \int\limits_C \varphi(\tau) K(t,\tau) d\tau \,, \quad t \in C \,. \tag{3.26}$$

In summary, the function $K(\zeta,\tau)$, a quasi-automorphic analogue of the Cauchy kernel for the symmetric Schottky group Θ, has been presented in terms of a logarithmic derivative of the Schottky–Klein prime function on the Schottky double of the associated planar domain.

This result is significant in light of the fact that Crowdy and Marshall [13] have found a robust numerical scheme for the calculation of the prime function which does not employ a representation that is a sum or product over the elements of a Schottky group. In [6, 7] such an infinite sum representation of the Cauchy kernel is given. Now, when this infinite sum converges slowly or is absolutely divergent, (3.4) and (3.25) for $K(\zeta,\tau)$ in terms of the Schottky–Klein prime function can be used for its evaluation, together with the numerical algorithm of [13] for the evaluation of the prime function.

It is also important to point out that, in the construction of the prime function as presented in [13], the subsidiary functions $\{v_j(\zeta) | j = 1,\ldots,M\}$ are also computed as part of the algorithm. It follows from (3.12) that the algorithm in [13] not only provides a way of evaluating the prime function $\omega(\zeta,\tau)$ but, at the same time, a method for computing the cyclic terms associated with the kernel $K(\zeta,\tau)$.

In an appendix it is shown how to explicitly retrieve the Chibrikova–Silvestrov series representation [6] for the kernel $K(\zeta,\tau)$ when an infinite product representation of the prime function $\omega(\zeta,\tau)$ (given, for example, in [4]) is substituted into (3.25).

4. RH problem on a Riemann surface

Let $C = C_1 \cup C_2 \cup \ldots \cup C_M$ and $C' = C'_1 \cup C'_2 \cup \ldots \cup C'_M$. Let Λ be the set of the limit points of the group Θ. Then $\bar{\mathbb{C}} = \Omega \cup \Lambda$, where $\bar{\mathbb{C}}$ is the extended complex plane, and $\Omega = \cup_{\theta\in\Theta}\theta(F)$. Let $\Phi(\zeta)$ be a holomorphic function in the domain D_ζ. Extend its definition for the whole domain Ω by

$$\begin{aligned} &\Phi(\zeta) = \overline{\Phi\big(T(\zeta)\big)}\,, \quad \zeta \in T(D_\zeta)\,, \\ &\Phi(\zeta) = \Phi\big(\theta^{-1}(\zeta)\big)\,, \quad \zeta \in \theta\big(D_\zeta \cup T(D_\zeta)\big)\,, \quad \theta \in \Theta\,. \end{aligned} \tag{4.1}$$

Note that the first expression in (4.1) extends the definition of $\Phi(\zeta)$ to the other "half" of the fundamental region F while the second expression extends $\Phi(\zeta)$ to all regions of the plane equivalent to F under the group action. With this definition, $\Phi(\zeta)$ is a piecewise holomorphic and Θ-automorphic function and satisfies the symmetry condition

$$\overline{\Phi\big(\theta_j T^{-1}(\zeta)\big)} = \Phi(\zeta)\,, \quad \zeta \in \theta_j\big(T(D_\zeta)\big), \quad j = 1,2,\ldots,M\,. \tag{4.2}$$

All circles $\theta(C)$ including C are the discontinuity lines of the function $\Phi(\zeta)$. Let $\Phi^+(\zeta)$ and $\Phi^-(\zeta)$ be the boundary values of the function $\Phi(\zeta)$ from the interior and the exterior of the circles $\theta(C)$, $\theta \in \Theta$, respectively. Consider the following RH problem (for simplicity, we confine ourselves to analyzing the homogeneous problem).

Problem 4.1. *Find all piecewise holomorphic and Θ-automorphic functions bounded at infinity which meet the symmetry condition* (4.2) *and on the discontinuity line C of the group Θ satisfy the linear relation*

$$\Phi^+(\tau) = \Pi(\tau)\Phi^-(\tau)\,, \quad \tau \in C\,. \tag{4.3}$$

The function $\Pi(\tau)$ satisfies the solvability condition of the problem $|\Pi(\tau)| = 1$, it is H-continuous on the contour C everywhere but at a finite set of singular points $\Sigma = \{t_{\nu 1}, t_{\nu 2}, \ldots, t_{\nu m_\nu}; \nu = 1, 2, \ldots, M\}$, where it is finite and discontinuous. The function $\Phi(\zeta)$ is H-continuous in the domain D_ζ up to the contour C apart from the singular points, where it may have integrable singularities.

