ON THE POSITIVITY CONJECTURE: A DIGEST OF MASON’S COUNTEREXAMPLE

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1. Introduction

In this note, we give an explicit construction of a counterexample to the positivity conjecture on the second Frobenius-Schur indicators of a modular category. The positivity conjecture can be formulated in the following way [Wan10, Conjecture 4.26]:

Let \( C \) be a modular category, and let \( X, Y \) be simple objects of \( C \). Then \( N^Y_{X,X} > 0 \) implies \( \nu_2(Y) = 1 \).

We will borrow ideas from Mason’s preprint [Mas17], but our example comes from a smaller group than that in [Mas17]. In Section 3, we give the explicit character data of the counterexample implemented in GAP.

We thank Professor Richard Ng for helpful discussions and suggestions.

2. Construction of a counterexample

Let \( Q := Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle \) be the quaternion group. It is easy to check that the assignment \( \alpha : Q \rightarrow \text{GL}(2, 3) \), given on generators by

\[
\begin{align*}
    a &\mapsto \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \\
    b &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

embeds \( Q \) into \( \text{GL}(2, 3) \) as a subgroup. Therefore, \( Q \) acts faithfully on \( H := (\mathbb{Z}/3\mathbb{Z})^2 \) by (left) matrix multiplication (with column vectors). For any \( q \in Q \) and any \( h \in H \), we denote this action by \( q \cdot h \).

Let \( G := H \rtimes_\alpha Q \) be the semidirect product of \( H \) and \( Q \) with respect to \( \alpha \). As subgroups of \( G \), \( Q \) acts on \( H \) by conjugation (in \( G \)). Note that \( G \) is of order 72.

Let \( \lambda : H \rightarrow \mathbb{C}^\times \) be a character of \( H \) defined by

\[
\lambda(x, y) := \omega^{y-x}
\]

for any \((x, y) \in H = (\mathbb{Z}/3\mathbb{Z})^2\), where \( \omega = \exp(\frac{2\pi i}{3}) \). Let \( \chi := \text{ind}_H^G(\lambda) \) be the representation of \( G \) induced by \( \lambda \).

**Proposition 1.** The representation \( \chi \) is irreducible.

**Proof.** We prove the irreducibility of \( \chi \) by computing its character. By the character formula of induced representations ([Ser77, Theorem 12]), for any \( g \in G \), we have

\[
\text{char}(\chi)(g) = \sum_{q \in Q \atop q^{-1}gg \in H} \lambda(q \cdot g) = \begin{cases}
0, & \text{if } g \notin H \\
-1, & \text{if } g \in H \setminus \{e_G\} \\
8, & \text{if } g = e_G
\end{cases}
\]

where \( e_G \) stands for the identity of \( G \).
Therefore, the character inner product (following notations in [Ser77]) of $\chi$ with itself is given by

\[
(char(\chi)|char(\chi)) = \frac{1}{72} \sum_{g \in G} |char(\chi)|^2 = \frac{1}{72} \times (8^2 + (-1)^2 \times 8) = 1.
\]

Hence, by [Ser77, Theorem 3], $\chi$ is irreducible. \hfill \Box

Remark 1. We can also apply Mason’s idea to give an alternative proof of the irreducibility of $\chi$, which goes as follows. By a corollary of Mackey’s irreducibility criterion of induced representations ([Ser77, Corollary 7.4.23]), $\chi$ is irreducible if and only if

1. $\lambda$ is irreducible;
2. $g\lambda \neq \lambda$ for every $g \notin H$.

Here, $g\lambda : H \to \mathbb{C}^\times$ is defined by $g\lambda(h) := \lambda(g^{-1}hg)$ for any $h \in H$. It is clear that $\lambda$ is irreducible, so it remains to prove (2), which is the result of direct computation using the expression for $\lambda$.

