Degree of a map and cellular homology (Lecture XIII)

Recall: $\forall$ cell e_α^n $\exists$ characteristic map

$$\Phi_\alpha : (D^n, \partial D^n) \to (X^n, X^{n-1})$$

$\phi_\alpha = \Phi_\alpha|_{\partial D^n} \to X^{n-1}$ is attaching map of e_α^n

$$C_n^{CW}(X) = H_n(X^n, X^{n-1})$$

Choose generator $[D^n]$ for $H_n(D^n, \partial D^n) \cong \mathbb{Z}$

Set $[e_\alpha^n] = \Phi_{\alpha *}[D^n] \in H_n(X^n, X^{n-1}) = C_n^{CW}(X)$

Then $\{[e_\alpha^n] : e_\alpha^n$ n-cell of $X\}$ is a basis for $C_n^{CW}(X)$

Since $\partial_n^{CW} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ $\mathbb{Z}$-linear

$$\partial_n^{CW}[e_\alpha^n] = \sum_\beta d_{\alpha\beta}[e_\beta^{n-1}] \quad \text{for some } d_{\alpha\beta} \in \mathbb{Z}$$

Lemma $d_{\alpha\beta}$ is the degree of the map

$$S^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{q_\beta} X^{n-1}/X^{n-1}\setminus e_\beta^{n-1} \cong S^{n-1}$$

Proof

$$\begin{tikzcd}
(D^n, \partial D^n) \arrow[r, "\Phi_\alpha"] & (X^n, X^{n-1}) & & \\
S^{n-1} \arrow[u, hook] \arrow[r, "\phi_\alpha"] & X^{n-1} \arrow[u, hook] \arrow[r] \arrow[dr, "q_\beta"'] & (X^{n-1}, X^{n-2}) \arrow[r] & (X^{n-1}, X^{n-1}\setminus e_\beta^{n-1}) \arrow[d] \\
 & (D^{n-1}, S^{n-2}) \arrow[urr, "\Phi_\beta"] & S^{n-1} \cong D^{n-1}/S^{n-2} \arrow[r] & X^{n-1}/(X^{n-1}\setminus e_\beta^{n-1}\text{pt})
\end{tikzcd}$$

good pair

Naturality:

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\Phi_{\alpha *}"] \arrow[d, "\partial_*", "\cong"'] & H_n(X^n, X^{n-1}) \arrow[d, "\partial_*"] \\
H_{n-1}(S^{n-1}) \arrow[r, "\phi_{\alpha *}"] & H_{n-1}(X^{n-1})
\end{tikzcd}$$

$$\partial_*[e_\alpha^n] = \phi_{\alpha *}[S^{n-1}]$$

$$H_{n-1}(X^{n-1}, X^{n-2}) \cong \bigoplus_\beta \langle [e_\beta] \rangle \to [e_\beta]$$

$q_{\beta *}\phi_{\alpha *}[S^{n-1}]$ $\subset$ β-component $[e_\beta]$

Example $X = S^n \cup_\phi e^{n+1}$ $\quad \phi: S^n \to S^n \quad \deg \phi = d$

$= e^0 \cup e^n \cup_\phi e^{n+1}$

$C_{n+1} \xrightarrow{\partial} C_n \to 0 \to 0$

$\mathbb{Z} \xrightarrow{d} \mathbb{Z} \to 0$

$H_n(X) \cong \mathbb{Z}/d\mathbb{Z}$

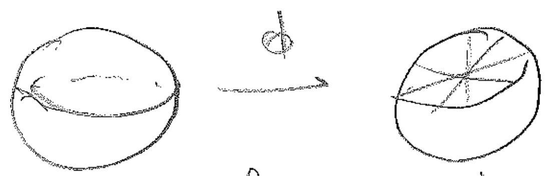
Example $X = \mathbb{R}P^{n+1} = \mathbb{R}P^n \cup_\phi e^{n+1} = \mathbb{R}P^{n-1} \cup e^n \cup_\phi e^{n+1}$

calculate the boundary map $\partial : C_{n+1} \to C_n$

$\phi : S^n \to \mathbb{R}P^n = S^n/(u \sim -u \ \forall u \in S^n)$

double covering map

$\partial(e^{n+1}) = d_n [e^n]$ where $d_n = \deg f_n = \deg q\phi$

S^n  $\xrightarrow{\phi}$ $\mathbb{R}P^n = S^n/(u \sim -u) \ \forall u \in S^n$

$f = f_n \qquad q \downarrow \ \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$

degree of this map?

