Lecture XI Eilenberg–Steenrod axioms

A homology theory assigns to each pair (X,A) sequence of Abelian groups $H_n(X,A)$ $n\in\mathbb{Z}$ and homomorphisms: $\partial_*: H_n(X,A) \to H_{n-1}(A)$ ($\overset{=}{\text{def}}\ H_{n-1}(A,\emptyset)$) to each map $f:(X,A)\to(Y,B)$ homomorphisms $f_*: H_n(X,A)\to H_n(Y,B)$

such that:

(1) $1_* = 1$, (2) $(gf)_* = g_* f_*$

(3) Naturality: $\forall\, f:(X,A)\to(Y,B)$,

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial_*"] & H_n(Y,B) \arrow[d, "\partial_*"] \\ H_{n-1}(A) \arrow[r] & H_{n-1}(B) \end{tikzcd}$$

(4) Exactness:

$\forall\ (X,A)$

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to H_{n-1}(X)$$

(5) Homotopy: If $f \simeq g$ $(X,A)\to(Y,B)$

Then $f_* = g_* : H_n(X,A)\to H_n(Y,B)$

(6) Excision: If Z open in X, $\overline{Z}\subseteq \operatorname{Int} A \subseteq A \subseteq X$

Then $i_*: H_n(X\setminus Z, A\setminus Z) \xrightarrow{i_*} H_n(X,A)$ is an isom.

where $i:(X\setminus Z, A\setminus Z)\to(X,A)$ is the inclusion

(7) Dimension: $H_n(pt) = \begin{cases} \mathbb{Z}, & \text{if } n=0 \\ 0, & \text{if } n\neq 0 \end{cases}$

<u>Theorem</u> E–S Any homology theory sat. (1)–(7) agrees with singular homology in CW-complexes

Remark (1) Axioms (1)-(6) define "generalized" homology theories
K-theory, cobordism theory
(2) Homotopy groups satisfy many of these axioms except excision

For singular homology it remains to verify (3), (6)

Lemma Naturality:

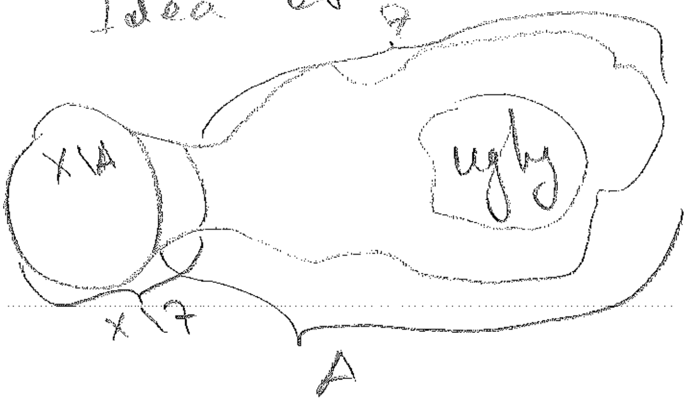
$$
\begin{array}{ccccccccc}
0 & \to & C_n(A) & \xrightarrow{i_\#} & C_n(X) & \xrightarrow{j_\#} & C_n(X,A) & \to & 0 \\
 & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\
0 & \to & C_n(B) & \xrightarrow{i_\#} & C_n(Y) & \xrightarrow{j_\#} & C_n(Y,B) & \to & 0
\end{array}
$$

For $z \in Z_n(X,A)$, $z = j_\# y$, $y \in C_n(X)$ $\partial y = i_\# x$

$\partial_* [z] = [x] \in H_{n-1}(A)$ $f_\# z = j_\# f_\# y$, $\partial f_\# y = i_\# f_\# x$

$\partial_* [f_\# z] = [f_\# x] = f_*[x] \in H_{n-1}(B)$

Idea of excision axiom



$$H_n(X,A) \underset{i_*}{\cong} H_n(X \setminus Z, A \setminus Z)$$

Lemma Exact sequence of triples: (X,Y,Z)

$Z \subseteq Y \subseteq X$

$$H_n(Y,Z) \xrightarrow{i_*} H_n(X,Z) \xrightarrow{j_*} H_n(X,Y) \to H_{n-1}(Y,Z) \to H_{n-1}(X,Z)$$

wher i_*, j_*:

$$\begin{tikzcd}
H_n(X,Y) \arrow[r, "\partial_*"] \arrow[dr, "\partial_*"'] & H_{n-1}(Y,Z) \\
& H_{n-1}(Y) \arrow[u, "j_*"']
\end{tikzcd}$$

commutes

Proof:

$$0 \to C_n(Y,Z) \xrightarrow{i_\#} C_n(X,Z) \xrightarrow{j_\#} C_n(X,Y) \to 0$$

snake lemma

Alternative E-S axioms:

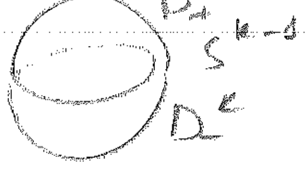
$$\begin{tikzcd}
& H_n(X,Y) \arrow[dr] \arrow[rr, bend left, "\partial_*"] & & H_{n-1}(Y,Z) \arrow[r, "\partial_*"] \arrow[dr] & H_{n-2}(Z) \\
H_n(X,Z) \arrow[ur] \arrow[dr] & & H_{n-1}(Y) \arrow[ur, "i_*"] \arrow[dr] & & H_{n-1}(X,Z) \\
& H_{n-1}(Z) \arrow[ur] & & H_{n-1}(X) \arrow[ur] \arrow[r] & H_{n-1}(X,Y)
\end{tikzcd}$$

If three are exact, the fourth is exact too.