We proceed with the following known facts. Firstly, $Q$ has four 1-dimensional (denoted by $\gamma_1, \ldots, \gamma_4$) and one 2-dimensional irreducible representation (denoted by $\phi$). Moreover, direct computation shows that $\nu_2(\phi) = -1$. In addition, by dimension counting, $\chi$ is the unique 8-dimensional representation of $G$ with $\nu_2(\chi) = 1$.

Since $\text{Rep}(Q)$ is a braided (symmetric) spherical fusion full subcategory of $\text{Rep}(G)$, any irreducible representation of $Q$ can be viewed as an irreducible representation of $G$ by pre-composing the quotient map $G \twoheadrightarrow Q$. In addition, for any $X \in \text{Rep}(Q)$, its dimension and its second Frobenius-Schur indicator $\nu_2(X)$ computed in $\text{Rep}(Q)$ is the same as $\nu_2(X)$ computed in $\text{Rep}(G)$. More precisely, let $\xi \in \text{Rep}(Q)$, by definition, $\nu_2(\xi)$ in $\text{Rep}(Q)$ and in $\text{Rep}(G)$ are given respectively as

\[
\nu_2(\xi)_{\text{Rep}(Q)} = \frac{1}{|Q|} \sum_{q \in Q} \text{char}(\xi)(q^2),
\]
and

\[
\nu_2(\xi)_{\text{Rep}(G)} = \frac{1}{|G|} \sum_{g \in G} \text{char}(\xi)(g^2).
\]

By definition, when we view $\xi$ as in $\text{Rep}(G)$, we have $\xi(gh) = \xi(g)$ for any $g \in G$ and $h \in H$. Therefore, by the fact that $Q \cong G/H$, we have

\[
\nu_2(\xi)_{\text{Rep}(G)} = \frac{1}{|G|} \sum_{g \in G/H} \text{char}(\xi)(g^2)|H| = \frac{1}{|Q|} \sum_{q \in Q} \text{char}(\xi)(q^2) = \nu_2(\xi)_{\text{Rep}(Q)}.
\]

Similar argument holds for $\text{Rep}(D(G))$. More precisely, since both $\text{Rep}(Q)$ and $\text{Rep}(G)$ are braided (symmetric) spherical fusion full subcategory of $\text{Rep}(D(G))$, for any $X \in \text{Rep}(Q)$, and for any $Y \in \text{Rep}(G)$, we have

\[
\nu_2(X)_{\text{Rep}(Q)} = \nu_2(X)_{\text{Rep}(D(G))}
\]
and

\[
\nu_2(Y)_{\text{Rep}(G)} = \nu_2(Y)_{\text{Rep}(D(G))}.
\]

In particular, we have

1. $\nu_2(\phi)_{\text{Rep}(D(G))} = \nu_2(\phi)_{\text{Rep}(Q)} = -1$. 

Let $\rho_G$ be the regular representation of $G$. It is standard that

$$\rho_G = \bigoplus_{j=1}^{4} \gamma_j \oplus 2\phi \oplus 8\chi.$$  

**Theorem 1.** The 2-dimensional representation $\phi$ is a constituent of $\chi \otimes \chi$ in $\text{Rep}(G)$.

**Proof.** It is well-known (or see Appendix) that

$$\rho_G \otimes \phi = \rho_G^{\oplus \text{deg}(\phi)} = \rho_G \oplus \rho_G = 2 \bigoplus_{j=1}^{4} \gamma_j \oplus 4\phi \oplus 16\chi.$$  

Decomposing the left hand side, we have

$$\rho_G \otimes \phi = 4\phi \oplus 2 \bigoplus_{j=1}^{4} \gamma_j \oplus (8\chi \otimes \phi),$$

where the first two summands are derived from the familiar representation theory of $Q$. Comparing both sides of Equation (2), we have

$$(3) \quad \chi \otimes \phi = 2\chi.$$  

In other words, $\text{Hom}_{\text{Rep}(G)}(\chi \otimes \phi, \chi) \neq 0$, which implies that $\text{Hom}_{\text{Rep}(G)}(\chi \otimes \chi, \phi) \neq 0$, as both $\chi$ and $\phi$ are self-dual. $\square$

