Lecture XIII

Cellular homology Goal: calc. homology of CW complex X from a chain complex with one basis element from each cell of X

Def. For each CW-complex $C_n^{CW}(X) = H_n(X^n, X^{n-1})$ (sing. homology)

X^n, X^{n-1} – $n, n-1$ skeletons, $X^{-1} = \emptyset$

Boundary $\partial_n : C_n(X) \to C_{n-1}(X)$

$$C_n^{CW}(X) \xrightarrow{\partial_n^{CW}} C_{n-1}^{CW}(X)$$

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[rr, "\partial_*"] \arrow[dr, "\partial_*"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_{n-1}(X^{n-1}) \arrow[ur, "j^*"'] &
\end{tikzcd}$$

Lemma $\partial_n^{CW} \partial_{n-1}^{CW} = 0$

$$\begin{tikzcd}
& & & H_{n-1}(X^{n-1}) \arrow[dr, "j_*"] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[rr, "\partial_{n+1}^{(CW)}"] \arrow[dr, "\partial_*"'] & & H_n(X^n, X^{n-1}) \arrow[ur, "\partial_*"] \arrow[rr, "\partial_n^{CW}"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_n(X^n) \arrow[ur, "j^*"'] & & &
\end{tikzcd}$$

diagram commute

$$\partial_n^{CW} \partial_{n+1}^{CW} = j_* \underbrace{\partial_* j_*}_{} \partial_* = 0$$

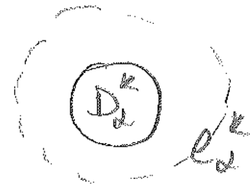
Def. $H_n^{(CW)}(X) = \dfrac{\ker \partial_n^{CW}}{\ker \partial_{n+1}^{CW}} = H_n(C^{CW}(X))$

Lemma $\forall$ CW complex X $H_n(X^k, X^{k-1}) = 0$ if $n \neq k$

$H_n(X^n, X^{n-1})$ – free ab. group, gen. by cells

<u>Proof</u> $X^k = X^{k-1} \cup \bigcup_{\alpha \in A} e_\alpha^k$

Consider k-disk $D_\alpha^k \subset e_\alpha^k$



$[\text{s.t. } (e_\alpha^k, D_\alpha^k) \cong (\mathbb{R}^k, D^k)]$

X^{k-1} is a def. retract of $X^k \setminus \bigcup_{\alpha \in A} \operatorname{Int} D_\alpha^k$

$H_n(X^k, X^{k-1}) \cong H_n(X^k, X^k \setminus \bigcup_{\alpha \in A} \operatorname{Int} D_\alpha^k) \underset{\text{excise } X^{k-1}}{\cong}$

$H_n(\bigsqcup_{\alpha \in A} e_\alpha^k, \bigsqcup_{\alpha \in A} (e_\alpha^k \setminus \operatorname{Int} D_\alpha^k))$

$[X^{k-1} \text{ closed} \subseteq X^k \setminus \bigcup \operatorname{Int} D_\alpha$ $\cong \bigoplus_{\alpha \in A} H_n(e_\alpha^k, e_\alpha^k \setminus \operatorname{Int} D_\alpha^k)$

$\underset{\text{homotopy}}{\cong} \bigoplus_{\alpha \in A} H_n(D_\alpha^k, \partial D_\alpha^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$

<u>Lemma</u> $H_n(X^k, X^\ell) \xrightarrow{\cong} H_n(X^{k+1}, X^\ell)$ if $n \neq k, k+1$

$H_n(X^k, X^{\ell-1}) \xrightarrow{\cong} H_n(X^k, X^\ell)$ if $n \neq \ell, \ell+1$

<u>Proof.</u> Exact sequence of triple (X^{k+1}, X^k, X^ℓ)

$\underbrace{H_{n+1}(X^{k+1}, X^k)}_{0 \text{ if } n \neq k} \xrightarrow{\cong} H_n(X^k, X^\ell) \xrightarrow{i_*} H_n(X^{k+1}, X^\ell) \to \underbrace{H_n(X^{k+1}, X^k)}_{0 \text{ if } n \neq k+1}$

Triple $(X^k, X^\ell, X^{\ell-1})$ – second isom

<u>Cor.</u> If $n < k$ then $H_n(X^k) \cong H_n(X^{n+1}, X^{n-2})$

<u>Proof:</u> $H_n(X^{n+1}, X^{n-2}) \cong H_n(X^{n+1}, X^{n-3}) \cong H_n(X^{n+1}, \emptyset)$

and using first result of prev. lemma: $H_n(X^k) \cong H_n(X^{k+1})$ if $n < k$

Cor. If X is finite dim, then

$$H_n(X) \cong H_n(X^{n+1}, X^{n-2})$$

Proof. $X = X^k$ for some k

Lemma If X is a finite-dim CW complex then $H_n^{CW}(X) \cong H_n(X)$ (sing.)