4.1. The automorphic kernel $A(\zeta, \tau)$

For the solution of Problem 4.1, it will be convenient to use the automorphic kernel [7]

$$A(\zeta, \tau) = K(\zeta, \tau) - \sum_{j=1}^{M} \hat{\omega}_j(\tau) K(\zeta, c_j)\,, \tag{4.4}$$

where $K(\zeta, \tau)$ is the logarithmic derivative of the Schottky–Klein prime function introduced in (3.25). The points $c_1, \ldots, c_M \in D_\zeta$ are fixed arbitrarily subject only to the condition that

$$\Delta_0 \equiv \det ||\eta_j(c_\nu)||_{j,\nu=1,\ldots,M} \neq 0\,. \tag{4.5}$$

The functions $\hat{\omega}_1(\tau), \ldots, \hat{\omega}_M(\tau)$ solve the following linear system of algebraic equations

$$\sum_{j=1}^{M} \eta_j(c_j)\hat{\omega}_j(\tau) = \eta_\nu(\tau)\,, \quad \nu = 1, \ldots, M\,. \tag{4.6}$$

This choice guarantees the automorphicity of the kernel: $A(\zeta, \tau) = A(\theta(\zeta), \tau)$. In the domain D_ζ, in addition to the simple pole at $\zeta = \tau$, the function $A(\zeta, \tau)$ has M simple poles at the points $c_1, \ldots, c_M$. Note that because of (4.6) the residues of the kernel at the points c_ν, the functions $\hat{\omega}_\nu(\tau)$ ($\nu = 1, \ldots, M$) are linear combinations of the cyclic terms $\eta_\nu(\tau) = 2\pi i v'_\nu(\tau)$.

The kernel $A(\zeta, \tau)$ is an automorphic analogue of the Weierstrass kernel on a compact Riemann surface and can be presented as follows [7]

$$A(\zeta, \tau) = \frac{\Delta(\zeta, \tau)}{\Delta_0}\,, \tag{4.7}$$

where

$$\Delta(\zeta,\tau) = \begin{vmatrix} K(\zeta,\tau) & K(\zeta,c_1) & \dots & K(\zeta,c_M) \\ \eta_1(\tau) & \eta_1(c_1) & \dots & \eta_1(c_M) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_M(\tau) & \eta_M(c_1) & \dots & \eta_M(c_M) \end{vmatrix} . \tag{4.8}$$

4.2. Factorization on the Schottky double

Let $\mathfrak{R}$ be the Schottky double associated with the domain D_ζ. It is formed by gluing the two halves D_ζ and D'_ζ of the fundamental domain F along their boundaries C and C', (the a-cycles), respectively (Figure 1). The b-cycles lie on both sheets of the surface (the broken line corresponds to the part which lies on the second sheet D'_ζ). The loop b_j crosses the cycle a_j from right to the left and does not cross the other loops a_ν and b_ν $(\nu \neq j)$. A point q with affix ζ on this two-sheeted surface $\mathfrak{R}$ of genus-M is ζ on the first sheet D_ζ and $T(\zeta)$ on the second sheet D'_ζ. Introduce a new function $\mathfrak{F}(q)$

$$\mathfrak{F}(q) = \begin{cases} \Phi(\zeta), & q \in D_\zeta \\ \overline{\Phi(T(\zeta))}, & q \in D'_\zeta \end{cases} \tag{4.9}$$

which is holomorphic and Θ-automorphic on the surface $\mathfrak{R}$. The RH problem (4.3) can be written as the following RH problem on the surface $\mathfrak{R}$.

4.2.1. Problem 4.2.

Find all meromorphic functions on $\mathfrak{R} \setminus \mathfrak{C}$ which admit an H-continuous extension to $\mathfrak{C} \setminus \Sigma$ and whose boundary values satisfy the relation

$$\mathfrak{F}^+(p) = \mathfrak{G}(p)\mathfrak{F}^-(p)\,, \quad p \in \mathfrak{C}\,, \tag{4.10}$$

where

$$\mathfrak{G}(p) = \begin{cases} \Pi(\tau), & p \in C \\ \overline{[\Pi(T(\tau))]^{-1}}, & p \in C' \end{cases}, \qquad p = q(\tau)\,, \tag{4.11}$$

and $\mathfrak{C} = C \cup C'$ is a contour on the surface $\mathfrak{R}$, $C \subset D_\zeta$, $C' \subset D'_\zeta$.

