Projective modules and Ext functor (Lecture IV) 1

Extensions:

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$$

Equivalence of $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C \to 0$ when $B \cong B'$ and diagr. comm.

A, B, C - R-modules for a ring R

<u>Def.</u> $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of R-modules. The sequence is known to be split if there is an R-module complement to $\psi(A)$ in B.

Then $B = A \oplus C$.

<u>Proposition</u> Let D, L and M be R-modules and let $\psi: L \to M$ be an R-module hom.

$$\psi': \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M)$$
$$f \to f' = \psi \circ \varphi$$
is a hom of abelian groups

If $0 \to L \xrightarrow{\psi} M$ is exact, then $0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M)$ is exact

<u>Theorem</u> D, L, M, N be R-modules

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$
$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N)$$

<u>Example</u>

$$\overset{L}{0 \longrightarrow \mathbb{Z}} \xrightarrow{2} \overset{M}{\mathbb{Z}} \xrightarrow{\pi} \overset{N}{\mathbb{Z}/2\mathbb{Z}} \to 0 \quad \text{Let } D = \mathbb{Z}/2\mathbb{Z}$$

Let f - identity map $f : D \to N$

Any $f \in \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 \Rightarrow \pi \circ F$ maps $D \to 0$ $\forall f$

<u>Prop-</u> $\mathrm{Hom}_R(D, L \oplus N) = \mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$

$\mathrm{Hom}_R(L \oplus N, D) = \mathrm{Hom}(\ ) \oplus \mathrm{Hom}(\ )$

<u>Prop.</u> P - R-module. The following are eq:

1) $\forall$ L, M, N

short exact L, M, N $\Rightarrow$

$$0 \to \mathrm{Hom}(P, L) \longrightarrow \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$$

short exact

2) $\forall$ M, N if $M \xrightarrow{\varphi} N \to 0$ is exact $\Rightarrow$ every R-module hom. $P \to N$ lifts to M

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "f"] & \\ M \arrow[r, "\psi"] & N \arrow[r] & 0 \end{tikzcd}$$

3) If P is a quotient of the R-module M

$\Rightarrow 0 \to L \to M \to P \to 0$ splits.

4) P is a direct summand of a free R-module

<u>Corollary</u> Free modules are projective. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module.

Every projective module over principal ideal domain is free.

Example $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ Submodule: $\mathbb{Z}/2\mathbb{Z}$ - direct summand

Finally, let us describe Ext:

Def. Let A be any R-module. A projective resolution of A is an exact sequence

$$\to P_n \xrightarrow{d_n} P_{n-1} \to \dots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

Ex. 1) $A = \mathbb{Z}/m\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

2) $R = \mathbb{Z}/m\mathbb{Z}$ d - divisor of m:

$$\xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \dots \to \mathbb{Z}/d\mathbb{Z} \to 0$$

Prop. $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$$

exact

Then we have:

$$0 \to \mathrm{Hom}_R(A, D) \xrightarrow{\varepsilon} \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, D) \to$$

$\to \dots$ etc.

Def. $d_n: \mathrm{Hom}_R(P_{n-1}, D) \to \mathrm{Hom}_R(P_n, D)$ $n \geq 1$

$$\mathrm{Ext}^n_R(A, D) = \ker d_{n+1} / \mathrm{Im}\, d_n$$

Proposition $\operatorname{Ext}^0_R(A,D) \cong \operatorname{Hom}_R(A,D)$

Proof. $0 \to \operatorname{Hom}_R(A,D) \to \operatorname{Hom}_R(P_0,D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1,D)$

$\operatorname{Ker} d_1 = \operatorname{Im} \epsilon = \operatorname{Hom}_R(A,D)$

Ex. Compute Ext for $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$

$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},D) \xrightarrow{m} \operatorname{Hom}(\mathbb{Z},D) \to 0$

$\operatorname{Ext}^0(\mathbb{Z}/m\mathbb{Z}, D) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD = \{d \in D \mid md = 0\}$

$D \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},D) \Rightarrow \operatorname{Ext}^1(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$

Compare resolutions

Prop. If $f : A \to A'$ - hom of R-modules Then there is a lift to chain maps f_n of proj. resolutions

Prop. $f : A \to A'$ - hom. of R-modules and take proj. resolutions $\Rightarrow \varphi_n : \operatorname{Ext}^n_R(A',D) \to \operatorname{Ext}^n_R(A,D)$
dep. only on f.

