Lecture X

$f: X \to Y$ - continuous map

$f_\#: C(X) \to C(Y) \Rightarrow f_{\# n} = f_n: C_n(X) \to C_n(Y)$

<u>Lemma</u> Let $C^\cdot, D^\cdot$ - chain complexes, $\phi: C \to D$ is a chain map. Then $\exists$ homomorphism $\phi_*: H_n(C) \to H_n(D)$ such that $\phi_*[z] = [\phi_n z]$ $\forall z \in Z_n(C)$. There $[z]$ denotes the element of $H_n(C) = Z_n(C)/B_n(C)$

<u>Proof</u> $\phi_n(Z_n(C)) \subseteq Z_n(D)$ since $\partial_n \phi_n = \phi_{n-1} \partial_n$

$\phi_n(B_n(C)) \subseteq B_n(D)$ similarly

Therefore well-defined map.

<u>Cor</u> Composition $C \xrightarrow{\phi} D \xrightarrow{\psi} E$ $(\psi\phi)_* = \psi_* \phi_*$

<u>Theorem</u> If $f, g: X \to Y$ are homotopic, then $f_* = g_*$

<u>Def.</u> $\phi, \psi: C \to D$ are chain homotopic $\phi \simeq \psi$ if $\exists$ hom. $P_n: C_n \to D_{n+1}$

$$\psi_n - \phi_n = \partial_{n+1} P_n + P_{n-1} \partial_n$$

$$\begin{tikzcd}
\to C_{n+1} \arrow[r] \arrow[d, "\psi_{n+1}"] & C_n \arrow[r] \arrow[d, "\psi_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \\
\to D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & {}
\end{tikzcd}$$

<u>Lemma</u> If $\phi, \psi : C \to D$ are chain homotopic then $\phi_* = \psi_* : H_n(C) \to H_n(D)$

<u>Proof</u> $\forall z \in Z_n(C)$

$$\psi_*[z] - \phi_*[z] = [\psi_n z - \phi_n z] = [(\partial P + P\partial) z] = [\partial P z] = 0 \quad \boxtimes$$

<u>Lemma</u> If $f, g : X \to Y$ are homotopic, then $f_\#, g_\# : C(X) \to C(Y)$ are chain homotopic

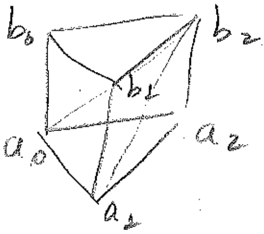
<u>Proof</u> $H : X \times I \to Y$ - homotopy from f to g.

Define $P_n : C_n(X) \to D_{n+1}(Y)$

for singular n-simplex $\sigma : [a_0 \ldots a_n] \to X$

Set $\bar{\sigma} = H(\sigma \times I) : [a_0 \ldots a_n] \times I \to Y$

$(z, t) \to H(\sigma(z), t)$



a_i for $(a_i, 0)$

b_i for $(a_i, 1)$

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i \bar{\sigma} | [a_0 \ldots a_i b_i \ldots b_n] \in C_{n+1}(Y)$$

$$\partial P_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i-1} (-1)^j \bar{\sigma} | [a_0 \ldots \hat{a}_j \ldots a_i b_i \ldots b_n] +$$

$$+ \sum_{i=0}^{n} (-1)^i \sum_{j=i+1}^{n} (-1)^{j+1} \bar{\sigma} | [a_0 \ldots a_i b_i \ldots \hat{b}_j \ldots b_n] +$$

$$+ \sum_{i=0}^{n} (-1)^i \left(\bar{\sigma} | [a_0 \ldots a_{i-1} b_i \ldots b_n] - \bar{\sigma} | [a_0 \ldots a_i b_{i+1} \ldots b_n] \right)$$

$$= \sum_{j<i} (-1)^{i+j} \bar{\sigma} | [a_0 \ldots \hat{a}_j \ldots a_i b_i \ldots b_n] - \sum_{j>i} (-1)^{i+j} \bar{\sigma} | [a_0 \ldots a_i b_i \ldots \hat{b}_j \ldots b_n]$$

$$+ \bar{\sigma} | [b_0 \ldots b_n] - \bar{\sigma} | [a_0 \ldots a_n] = -P_{n-1} \partial \sigma + g\sigma - f\sigma$$

Corollary If f - homotopy eq.