Proof. Triple (X^n, X^{n-1}, X^{n-2}):

$$\begin{array}{ccccccc} H_n(X^{n-1}, X^{n-2}) & \to & H_n(X^n, X^{n-2}) & \to & H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n^{CW}} & H_{n-1}(X^{n-1}, X^{n-2}) \\ 0 & & Z_n^{CW}(X) & \longrightarrow & C_n^{CW}(X) & & C_{n-1}^{CW}(X) \end{array}$$

Triple (X^{n+1}, X^n, X^{n-2})

$$\begin{array}{ccccccc} H_{n+1}(X^{n+1}, X^n) & \to & H_n(X^n, X^{n-2}) & \to & H_n(X^{n+1}, X^{n-2}) & \to & H_n(X^{n+1}, X^n) \\ C^{CW}_{n+1}(X) & \longrightarrow & Z_n^{CW}(X) & \longrightarrow & H_n^{CW}(X) & & 0 \\ & & & & \uparrow_{sing} & & \end{array}$$

Lemma If $n < k$ then $H_n(X^k) \cong H_n(X)$

Proof Comm. diagram $n < k < l$

$$\begin{tikzcd} H_n(X^k) \arrow[rr, "i_{kl*}"] \arrow[dr, "i_{k*}"'] & & H_n(X^l) \arrow[dl, "i_{l*}"] \\ & H_n(X) & \end{tikzcd}$$

i_{kl}, i_l inclusions

If $\xi \in H_n(X)$, ξ represented by $z = r_1 \sigma_1 + \dots + r_t \sigma_t$

$\sigma_i : \Delta^n \to X$

$\sigma_i(\Delta^n) \subseteq$ finite subcomplex of X $\quad \sigma_i(\Delta^n) \subseteq X^{m_i}$

Set $m = \max(m_1, \dots, m_t)$. Then $\xi \in i_{m*} H_n(X^m)$

$i_{m*} : H_n(X^m) \to H_n(X)$ onto m large

i_{k*} onto for $k > n$ since $i_{k\ell *}$ isom.

Similarly, i_{m*} 1-1 (m-large)

i_{k*} 1-1 for $k > n$

<u>Theorem</u> $\forall$ CW complex X $\quad H_n^{CW}(X) \cong H_n(X)$ (sing.)

<u>Ex.</u> $\mathbb{C}P^k = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2k}$

$$H_n(\mathbb{C}P^k) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le n \le 2k,\ n\text{-even} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{C}P^\infty = \bigcup_{k \in \mathbb{N}} \mathbb{C}P^k = e^0 \cup e^2 \cup \dots$ similarly

<u>Ex.</u> $S^2 \vee S^4 \vee \dots \vee S^{2k} = e^0 \cup e^2 \dots \cup e^{2k}$

$\cong H_n(\mathbb{C}P^k)$ not homotopic

<u>Ex</u> $\mathbb{R}^k = e^0 \cup e^1 \cup \dots \cup e^k$

$$C_n^{CW}(\mathbb{R}P^k) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le n \le k \\ 0 & \text{otherwise} \end{cases}$$

$0 \to C_k \to C_{k-1} \quad \dots \quad C_0 \to 0$

<u>Lemma</u> $\forall$ hom. $\phi : \mathbb{Z} \to \mathbb{Z}$ $\exists\, d \in \mathbb{Z}$ $\quad \phi(z) = dz$

$\phi(1) = d$

<u>Def.</u> for a cont. map $f : S^n \to S^n$ <u>Degree</u>

$$\underset{\mathbb{Z}}{\tilde{H}_n(S^n)} \xrightarrow{f_*} \underset{\mathbb{Z}}{\tilde{H}_n(S^n)}$$