Homework II

1. Regarding S^1 as a unit circle in the complex plane, let $f_n : S^1 \to S^1$ be the map sending z to z^n. Describe all the connected covering spaces of $X = S^1 \cup_{f_3} D^2$ and $Y = S^1 \cup_{f_4} D^2$

2. Let X, Y be spaces with base points x_0, y_0 and let $f : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be any homomorphism. Prove that if X is a 2-dimensional CW-complex, then there is a map $\phi : X \to Y$ such that $\phi(x_0) = y_0$ and $\phi_* = f : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

3. Show that if $n>1$ then every map from $\mathbb{RP}^n$ to T^n (n-torus) is null-homotopic

4. Let X be the space obtained from a tetrahedron with vertices a, b, c, d by identifying faces in pairs according to the scheme $abc \sim bcd$, $cad \sim adb$ (e.g. abc is identified with bcd so that a is identified with b, b with c, c with d). Calculate $\pi_1(X)$.

(5) Let X be a connected and locally path-connected space that has a universal covering $p: \tilde{X} \to X$. Suppose $f: X \to X$ and $\tilde{f}: \tilde{X} \to \tilde{X}$ are maps such that $p\tilde{f} = fp$. Prove that for each covering transformation $\alpha: \tilde{X} \to \tilde{X}$ there is exactly one covering transformation $\beta: \tilde{X} \to \tilde{X}$ such that $\tilde{f}\alpha = \beta\tilde{f}$. Show by example that β is not necessarily equal to α. (Maps are written on the left, so $\tilde{f}\alpha$ maps t to $\tilde{f}(\alpha(t))$).

(6) For each topological space X below, write next to it the universal covering space of X and the fundamental group $\pi_1(X)$

$\mathbb{R}P^3$, $\mathbb{R}$, $[0,1]$, Möbius band, T^2 (torus), $\mathbb{R}P^2 \times S^3$

(7) Show an example of a degree 4 covering space of $S^1 \vee S^1$.