Projective modules and Ext functor (Lecture D) [.	<u> </u>
Extensions: 0 -> 4 -> 4 -> 4 -> 4/24 -> 0	
0 - 2 - 2 - 2 - 0	
Equivalence of O-A-B-3C-30 when O-A-B-3C-30 BSB' and dig	V.
Det. 0 - A - B - C - 0 The a short exact sequence of R - modules The sequence is known to be split if there is The sequence is known to be split if B.	
Proposition Let D, L and M be R-module hom. Proposition Let D, L and M be R-module hom. and let 4: La M be an R-module hom. and let 4: La M be an R-module hom. 4: Homp (D,L) a Homp (D,M) 4: Homp (D,L) a Homp (D,M)	
Then 0-2 Home (D, N) & Home (D, M) is a read then 0-3 Home (D, N) & Home (D, M) is a read	
Theorem D, A, M, N, N & N & N & N & Home (0, M) Littoure (1))u

Example / M N 0 - 7 = 2 = 1/2 + 0 Let D = 1/2 7 Let I - identity map I: D - N Anytetten (4/62, 3) = 0 = NOF mare D - 0 Prop- Ham (D, LON) = How (D, L) & How (D, N) Home (LON, D) = Hom(10 Hom () Prop. P-R-module. The following are eq: short exact LMN 1) Y L,M,N 0 -> Hom (P,d) -> Hom (P,N) -> Hom (P,N) -> O 2) YM, N 2f M PN-20 is exact =) every R-module hom. P-N lifts to M M'EN -O 3) If Pis a quotient of the R-module M => 0-2 -> M-P-O spliks y) Pis a direct summand of a free R-mobile Corolary Free modules are projective. A finitely generated module is projective if and only it generated it is a direct cummend of a finitely generated

Every projective module over porincipal ideal domain is free. Example R= 7/24× 4/24 Submodule: 4/24-direct Finally, let us describe Ext: Det. Let A he any R-module. A projective resolution of A is an exact sequence. -> Pr da Pr-1 -> ds Po = A - 0 Ex. DA=7/24 0-74 23 - 7/29-20 2) R= H/m7 d-divisor of m: me that in that in a that in 0-12 M-3.N-0 O -> Hamp (N,D) -> Hamp (M,D) -> Hamp (4,0) Then we have -O- Home (A,D) Es Home (B,D) - Home (B,D) o etc. Pel. d.: Home (Part. D) -> Home (Pa.D) 122 Exte(A,D) = xerbuel/emdu

Proposition Exte (A,N) = Home (A,O) Proof. O stomp (A,D) -> Home (Po,D) & Home (ED) Rerde = In E = Homp (A,D) Ex- Companie Ext for 0-24 57 24-20 0 - Hong (4/mx, D) - Hong (7, D) = Hon(4, D) = Exto(9/mx, D) = Houng (4/mx, D) = mD = 4de N(udeo) D= Homy (8,D) => Ex+2(8/24,D)=D/20 Compare resolutions Prop. It f: A > A' - hom of R-modrles
Then there is a lift to chain maps In of Prop- f. A. A'-hom. of R-modules and take proj. resolutions => Pu: Extr(A',B)
Jep.only Extr(A,D)

onf. Extig(A,D) dep. only on A,D 0 -> L-> M-> N=0 Exact, then
0 -> Ham (M,D) -> Ham (M,D)
1 Extr(N,D) - Extr(H,D) SExtr(N,D)+

0 - 1 - 1 - 1 - 0 0 -> Hom (D,2) -> Hom (D,M) -> Hom (D,N) CEXTE(D,L) - EXTE(D,M) - EXTE(D,L)

Extis a right derived functor perfection of the property of th

Tor: O=1-N=0 règht exact JDOLJDOM JDONJO

Tor (D, L)

Universal weff theorem for Homology 0-> Hu(c) 0G -> Hu(C;G) -> Tors(Hu=(c),G)-0

Universal Coefficient Theorem for Cohomology

We present a direct proof of the universal coefficient theorem for cohomology. It is essentially dual to the proof for homology.

THEOREM 1 Given a chain complex C in which each C_n is free abelian, and a coefficient group G, we have for each n the natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C), G) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Hom}(\dot{H_n}(C), G) \longrightarrow 0, \qquad (2)$$
which splits (but there is no natural splitting).

In particular, this applies immediately to singular cohomology.

THEOREM 3 Given a pair of spaces (X, A) and a coefficient group G, we have for each n the natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X,A),G) \longrightarrow H^n(X,A;G) \longrightarrow \operatorname{Hom}(H_n(X,A),G) \longrightarrow 0,$$

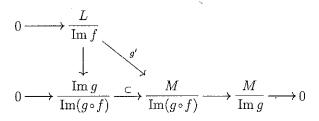
which splits (but there is no natural splitting). \square

We shall derive diagram (2) as an instance of the following elementary result.