We next specify the branches of the integrals

$$v_j(\zeta) = \int_{s_0}^{\zeta} dv_j(\tau)\,, \quad j = 1,\dots,M\,, \tag{4.12}$$

which are single-valued in the domain $\hat{D}_\zeta$ cut along the curves $\tilde{b}_j$ (the parts of the loops b_j which lie in D_ζ). Here s_0 is an arbitrarily fixed point on the unit circle C_0 which does not lie on the b-loops. Since $\operatorname{Im} dv_j(\tau) = 0$, $\tau \in C_j$, the relation

$$dv_j(\zeta) = \overline{dv_j\big(T(\zeta)\big)}\,, \quad \zeta \in D_\zeta\,, \tag{4.13}$$

continues analytically the differentials $dv_j(\zeta)$ into the domain D'_ζ. These differentials are the abelian differentials of the first kind of the Schottky double $\mathfrak{R}$. The basis of the differentials can be normalized

$$\int_{a_j} dv_\nu = 2\,\mathrm{Re}\int_{C_j} dv_\nu = \delta_{\nu j}\,,$$

$$\int_{b_j} dv_\nu = 2i\,\mathrm{Im}\int_{\tilde{b}_j} dv_\nu = B_{\nu j}\,, \quad \nu, j = 1, \ldots M\,. \tag{4.14}$$

The matrix $B = ||B_{\nu j}||_{\nu,j=1,\ldots,M}$ is symmetric, $\mathrm{Re}\,B = 0$, and $\mathrm{Im}\,B$ is a positive definite matrix.

Now factorize the coefficient $\Pi(t)$ so that

$$\Pi(t) = \frac{\Psi^+(t)}{\Psi^-(t)}\,, \quad t \in C\,, \tag{4.15}$$

where

$$\Psi(\zeta) = \exp\left\{\Gamma(\zeta) + \overline{\Gamma\big(T(\zeta)\big)}\right\}, \tag{4.16}$$

and

$$\Gamma(q) = \frac{1}{8\pi i}\int_{\mathfrak{C}} \log\mathfrak{G}(p)A(q,p)dp + \sum_{j=1}^{M}\left(\kappa_j\int_{t_{j1}p_j} A(q,p)dp\right.$$

$$\left.+\int_{r_jq_j} A(q,p)dp + m_j\oint_{a_j} A(q,p)dp + n_j\oint_{b_j} A(q,p)dp\right). \tag{4.17}$$

$\Psi(\zeta)$ will be referred to as the *canonical solution*. Here $t_{j1}p_j \subset D_\zeta$ and $r_jq_j \subset \mathfrak{R}$. The curves r_jq_j pass through the point $s_0 \in C_0$ and do not cross each other and the a- and b-cycles. The points $r_j \in D_\zeta$ and $p_j \in D_\zeta$ are fixed arbitrarily (Figure 1). The points $q_j \in \mathfrak{R}$ and the integers m_j and n_j are to be determined. If it turns out that $q_j \in D_\zeta$, then the curves r_jq_j lie on the first sheet D_ζ. Otherwise, if $q_j \in D'_\zeta$, then the contours r_jq_j consist of two parts $r_js_0 \subset D_\zeta$ and $s_0q_j \subset D'_\zeta$. A single branch of the function $\log\Pi(\tau)$ is fixed (arbitrarily) by imposing the condition $\arg\Pi(t^+_{\nu 1}) = \varphi_{\nu 1}$, $\nu = 1, \ldots, M$. Let $\Delta_{\nu j}$ be the increment of $\arg\Pi(\tau)$ as the point τ traverses the arc $t_{\nu j}t_{\nu j+1}$, $j = 1, \ldots, m_\nu$, and $t_{\nu m_\nu}t_{\nu m_\nu+1}$ is assumed to be $t_{\nu m_\nu}t_{\nu 1}$. Then $\arg\Pi(t^-_{\nu j+1}) = \varphi_{\nu j} + \Delta_{\nu j}$, $j = 1, \ldots, m_\nu$, where $\varphi_{\nu j} = \arg\Pi(t^+_{\nu j})$. The arguments $\varphi_{\nu j}$ ($\nu = 2, \ldots, m_\nu$) cannot be chosen to be arbitrary but are defined by the class of solutions.