Theorem $\operatorname{Ext}^n_R(A,D)$ dep. only on A, D

$0 \to L \to M \to N \to 0$ exact, then

Theorem $0 \to \operatorname{Hom}(N,D) \to \operatorname{Hom}(M,D) \to \operatorname{Hom}(L,D)$
$\xrightarrow{\delta_0} \operatorname{Ext}^1_R(N,D) \to \operatorname{Ext}^1_R(M,D) \xrightarrow{\delta_1} \operatorname{Ext}^2_R(N,D) \to$

Theorem $0 \to L \to M \to N \to 0$

$$0 \to Hom(D,L) \to Hom(D,M) \to Hom(D,N)$$
$$\to Ext^1_R(D,L) \to Ext^1_R(D,M) \to Ext^2_R(D,L)$$

Ext is a right derived functor ← left exact

$0 \to A \to B \to C \to 0 \quad F: \ 0 \to F(A) \to F(B) \to F(C)$

$R^iF: \quad 0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \cdots$

Tor: $0 \to L \to M \to N \to 0$ right exact

$\to D \otimes L \to D \otimes M \to D \otimes N \to 0$

$Tor^n(D,L)$

Universal coeff theorem for Homology

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to Tor_1(H_{n-1}(C),G) \to 0$$

Universal Coefficient Theorem for Cohomology

We present a direct proof of the universal coefficient theorem for cohomology. It is essentially dual to the proof for homology.

THEOREM 1 *Given a chain complex C in which each C_n is free abelian, and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C), G) \longrightarrow H^n(\operatorname{Hom}(C,G)) \longrightarrow \operatorname{Hom}(H_n(C), G) \longrightarrow 0, \qquad (2)$$

which splits (but there is no natural splitting).

In particular, this applies immediately to singular cohomology.

THEOREM 3 *Given a pair of spaces (X, A) and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X,A), G) \longrightarrow H^n(X, A; G) \longrightarrow \operatorname{Hom}(H_n(X,A), G) \longrightarrow 0,$$

which splits (but there is no natural splitting). □

We shall derive diagram (2) as an instance of the following elementary result.

LEMMA 4 *Given homomorphisms $f\colon K \to L$ and $g\colon L \to M$ of abelian groups, with a splitting homomorphism $s\colon M \to L$ such that $s \circ g = \mathrm{id}_L$, we have a split short exact sequence*

$$0 \longrightarrow \operatorname{Coker} f \xrightarrow{g'} \operatorname{Coker}(g \circ f) \longrightarrow \operatorname{Coker} g \longrightarrow 0. \qquad (5)$$

Proof We write each cokernel, such as $\operatorname{Coker} f$, as $L/\operatorname{Im} f$ etc. Then the sequence (5) appears as the upper edge of the following diagram; we shall identify it with the bottom row, which is the canonical short exact sequence formed from the triple $\operatorname{Im}(g \circ f) \subset \operatorname{Im} g \subset M$.

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{L}{\operatorname{Im} f} \arrow[d] \arrow[dr, "g'"] \\
0 \arrow[r] & \dfrac{\operatorname{Im} g}{\operatorname{Im}(g \circ f)} \arrow[r, "\subset"] & \dfrac{M}{\operatorname{Im}(g \circ f)} \arrow[r] & \dfrac{M}{\operatorname{Im} g} \arrow[r] & 0
\end{tikzcd}$$

Since $g(\operatorname{Im} f) = \operatorname{Im}(g \circ f)$, g induces a homomorphism g'. Since $s(\operatorname{Im}(g \circ f)) = \operatorname{Im}(s \circ g \circ f) = \operatorname{Im} f$, $s\colon M \to L$ induces a homomorphism s' which splits g', $s' \circ g' = \mathrm{id}$; thus g' is injective. But g' clearly factors through a surjective homomorphism $L/\operatorname{Im} f \to \operatorname{Im} g/\operatorname{Im}(g \circ f)$, which is therefore an isomorphism. □