As pointed out before, we can view $\chi$ and $\phi$ as objects in the modular category $\text{Rep}(D(G))$. Since $\text{Rep}(G)$ is a fusion full subcategory of $\text{Rep}(D(G))$, we will still have $N_{\chi,\chi}^\phi = 2$ in $\text{Rep}(D(G))$. Together with Equation (1) and Theorem 1 we have

**Theorem 2.** In the modular category $\text{Rep}(D(G))$, there exist irreducible representations $\chi, \phi \in \text{Irr}(\text{Rep}(D(G)))$ such that $N_{\chi,\chi}^\phi = N_{\chi,\chi}^{\phi^*} = 2$ and $\nu_2(\phi) = -1$. $\square$

The above theorem nullifies the positivity conjecture.

**Remark 2.** Equation (3) implies that $\nu_2$ is not a fusion character. Indeed, by definition and the linearity of $\nu_2$, we have

$$\nu_2(\chi \otimes \phi) = \nu_2(2\chi) = 2\nu_2(\chi) = 2,$$

while

$$\nu_2(\chi) \times \nu_2(\phi) = 1 \times (-1) = -1.$$

3. GAP IMPLEMENTATION

In fact, we can identify $G$ with $\text{PSU}(3, 2)$ whose GAP ID is $\text{SmallGroup}(72, 41)$. We use the following code in GAP to get the information we need.

```gap
G:=SmallGroup(72,41);;
Irr(G);
```

The output is
We can see that among the 6 irreducible representations, there is a unique 8-dimensional representation, which is denoted by $\chi$ in the previous section.

Next, we compute the second Frobenius-Schur indicator of the above irreducible representations

$\text{Indicator}(\text{CharacterTable}(G), 2)$;

The output is

$[1, 1, 1, 1, -1, 1]$.

This means the 2-dimensional irreducible representation of $G$ has -1 as its $\nu_2$.

Finally, we decompose $\chi \otimes \chi$ into irreducible representations

$\text{ConstituentsOfCharacter}(\text{Irr}(G)[6] \ast \text{Irr}(G)[6])$;

The output is

$[\text{Character}(\text{CharacterTable}(G), [1, -1, 1, 1, -1, 1]),$
$\text{Character}(\text{CharacterTable}(G), [1, -1, 1, 1, 1, -1]),$
$\text{Character}(\text{CharacterTable}(G), [1, 1, 1, -1, 1, -1]),$
$\text{Character}(\text{CharacterTable}(G), [1, 1, 1, 1, 1, 1]),$
$\text{Character}(\text{CharacterTable}(G), [2, 0, 0, -2, 2, 0]),$
$\text{Character}(\text{CharacterTable}(G), [8, 0, 0, 0, -1, 0])]$.

We can see that the 2-dimensional irreducible representation of $G$ is indeed a constituent of $\chi \otimes \chi$.

**Appendix**

For any braided spherical fusion category $C$, let $\text{Irr}(C)$ denote the set of isomorphism class of simple objects, and let $d_X$ be the categorical dimension of $X \in C$. Let $R := \sum_{X \in \text{Irr}(C)} d_X X$ be the regular element in the Grothendieck algebra of $C$.

**Lemma 1.** For any $V \in C$, we have the equality in the Grothendieck algebra of $C$

$RV = d_V R$. 

Proof:

\[ RV = \left( \sum_{X \in \text{Irr}(\mathcal{C})} d_X X \right) V \]
\[ = \sum_{X \in \text{Irr}(\mathcal{C})} d_X \sum_{Y \in \text{Irr}(\mathcal{C})} N_{X,Y}^V Y \]
\[ = \sum_{Y \in \text{Irr}(\mathcal{C})} Y \sum_{X \in \text{Irr}(\mathcal{C})} N_{X,Y}^V d_X \]
\[ = \sum_{Y \in \text{Irr}(\mathcal{C})} d_Y d_Y Y = d_Y R. \]

\[ \square \]

References

