Lecture XIV Cohomology

More on cell complexes:

How does the differential $H_1(X^1, X^0) \xrightarrow{\partial_*} H(X^0)$ act?

Remember that $\partial_*[z]$ is defined as follows:

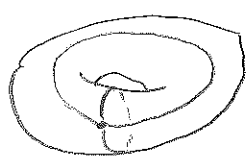
$\partial z = 0$, $z = j_* y$, $\partial j_* y = j_* \partial y \Rightarrow \partial y \in \ker j_* = \operatorname{Im} i_* \Rightarrow$

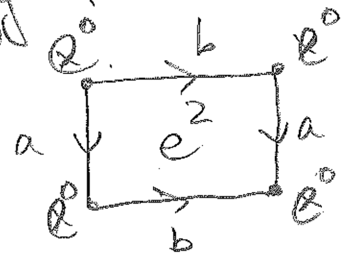
$\Rightarrow \partial y = i_* x$, $x \in H_0(X^0)$

$$e^0_1 \xrightarrow{e^1} e^0_2$$

Therefore for 1-cell e^1 we can look at it as 1-simplex $\partial e^1 = e^0_2 - e^0_1$

Compute homology of a torus





$\partial a = e^0 - e^0 = 0$

$\partial b = e^0 - e^0 = 0$

$\partial e^2 = a + b - a - b = 0$

Ex. a)

b) Klein bottle

c) Homology of $S^{n_1} \vee \dots \vee S^{n_n}$

d) Homology of a Riemann surface of genus g with n punctures and m boundaries

For a given chain complex $(C_n, \partial_n)_{n\in\mathbb{Z}}$
and abelian group G (coeff. group)

$C^n(C,G) = \mathrm{Hom}(C_n, G)$ - cochain group

$\delta^n : C^n(C,G) \to C^{n+1}(C,G)$ dual to $\partial_{n+1}: C_{n+1} \to C_n$

$\delta^n \delta^{n-1} = 0$

$$\begin{tikzcd} \partial_{n+1} : C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "\delta^n \gamma"'] & C_n \arrow[r, "\gamma"] & G \end{tikzcd}$$

$$H^n(C,G) = \frac{\ker \delta^n}{\ker \delta^{n-1}}$$

For space X

$C(X) = (C_n(X), \partial_n)$ $\quad C^n(X;G) = \mathrm{Hom}(C_n(X), G)$

$\gamma : S_n(X) \to G$ - functions on the set of simplices with values in G

$C^n(X,A;G) = \mathrm{Hom}(C_n(X,A); G)$

<u>Lemma</u> $\quad 0 \to C^n(X,A;G) \xrightarrow{j^\#} C^n(X;G) \xrightarrow{i^\#} C^n(A,G) \to 0$

$$\begin{tikzcd} C_n(A) \arrow[r, "i_\#"] \arrow[dr, "i^\# \gamma"'] & C_n(X) \arrow[r, "j_\#"] \arrow[d, "\gamma"] & C_n(X,A) \arrow[r] & 0 \\ & G & & \end{tikzcd}$$

$$\to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \to H^{n+1}(X,A;G) \xrightarrow{j^*} H^{n+1}(X;G) \xrightarrow{i^*} \cdots$$

<u>Notice:</u> contravariant functor:

$f : (X,A) \to (Y,B)$

$f^* : C^n(Y,B;G) \to C^n(X,A,G)$

$(gf)^* = f^* g^*$

<u>Naturality</u> $f : (X,A) \to (Y,B)$

$$\begin{array}{ccc} H^n(B;G) & \xrightarrow{f^*} & H^n(A;G) \\ \downarrow \delta^* & & \downarrow \delta^* \\ H^{n+1}(Y,B;G) & \xrightarrow{f^*} & H^{n+1}(X,A;G) \end{array}$$

commutes

<u>Homotopy:</u> $f \simeq g : (X,A) \to (Y,B) \Rightarrow$

$f^* = g^* : H^n(Y,B;G) \to H^n(X,A;G)$

(dualize chain homotopy)

<u>Excision</u> If $\bar{Z} \subset \operatorname{Int} A \subseteq A \subseteq X$ then

$i^* : H^n(X,A;G) \to H^n(X \setminus Z, A \setminus Z, G)$ isom.