LEMMA 4 Given homomorphisms $f: K \to L$ and $g: L \to M$ of abelian groups, with a splitting homomorphism $s: M \to L$ such that $s \circ g = \mathrm{id}_L$, we have a split short exact sequence

$$0 \longrightarrow \operatorname{Coker} f \xrightarrow{g'} \operatorname{Coker} (g \circ f) \longrightarrow \operatorname{Coker} g \longrightarrow 0. \tag{5}$$

Proof We write each cokernel, such as Coker f, as $L/\operatorname{Im} f$ etc. Then the sequence (5) appears as the upper edge of the following diagram; we shall identify it with the bottom row, which is the canonical short exact sequence formed from the triple $\operatorname{Im}(g \circ f) \subset \operatorname{Im} g \subset M$.



Since $g(\operatorname{Im} f) = \operatorname{Im}(g \circ f)$, g induces a homomorphism g'. Since $s(\operatorname{Im}(g \circ f)) = \operatorname{Im}(s \circ g \circ f) = \operatorname{Im} f$, $s : M \to L$ induces a homomorphism s' which splits g', $s' \circ g' = \operatorname{id}$; thus g' is injective. But g' clearly factors through a surjective homomorphism $L/\operatorname{Im} f \to \operatorname{Im} g/\operatorname{Im}(g \circ f)$, which is therefore an isomorphism. \square

Preliminaries We consider a chain complex C as in Theorem 1. We adopt the usual notation: Z_n for the group of n-cycles, B_n for the group of n-boundaries, and $H_n = H_n(C) = Z_n/B_n$. The key idea in our proof of Theorem 1 is to express $\partial: C_n \to C_{n-1}$ as the composite

$$\partial: C_n \longrightarrow \frac{C_n}{B_n} \longrightarrow \frac{C_n}{Z_n} \xrightarrow{\cong} B_{n-1} \xrightarrow{\subset} Z_{n-1} \xrightarrow{\subset} C_{n-1}.$$
 (6)

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By definition, we have the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{\subset} Z_n \longrightarrow H_n \longrightarrow 0 \qquad \qquad --\overline{\qquad (7)}.$$

for any n. As B_n and Z_n are subgroups of the free abelian group C_n and therefore free abelian, we recognize (7) as a free resolution of H_n .

Also by definition, from diagram (6) we have the short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\subset} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0, \tag{8}$$

which splits because B_{n-1} is free abelian. Using (7) and (8), we may rewrite the canonical short exact sequence

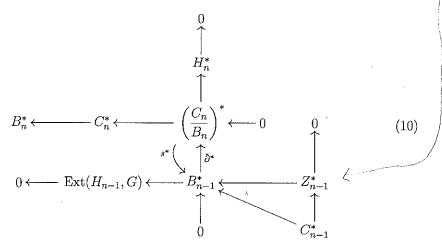
$$0 \longrightarrow Z_n/B_n \stackrel{\subset}{\longrightarrow} C_n/B_n \longrightarrow C_n/Z_n \longrightarrow 0$$

as

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \xrightarrow{\tilde{\partial}} B_{n-1} \longrightarrow 0, \tag{9}$$

where $\bar{\partial}$ is a quotient of ∂ . As B_{n-1} is free abelian, this short exact sequence also splits, and we may choose a splitting homomorphism $s: B_{n-1} \to C_n/B_n$.

Proof of Theorem 1 We dualize diagram (6) and the above short exact sequences to form the following diagram of exact sequences. To simplify, we write A^* for the G-dual Hom(A, G) of any abelian group A.



Both vertical sequences are exact, since (9) and (8) split. The upper horizontal sequence is exact, because the functor Hom(-,G) takes cokernels to kernels. The lower horizontal sequence is exact, as it defines $\text{Ext}(H_{n-1},G)$.

As usual, we write Z^n for the n-cocycles, etc., in the cochain complex C^* . Diagram (10) shows that $Z^{n-1} = \operatorname{Ker}[\partial^* : C_{n-1}^* \to C_n^*] = \operatorname{Ker}[C_{n-1}^* \to B_{n-1}^*]$. On replacing n-1 by n, we see that $Z^n = \operatorname{Ker}[C_n^* \to B_n^*] = (C_n/B_n)^*$. Also, $B^n = \operatorname{Im} \partial^* \cong \operatorname{Im}[Z_{n-1}^* \to (C_n/B_n)^*]$, so that $H^n = Z^n/B^n \cong \operatorname{Coker}[Z_{n-1}^* \to (C_n/B_n)^*]$.

We now apply Lemma 4 with $K = Z_{n-1}^*$, $L = B_{n-1}^*$ and $M = (C_n/B_n)^*$, and use the splitting $s^*: M \to L$. Diagram (10) identifies $\operatorname{Coker}[K \to L]$ with $\operatorname{Ext}(H_{n-1}, G)$ and $\operatorname{Coker}[L \to M]$ with H_n^* . These identifications reduce the split short exact sequence (5) to the desired (2). \square

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