Since $\Phi(\zeta)$ may have integrable singularities at the singular points, define $\varphi_{\nu j} = \arg\Pi(t^+_{\nu j})$ by

$$-2\pi < \arg\Pi(t^-_{\nu j}) - \arg\Pi(t^+_{\nu j}) < 0\,, \quad j = 2, \ldots, m_\nu\,. \tag{4.18}$$

We next introduce integers κ_ν ($\nu = 1, \ldots, M$) which are uniquely defined by the inequalities

$$-2\pi < \arg\Pi(t^-_{\nu 1}) - \arg\Pi(t^+_{\nu 1}) - 4\pi\kappa_\nu < 0\,, \tag{4.19}$$

or, equivalently,

$$-2\pi < \varphi_{\nu m_\nu} - \varphi_{\nu 1} + \Delta_{\nu m_\nu} - 4\pi\kappa_\nu < 0\,. \tag{4.20}$$

These conditions guarantee that the solution has integrable singularities at the initial points $t_{\nu 1}$.

It will now be shown that the boundary values of the function $\Psi(t)$ solve (4.15). Note that if $q=\zeta$, then the first integral in (4.17) is equal to

$$\frac{1}{4\pi i}\int_C \log\Pi(\tau)A(\zeta,\tau)d\tau\,. \tag{4.21}$$

Clearly, if $t\in C$, then $\Gamma^+(t)-\Gamma^-(t) = 1/2\log\Pi(t)$. It is directly verified that on the contours C_j, $\theta_j^{-1}(t) = T(t)$. Since the kernel $A(\zeta,\tau)$ is automorphic, it follows that

$$\overline{\Gamma^\pm\big(T(t)\big)} = \overline{\Gamma^\pm\big(\theta^{-1}(t)\big)} = \overline{\Gamma^\pm(t)}\,. \tag{4.22}$$

Furthermore, $|\Pi(\tau)| = 1$ on C, and thus the jump $\overline{\Gamma^+(T(t))} - \overline{\Gamma^-(T(t))}$ on the contour C is also $1/2\log\Pi(t)$.

Let us now examine the properties of the canonical solution $\Psi(\zeta)$ at the points c_ν, $\nu=1,\dots,M$. Since the kernel $A(\zeta,\tau)$ has simple poles at the points c_ν, the function $\Psi(\zeta)$ has essential singularities at these points. They become removable singular points if

$$\frac{1}{4\pi i}\int_C \log\Pi(\tau)\hat\omega_\nu(\tau)d\tau + \sum_{j=1}^{M}\left(\kappa_j\int_{t_{j1}p_j}\hat\omega_\nu(\tau)d\tau + \int_{r_jq_j}\hat\omega_\nu(\tau)d\tau \right.$$
$$\left. + \, m_j\oint_{a_j}\hat\omega_\nu(\tau)d\tau + n_j\oint_{b_j}\hat\omega_\nu(\tau)d\tau\right) = 0\,, \quad \nu=1,\dots,M\,. \tag{4.23}$$

Recognizing the integrals

$$u_\nu(q) = \int_{s_0q}\hat\omega_\nu(\tau)d\tau \tag{4.24}$$

as abelian integrals on the Riemann surface $\mathfrak{R}$ (single-valued on the surface $\mathfrak{R}$ cut along the loops a_j and b_j) and the differentials $\{\hat\omega_\nu(\tau)d\tau\}_{\nu=1,\dots,M}$ as abelian differentials, we can rewrite (4.23) as the following Jacobi inversion problem

$$\sum_{j=1}^{M}\left[u_\nu(q_j) + m_j\tilde A_{\nu j} + n_j\tilde B_{\nu j}\right] = d_\nu\,, \quad \nu=1,\dots,M\,, \tag{4.25}$$

where $\tilde A_{\nu j}$ and $\tilde B_{\nu j}$ $(\nu, j = 1,\dots,M)$ form the A and B period matrices, and

$$d_\nu = -\frac{1}{4\pi i}\int_C \log\Pi(\tau)\hat\omega_\nu(\tau)d\tau - \sum_{j=1}^{M}\left(\kappa_j\int_{t_{j1}p_j}\hat\omega_\nu(\tau)d\tau + \int_{r_js_0}\hat\omega_\nu(\tau)d\tau\right). \tag{4.26}$$