$f_*: H_n(X) \to H_n(Y)$ - isomorphism

Cor If X is contractible, $X \simeq \{pt\}$ $H_n(X) = 0$ ($n \geq 1$) $H_0(X) = \mathbb{Z}$

Exact Sequences and Relative Homology

Def. (X, A) - pair $C_n(A) \subseteq C_n(X)$

$$C_n(X, A) = C_n(X)/C_n(A)$$

Short exact sequence:

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

Since $i_\#: C_n(A) \to C_n(X)$ is a chain map:

$\exists!$ homomorphism $\partial_n: C_n(X, A) \to C_{n-1}(X, A)$

$$\begin{array}{ccccccccc}
0 & \to & C_n(A) & \xrightarrow{i_\#} & C_n(X) & \to & C_n(X,A) & \to & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n & & \\
0 & \to & C_{n-1}(A) & \xrightarrow{i_\#} & C_{n-1}(X) & \to & C_{n-1}(X,A) & \to & 0
\end{array}$$

and $C_{n+1}(X, A) \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A)$ is 0.

Def $C(X, A) = (C_n(X, A), \partial_n)_{n \geq 0}$

is the relative (singular) chain complex

Def. Map of pairs $f: (X, A) \to (Y, B)$ is

a cont. map $f: X \to Y$ $f(A) \subseteq B$

$$(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$$

$f:(X,A)\to(Y,B)$ induces hom. $f_\#: C_n(X,A)\to C_n(Y,B)$

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "f_\#"] & C_n(X) \arrow[r] \arrow[d, "f_\#"] & C_n(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\ 0 \arrow[r] & C_n(B) \arrow[r] & C_n(Y) \arrow[r] & C_n(Y,B) \arrow[r] & 0 \end{tikzcd}$$

commute

$f_\#: C_n(X,A)\to C_n(Y,B)$ is a chain map ($f_\#\partial=\partial f_\#$)

the chain map induced by $f:(X,A)\to(Y,B)$

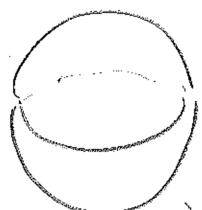
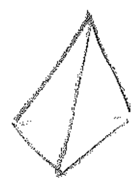
Induced homomorphism $f_*: H_n(X,A)\to H_n(Y,B)$

Ex. Short exact seq. (for (X,A))

$$0\to C_n(A)\xrightarrow{i_\#}C_n(X)\xrightarrow{j_\#}C_n(X,A)\to 0$$

(1) let $c\in C_n(X) \Rightarrow j_\# c\in Z_n(X,A) \Leftrightarrow \partial c\in C_{n-1}(A)$

(2) $j_\# c\in B_n(X,A) \Leftrightarrow \exists\, b\in C_{n+1}(X)$ s.t. $c-\partial b\in C_n(A)$

Ex.  $\cong$ 

(D^n, S^{n-1})

$X=\Delta^n$, A = union of its proper faces

$(X,A)\cong(D^n,S^{n-1})$

$\sigma:\Delta^n\to X$ is a basis elt

$\sigma\in C_n(X)$

$\partial\sigma\in C_{n-1}(A)$ $\quad j_\#\sigma\in C_n(X,A)$ cycle, repr.

$H_n(X,A)\cong H_n(D^n,S^{n-1})$

Snake Lemma let C, D, E be chain complexes

$\phi: C\to D$, $\psi: D\to E$ so that $0\to C_n\xrightarrow{\phi_n}D_n\xrightarrow{\psi_n}E_n\to 0$ exact, then

$\to H_n(C)\xrightarrow{\phi_*}H_n(D)\to H_n(E)\to$

$\to H_{n-1}(C)\xrightarrow{\phi_*}H_{n-1}(D)\to\cdots$

Corollary:

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{i_*}$$

$$\dots H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{i_*} H_0(X,A) \to 0$$

Proof of Snake

1) $\partial_* : H_n(E) \to H_{n-1}(C)$

Let $[z] \in H_n(E)$, $z \in Z_n(E)$ $\partial z = 0 \Rightarrow z = \psi y$ for some $y \in D_n$ $\underset{0}{\underset{\shortparallel}{\partial z}} = \partial(\psi y) = \psi \partial y \Rightarrow y \in \ker\psi = \operatorname{Im}\phi$

$\partial y = \phi x$ for $x \in C_{n-1}'$

$\phi \partial x = \partial \phi x = \partial\partial y = 0 \Rightarrow \partial x = 0$ (since ϕ - monomorph.)

$x \in Z_{n-1}(C)$

2) Indep. on choice of y, indep. on z