Preliminaries We consider a chain complex C as in Theorem 1. We adopt the usual notation: Z_n for the group of n-cycles, B_n for the group of n-boundaries, and $H_n = H_n(C) = Z_n/B_n$. The key idea in our proof of Theorem 1 is to express $\partial\colon C_n \to C_{n-1}$ as the composite

$$\partial\colon C_n \longrightarrow \frac{C_n}{B_n} \longrightarrow \frac{C_n}{Z_n} \xrightarrow{\cong} B_{n-1} \xrightarrow{\subset} Z_{n-1} \xrightarrow{\subset} C_{n-1}. \qquad (6)$$

By definition, we have the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{\subset} Z_n \longrightarrow H_n \longrightarrow 0 \tag{7}$$

for any n. As B_n and Z_n are subgroups of the free abelian group C_n and therefore free abelian, we recognize (7) as a free resolution of H_n.

Also by definition, from diagram (6) we have the short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\subset} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0, \tag{8}$$

which splits because B_{n-1} is free abelian. Using (7) and (8), we may rewrite the canonical short exact sequence

$$0 \longrightarrow Z_n/B_n \xrightarrow{\subset} C_n/B_n \longrightarrow C_n/Z_n \longrightarrow 0$$

as

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \xrightarrow{\bar\partial} B_{n-1} \longrightarrow 0, \tag{9}$$

where $\bar\partial$ is a quotient of ∂. As B_{n-1} is free abelian, this short exact sequence also splits, and we may choose a splitting homomorphism $s\colon B_{n-1} \to C_n/B_n$.

Proof of Theorem 1 We dualize diagram (6) and the above short exact sequences to form the following diagram of exact sequences. To simplify, we write A^* for the G-dual $\operatorname{Hom}(A, G)$ of any abelian group A.

$$\begin{tikzcd}
 & & 0 & & \\
 & & H_n^* \arrow[u] & & \\
B_n^* & C_n^* \arrow[l] & \left(\dfrac{C_n}{B_n}\right)^* \arrow[l] \arrow[u] \arrow[d, bend right, "s^*"'] & 0 \arrow[l] & 0 \\
0 & \operatorname{Ext}(H_{n-1}, G) \arrow[l] & B_{n-1}^* \arrow[l] \arrow[u, "\bar\partial^*"'] & & Z_{n-1}^* \arrow[ll] \arrow[u] \\
 & & 0 \arrow[u] & & C_{n-1}^* \arrow[u] \arrow[ull]
\end{tikzcd} \tag{10}$$

Both vertical sequences are exact, since (9) and (8) split. The upper horizontal sequence is exact, because the functor $\operatorname{Hom}(-, G)$ takes cokernels to kernels. The lower horizontal sequence is exact, as it defines $\operatorname{Ext}(H_{n-1}, G)$.

As usual, we write Z^n for the n-cocycles, etc., in the cochain complex C^*. Diagram (10) shows that $Z^{n-1} = \operatorname{Ker}[\partial^*\colon C_{n-1}^* \to C_n^*] = \operatorname{Ker}[C_{n-1}^* \to B_{n-1}^*]$. On replacing $n-1$ by n, we see that $Z^n = \operatorname{Ker}[C_n^* \to B_n^*] = (C_n/B_n)^*$. Also, $B^n = \operatorname{Im}\partial^* \cong \operatorname{Im}[Z_{n-1}^* \to (C_n/B_n)^*]$, so that $H^n = Z^n/B^n \cong \operatorname{Coker}[Z_{n-1}^* \to (C_n/B_n)^*]$.

We now apply Lemma 4 with $K = Z_{n-1}^*$, $L = B_{n-1}^*$ and $M = (C_n/B_n)^*$, and use the splitting $s^*\colon M \to L$. Diagram (10) identifies $\operatorname{Coker}[K \to L]$ with $\operatorname{Ext}(H_{n-1}, G)$ and $\operatorname{Coker}[L \to M]$ with H_n^*. These identifications reduce the split short exact sequence (5) to the desired (2). $\square$