Theorem $H_n(D^k, S^{k-1}) \cong \begin{cases} \mathbb{Z} & \text{if } n=k \\ 0 & \text{if } n \neq k \end{cases}$

Proof $\partial D^{k+1} = S^k = D^k_+ \cup D^k_-$

$D^k_+ \cap D^k_- = S^{k-1}$



Exact seq. of a triple.

$$\underbrace{H_{n+1}(D^{k+1}, D_-^k)}_{0} \to H_{n+1}(D^{k+1}, S^k) \underset{\cong}{\xrightarrow{\partial_*}} H_n(S^k, D_-^k) \to \underbrace{H_n(D^{k+1}, D_-^k)}_{0}$$

$i : D_-^k \to D^{k+1}$ is a hom. equiv.

$$H_n(S^k, D_-^k) \underset{\text{excision}}{\cong} H_n(S^k \setminus z, D_-^k \setminus z) \cong H_n(D_+^k, S^{k-1}) \cong H_n(D^k, S^{k-1})$$

$$H_n(D^k, S^{k-1}) \cong H_{n-1}(D^{k-1}, S^{k-2}) \dots \dots H_{n-k}(D^0, \emptyset) \cong \begin{cases} \mathbb{Z} & \text{if } n-k=0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Corollary</u> $H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n=k,0 \\ 0 & \text{other cases} \end{cases}$

(D^{k+1}, S^k)

<u>Proof</u> $H_{n+1}(D^{k+1}) \to H_{n+1}(D^{k+1}, S^k) \xrightarrow{\text{isom}} H_n(S^k) \to H_n(D^{k+1})$

<u>Theorem</u> (Good pairs)
Suppose $A \subset X$ is a deformation retract of some neighborhood U of A in X, $A \neq \emptyset$
Then $H_n(X,A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X,A)$ reduced homology

<u>Proof</u>
$$\begin{array}{ccccc} H_n(X,A) & \underset{\text{homotopy}}{\cong} & H_n(X,U) & \underset{\text{excision}}{\cong} & H_n(X\setminus A, U\setminus A) \\ \downarrow & & \downarrow & & \downarrow \text{homeom.} \\ H_n(X/A, A/A) & \cong & H_n(X/A, U/A) & \cong & H_n(X/A - A/A, U/A - A/A) \end{array}$$

Since def. retraction of U on A induces —"— of U/A onto A/A

Theorem Mayer-Vietoris

Suppose $X = U \cup V$ and U, V - open

Then leq exact sequence:

$$\hookrightarrow H_n(U \cap V) \xrightarrow[i_* \oplus j_*]{} H_n(U) \oplus H_n(V) \longrightarrow H_n(X) \rightarrow$$

$$\hookrightarrow H_{n-1}(U \cap V) \longrightarrow$$

$$U \cap V \begin{matrix} \xrightarrow{i} U \xrightarrow{k} \\ \xrightarrow[j]{} V \xrightarrow[l]{} \end{matrix} X$$

Proof: Short exact sequence for

$$0 \to C_n(U \cap V) \longrightarrow C_n(U) \oplus C_n(V) \to C_n(X) \to 0$$

$$\omega \longrightarrow (i\omega, -j\omega) \qquad (u, v) \to ku + lv$$

Corollary $\forall X$, $H_n(X \times S^1) \cong H_n(X) \oplus H_{n-1}(X)$

Ex. $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$

Proof exercise.

$S^1 = U \cup V$ $\quad -1$  1 $\quad U = S^1 \setminus \{-1\}$, $V = S^1 \setminus \{1\}$

$$H_n(X \times (U \cap V)) \xrightarrow{\Theta_n} H_n((X \times U) \oplus (X \times V)) \to H_n(X \times S^1) \rightarrow$$

$H_n(X) \oplus H_n(X) \qquad H_n(X) \oplus H_n(X)$

$(a, b) \qquad (a+b, -a-b)$

$$\hookrightarrow H_{n-1}(X \times (U \cap V)) \xrightarrow{\Theta_{n-1}} H_{n-1}(X \times U) \oplus H_{n-1}(X \times V)$$

$H_{n-1}(X) \oplus H_{n-1}(X)$

$\{(a, -a)\}_{a \in H_{n-1}} \cong H_{n-1}$

$$0 \to \operatorname{coker} \Theta_n \to H_n(X \times S^1) \to \ker \Theta_{n-1} \to 0$$

$$H_n(X) = \frac{H_n(X)^2}{\{(c, -c)\} \in H_n(X)}$$