<u>Dim</u> $H^n(\{pt\}, G) = \begin{cases} 0 & n > 0 \\ G & n = 0 \end{cases}$

<u>Cellular cohomology</u>

$H^n_{CW}(X;G) = H^n(C^{CW}(X);G)$ is

isom. to sing. cohomology

<u>Ex.</u> $X = \mathbb{R}P^k$, $G = \mathbb{Z}$

$X = e^0 \cup e^1 \cup \ldots \cup e^k$

$$\begin{array}{ccccccccccccc} C^k & \xleftarrow{\delta} & C^{k-1} & \leftarrow & \ldots & \leftarrow & C^4 & \leftarrow & C^3 & \leftarrow & C^2 & \leftarrow C^1 \leftarrow & C^0 \\ \mathbb{Z} & \leftarrow & \mathbb{Z} & & & & & & & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\times 2} \mathbb{Z} \leftarrow & \mathbb{Z} \end{array}$$

$H^n(\mathbb{R}P^k;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \text{ if } n=0 & \\ H^i(X) = \mathbb{Z}/2\mathbb{Z} & i\text{-even} \\ H^i(X) = 0 & i\text{-odd} \end{cases}$

Note: $H^n(\mathbb{R}P^k;\mathbb{Z})$ not just $\operatorname{Hom}(H_n(\mathbb{R}P^k), \mathbb{Z})$

Universal coeff. theorem (4

$$H^n(X;G) \cong \mathrm{Hom}(H_n(X);G) \oplus \mathrm{Ext}(H_{n-1}(X),G)$$

<u>Kronecker product</u> For space X, Abelian group G

$$C_n(X) \times C^n(X;G) \xrightarrow{\langle\,,\,\rangle} G$$

$$\langle c,\gamma\rangle = \gamma(c) \in G$$

$$c \in C_n(X),\ \gamma \in C^n(X;G) = \mathrm{Hom}(C_n(X), G)$$

<u>Lemma</u> $\langle \partial c, \gamma\rangle = \langle c, \delta\gamma\rangle \quad \forall\, c \in C_{n+1}(X),\ \gamma \in C^n(X)$

<u>Corollary</u> $\langle B_n, Z^n\rangle = 0 = \langle Z_n, B^n\rangle$

<u>Cor.</u> Well-defined bilinear Kronecker product:

$$H_n(X) \times H^n(X) \to G$$

$$\langle [z], [\zeta]\rangle = \langle z, \zeta\rangle \quad z \in Z_n(X),\ \zeta \in Z^n(X)$$

<u>Cup products</u> R - ring

Define the cup product

$$\cup : C^p(X;R) \times C^q(X;R) \to C^{p+q}(X;R)$$

$$\phi \in C^p,\ \psi \in C^q$$

$$\sigma : [a_0 \ldots a_{p+q}] \to X$$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[a_0 \ldots a_p]) \underset{\text{mult. in } R}{\cdot} \psi(\sigma|[a_p \ldots a_{p+q}])$$

<u>Lemma</u> $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^p \phi \cup \delta\psi$

<u>Proof</u> calculate.

Proof $\sigma : [a_0 \ldots a_{p+q+1}] \to X$

$$\delta(\phi \cup \psi)\sigma = (\phi\cup\psi)(\partial\sigma) = (\phi\cup\psi)\sum_{i=0}^{p+q+1}(-1)^i \sigma|[a_0 \ldots \hat{a}_i \ldots a_{p+q+1}]$$

$$= \sum_{i=0}^{p+q+1} (-1)^i (\phi\cup\psi)\, \sigma|[a_0 \ldots \hat{a}_i \ldots a_{p+q+1}] =$$

$$= \sum_{i=0}^{p+q+1} (-1)^i (\phi\cup\psi)\left(\sigma|[a_0 \ldots \hat{a}_i \ldots a_{p+q+1}]\right) =$$

$$= \sum_{i\le p} (-1)^i \phi(\sigma|[a_0 \ldots \hat{a}_i \ldots a_{p+1}])\psi(\sigma|[a_{p+1} \ldots a_{p+q+1}])$$

$$+ \sum_{i>p} (-1)^i \phi(\sigma|[a_0 \ldots a_p])\psi(\sigma|[a_p \ldots \hat{a}_i \ldots a_{p+q+1}])$$

$$= \phi(\partial(\sigma|[a_0 \ldots a_{p+1}]))\psi(\sigma|[a_{p+1} \ldots a_{p+q+1}]) +$$

$$+ (-1)^p \phi(\sigma|[a_0 \ldots a_p])\psi(\partial(\sigma|[a_p \ldots a_{p+q+1}]))$$

cancel $\left\{\begin{array}{l} -(-1)^{p+1}\phi(\sigma|[a_0 \ldots a_p])\psi(\sigma|[a_{p+1} \ldots a_{p+q+1}]) \\ -(-1)^p \phi(\sigma|[a_0 \ldots a_p])\psi(\sigma|[a_{p+1} \ldots a_{p+q+1}]) \end{array}\right.$

$$= (\delta\phi\cup\psi)(\sigma) + (-1)^p(\phi\cup\delta\psi)(\sigma)$$

Corollary $Z^p \cup Z^q \subseteq Z^{p+q}$

$B^p \cup Z^q \subset B^{p+q} \supseteq Z^p \cup B^q$

Corollary Well-defined map:

$\cup : H^p(X;R) \times H^q(X,R) \to H^{p+q}(X,R)$

Ex. Compute for T^2