The basis of abelian differentials can be normalized: $d\mathbf{v} = \hat{A}\hat{\omega}d\tau$, where $d\mathbf{v} = \{dv_\nu(\tau)\}_{\nu=1,\dots,M}$ is the canonical basis previously introduced and $\hat{A} = \tilde{A}^{-1}$. Because of the property (4.14) of the differentials dv_ν, the problem (4.25) reduces to

$$\sum_{j=1}^{M}\left[v_\nu(q_j) + n_j B_{\nu j}\right] + m_\nu = \hat{d}_\nu\,, \quad \nu = 1,\dots,M\,, \tag{4.27}$$

where $B = \hat{A}\tilde{B}$, and

$$\hat{d}_\nu = \sum_{j=1}^{M}\hat{A}_{\nu j}d_j\,, \quad \nu = 1,\dots,M\,. \tag{4.28}$$

The analysis of (4.27) and the first formula in (4.17) shows that the canonical function $\Psi(\zeta)$ does not have essential singularities if the points q_j and the integers m_ν solve the real analogue of the Jacobi inversion problem

$$\sum_{j=1}^{M}\operatorname{Re} v_\nu(q_j) + m_\nu = \operatorname{Re}\hat{d}_\nu, \quad \nu = 1,\dots,M\,. \tag{4.29}$$

The solution to the classical inversion problem (4.27) always exists [18] and can be expressed through the M zeros of the associated Riemann theta-function of the surface $\mathfrak{R}$ [1–3]. Notice that since (4.29) does not depend on the integers n_ν, they can be chosen arbitrarily say, $n_1 = \cdots = n_M = 0$, provided the points q_ν and the integers m_ν solve the problem (4.27).

4.3. The general solution to the RH problem

Consider the properties of the function $[\Psi(\zeta)]^{-1}\Phi(\zeta)$. It has simple poles at the points $q_j \in D_\zeta \cup D'_\zeta$ ($j = 1,\dots,M$). Also, if the integers κ_j are positive, then the function $[\Psi(\zeta)]^{-1}\Phi(\zeta)$ has poles of orders κ_j at the points $p_j \in D_\zeta$. It is bounded if $\kappa_j = 0$ and has zeros of orders $-\kappa_j$ if $\kappa_j < 0$. By the generalized Liouville theorem, the general solution to Problem 4.1 has the form

$$\Phi(\zeta) = \Psi(\zeta)\sum_{j=1}^{M}\left[\frac{E_j}{f(\zeta) - f(q_j)} + \frac{P_{\kappa_j - 1}\big(f(\zeta)\big)}{\left[f(\zeta) - f(p_j)\right]^{\kappa_j}}\right], \tag{4.30}$$

where $f(\zeta)$ is the fundamental function of the symmetric Schottky group

$$f(\zeta) = \frac{A(\zeta, a) + \overline{A\big(T(\zeta), a\big)}}{2}\,, \tag{4.31}$$

and $a \in D_\zeta \cup D'_\zeta$ is an arbitrary fixed point. Clearly, it is an automorphic function: $f(\zeta) = f(\theta(\zeta))$, $\theta \in \Theta$, and meets the symmetry condition $f(\zeta) = \overline{f(T(\zeta))}$. The constants E_j are arbitrary real constants, and $P_{\kappa_j - 1}$ is an arbitrary polynomial of degree $\kappa_j - 1$ with real coefficients. If $\kappa_j \le 0$, it is assumed that $P_{\kappa_j - 1} \equiv 0$. In general, the function (4.30) has simple poles at the points $r_\nu \in D_\zeta$ ($\nu = 1, \dots, M$), the simple poles of the canonical function $\Psi(\zeta)$. To remove them, we require

$$\sum_{j=1}^{M} \frac{E_j}{f(r_\nu) - f(q_j)} + \frac{P_{\kappa_j - 1}\big(f(r_\nu)\big)}{\big[f(r_\nu) - f(p_j)\big]^{\kappa_j}} = 0\,, \quad \nu = 1, \dots, M\,. \tag{4.32}$$

Let $\kappa = \sum_{j=1}^{M} \kappa_j$. Summing up, we have shown the following result.

Theorem 4.1. *If the index κ of the homogeneous RH problem* (4.3) *is positive, then it has κ linearly independent solutions* (4.30). *For $\kappa \leq 0$, the problem only admits a trivial solution.*

Finally, note that the technique just described also solves the homogeneous Hilbert problem for multiply connected circular domains with discontinuous coefficients.