$\alpha : S^n \to S^n$ (antipodal) $\quad \alpha(u) = -u$

$f\alpha = q\phi\alpha = q\phi = f \qquad \alpha$ - covering transf $f: S^n \to S^n$

$\deg f = \deg(f\alpha) = \deg f \deg \alpha$. If $\deg \alpha \neq 1 \Rightarrow \deg f = 0$

What if $\deg \alpha = 1$??

Local degrees:

Let $a \in S^n$, let U be open set in neighborhood of a

Then $H_n(U, U \setminus a) \underset{\text{excision}}{\cong} H_n(S^n, S^n \setminus a) \cong \mathbb{Z}$

Choose generator $[S^n]$ for $\tilde{H}_n(S^n) \cong \mathbb{Z}$

Let $[U], [S^n]$ be the images of $[S^n] \in \tilde{H}_n(S^n)$ generating $H_n(U, U \setminus a)$, $H_n(S^n, S^n \setminus a)$ resp.

Def. Suppose $f: S^n \to S^n$, $a \in S^n$, $f(a) = b$, s.t. $\exists$ open neighborhood of a with $U \cap f^{-1}(b) = \{a\}$.

Then $f|(U, U \setminus a) \to (S^n, S^n \setminus b)$

inducing $f_*: H_n(U, U \setminus a) \to H_n(S^n, S^n \setminus b)$

$[U] \to \deg(f,a)[S^n]$

(def. of local degree

Lemma $S^n \xrightarrow{f} S^n \xrightarrow{g} S^n$ $f(a) = b$, $g(b) = c$

$$\deg(gf, a) = \deg(g, b) \deg(f, a)$$

Ex. deg indep. on choice of neighb.

Proof $\exists$ nbhd V of b, U of a, s.t. $V \cap g^{-1}(c) = \{b\}$

$U \cap f^{-1}(b) = \{a\}$, $f(U) \subset V$

Lemma If $f: S^n \to S^n$ maps some neighb. U of a homeom. onto a nbhd V of b, then $\deg(f,a) = \pm 1$

Proof f_* $H_n(U, U \setminus a) \to H_n(V, V \setminus b)$ isom.

$\cong H_n(S^n, S^n \setminus b)$

Theorem Suppose $f : S^n \to S^n$, $b \in S^n$ such that $f^{-1}(b) = \{a_1 \dots a_k\}$ is finite. Then $\deg(f) = \sum_{i=1}^{k} \deg(f, a_i)$

Proof $\exists$ disj. open sets $U_1 \dots U_k$ with $a_i \in U_i$

$U_i \cap f^{-1}(b) = U_i \cap \{a_1 \dots a_k\} = a_i$

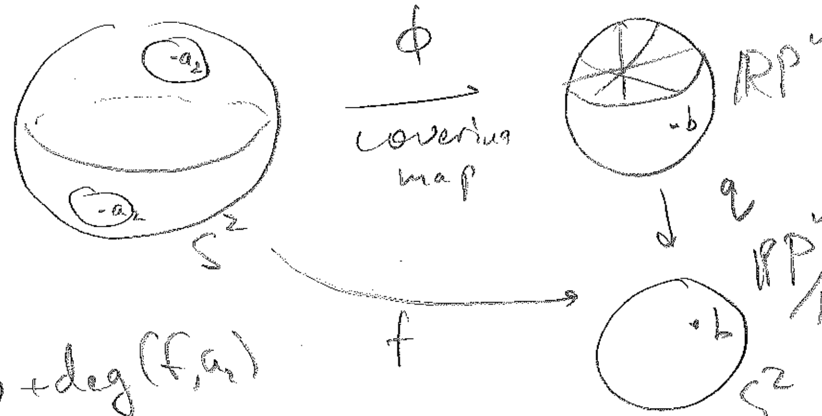
so $\deg(f, a_i)$ defined by $f_* : H_n(U_i, U_i \setminus a_i) \to H_n(S^n, S^n \setminus b)$ $\quad [U_i] \mapsto d\,[S^n]$