Problem 4.3. *Find all functions $\phi(\zeta) = u(\zeta) + iv(\zeta)$ holomorphic in D_ζ, H-continuous up to the contour C apart from some points $t_{\nu 1}, t_{\nu 2}, \dots, t_{\nu m_\nu}$ $\nu = 1, \dots, M$, where they may have integrable singularities and satisfying the boundary condition*

$$\alpha(\tau)u(\tau) + \beta(\tau)v(\tau) = 0\,, \quad \tau \in C\,. \tag{4.33}$$

Here $\alpha(\tau)$ and $\beta(\tau)$ are real functions which are H-continuous on the contour C everywhere but at the singular points $t_{\nu 1}, \dots t_{\nu m_\nu}$, where they are bounded and discontinuous.

This problem is equivalent [17] to the RH problem (4.3) with the coefficient

$$\Pi(\tau) = -\frac{\alpha(\tau) + i\beta(\tau)}{\alpha(\tau) - i\beta(\tau)}\,, \tag{4.34}$$

which satisfies the condition of solvability of Problem 4.1 $|\Pi(\tau)| = 1$.

Appendix A. Verification of infinite sum formulas

In this appendix we explicitly verify that an infinite sum formula for $K(\zeta, \tau)$ that appears in [1, 6] can be retrieved from the new formula (3.25) when an infinite product representation of the prime function $\omega(\zeta, \tau)$ converges absolutely. Note, importantly, that formula (3.25) for the Cauchy kernel in terms of the prime function $\omega(\zeta, \gamma)$ still holds *regardless* of whether the infinite product representation for the prime function converges absolutely.

Consider the kernel

$$K(\zeta, \tau) = \frac{d}{d\tau} \log\left(\frac{\omega(\zeta, \tau)}{\omega(\zeta_*, \tau)}\right). \tag{A.1}$$

For the first class groups Θ, the Schottky–Klein prime function $\omega(\tau,\zeta)$ admits the representation [4]

$$\omega(\tau,\zeta) = (\tau-\zeta)\prod_{\theta\in\Theta''}\frac{(\theta(\tau)-\zeta)(\theta(\zeta)-\tau)}{(\theta(\tau)-\tau)(\theta(\zeta)-\zeta)} \tag{A.2}$$

where, as introduced in Crowdy & Marshall [11], the notation Θ'' denotes all elements of the Schottky group *excluding* the identity and all inverses. By substituting the infinite product formula (A.2) for $\omega(\tau,\zeta)$ and $\omega(\tau,\zeta_*)$ into (A.1), we have

$$K(\zeta,\tau) = \frac{d}{d\tau}\log\left(\frac{\tau-\zeta}{\tau-\zeta_*}\prod_{\theta\in\Theta''}\frac{(\theta(\tau)-\zeta)(\theta(\zeta)-\tau)}{(\theta(\tau)-\zeta_*)(\theta(\zeta_*)-\tau)}\right). \tag{A.3}$$

To simplify (A.3), we make use of the identity

$$\theta(\zeta)-\tau = \left(\frac{c_\theta\tau-a_\theta}{c_\theta\zeta+d_\theta}\right)(\theta^{-1}(\tau)-\zeta) \tag{A.4}$$

which follows from the formulas for any map $\theta(\zeta)$ of the group Θ and the inverse $\theta^{-1}(\tau)$

$$\theta(\zeta) = \frac{a_\theta\zeta+b_\theta}{c_\theta\zeta+d_\theta}, \quad \theta^{-1}(\tau) = \frac{d_\theta\tau-b_\theta}{-c_\theta\tau+a_\theta}. \tag{A.5}$$

The identity (A.4) reduces (A.3) to

$$\begin{aligned} K(\zeta,\tau) &= \frac{d}{d\tau}\log\left(\frac{\tau-\zeta}{\tau-\zeta_*}\prod_{\theta\in\Theta''}\frac{(\theta(\tau)-\zeta)(\theta^{-1}(\tau)-\zeta)}{(\theta(\tau)-\zeta_*)(\theta^{-1}(\tau)-\zeta_*)}\right) \\ &= \frac{d}{d\tau}\log\left(\prod_{\theta\in\Theta}\frac{\theta(\tau)-\zeta}{\theta(\tau)-\zeta_*}\right). \end{aligned} \tag{A.6}$$

The final equality of (A.6) retrieves precisely the infinite sum formula for $K(\zeta,\tau)$

$$K(\zeta,\tau) = \sum_{\theta\in\Theta}\left(\frac{1}{\theta(\tau)-\zeta}-\frac{1}{\theta(\tau)-\zeta_*}\right)\theta'(\tau), \quad \zeta\in F, \quad \tau\in L, \tag{A.7}$$

presented in [1].

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