$$\begin{array}{ccc}
\tilde H_n(S^n) & \xrightarrow{f_*} & \tilde H_n(S^n) \\
\downarrow & & \downarrow \cong \\
H_n(S^n, S^n \setminus \{a_1, \dots a_k\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus b) \\
\cong ? & & \\
H_n\left(\bigcup_i U_i, \bigcup_i U_i \setminus \{a_1 \dots a_k\}\right) & &
\end{array}$$

disj. union

Return to $\mathbb{R}P^n$

$\partial_{n+1} : C_{n+1} \to C_n$
$\qquad \mathbb{Z} \qquad \mathbb{Z}$



$\deg(f) = \deg(f, a_1) + \deg(f, a_2)$

$\deg(f, a_i) = \pm 1$

$f \alpha = f \qquad \alpha$ - cov. transf. $\quad \alpha a_1 = a_2,\ \alpha a_2 = a_1$

$\deg(f, a_2) = \deg(f\alpha, a_2) = \deg(f, \alpha a_2) \deg(\alpha, a_2) =$
$= \deg(f, a_1) \deg \alpha$

$\deg f = \deg(f, a_1) + \deg(f, a_2) = (1 + \deg \alpha) \deg(f, a_1) =$
$= \pm(1 + \deg \alpha) \quad \alpha : S^n \to S^n$ - antip. map.

22

<u>Lemma</u> $S^n = \{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = 1\}$

Let $r_n : S^n \to S^n$ be defined by

$$r(x_1, x_2 \dots, x_{n+1}) = (-x_1, x_2 \dots, x_{n+1})$$

$$\deg r_n = -1$$

<u>Cor.</u> Antipodal map $\alpha : S^n \to S^n$ has deg. $(-1)^{n+1}$

<u>Proof.</u> α is a prod. of maps each. deg -1.

<u>Proof.</u> $r = r_n$ maps $D^n_\pm$ to itself, where

$$D^n_+ = \{(x_1, \dots, x_{n+1}) \in S^n : x_{n+1} \geq 0\}$$

$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[loop below, "r_n^*\ (S^n{,}D^n_-)"] \arrow[r, "\cong"] & H_n(S^n, D^n_-) \arrow[r, "\cong"] \arrow[r, "\text{excision}"'] & H_n(D^n_+, \partial D^n_-) \arrow[loop below, "r_{n*}"] \arrow[r, "\simeq"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[loop below, "(r_{n-1})_*"] \end{tikzcd}$$

$$\deg(r_n) = \deg(r_{n-1} : S^{n-1} \to S^{n-1}) = \deg r_0 : S^0 \to S^0 = -1$$

Therefore $\pm(1 + (-1)^{n+1}) = \pm(1 + \deg(\alpha))$

$$\deg \delta = 1 + (-1)^{n+1}$$

For $\mathbb{RP}^k = e^0 \cup \dots \cup e^{n-1} \cup e^n \dots \cup e^k$

$\partial_n : C^{cw}_n \to C^{cw}_{n-1}$ (all $\mathbb{Z}$) is multiplication by $1 + (-1)^n$

Cellular chain complex:

$$\begin{array}{l} 0 \to C_k \to C_{k-1} \to \dots \qquad \to C_0 \\ \mathbb{Z} \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \end{array} \quad H_n(\mathbb{RP}^k) \cong \begin{cases} \mathbb{Z} & n = 0 \text{ or } n = k,\ k \text{ odd} \\ \mathbb{Z}/2 & \text{if } 0 < n < k \\ & n \text{ odd.} \end{cases}$$