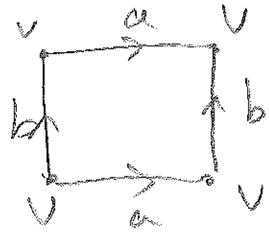
Lecture XVI - XVII

Cohomology ring, continued. Poincare duality

Ex. $X = $ Torus



$H_1(X) = \mathbb{Z}^2$ $\quad H^1(X,\mathbb{Z}) \cong \mathbb{Z}^2$

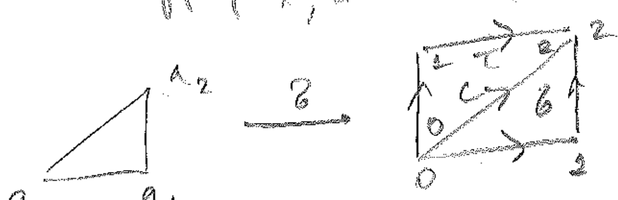
basis a, b

$\exists\ \alpha, \beta \in H^1(X,\mathbb{Z})$ s.t.

$\langle a, \alpha\rangle = 1 \quad \langle a, \beta\rangle = 0$

$\langle b, \alpha\rangle = 0 \quad \langle b, \beta\rangle = 1$

$H^1(X,\mathbb{Z}) \to \mathrm{Hom}(H^1(X,\mathbb{Z}))$ is onto



$\partial\sigma = b - c + a \quad z = \sigma - \tau \in Z_2(X)$

$\partial\tau = b - c + a \quad [z] \in H_2(X) \cong \mathbb{Z}$

$\langle [z], \alpha\cup\beta\rangle = (\alpha\cup\beta)(\sigma - \tau) = (\alpha\cup\beta)(\sigma) - (\alpha\cup\beta)(\tau)$

$= \alpha(a)\beta(b) - \alpha(b)\beta(a) = 1 \qquad \langle [z], \beta\cup\alpha\rangle = -1$

$\alpha\cup\beta$ generates $H^2(X;\mathbb{Z}) \cong \mathbb{Z} \qquad \alpha\cup\alpha,\ \beta\cup\beta = 0$

Lemma If $f: X \to Y$ is continuous, then

$f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta) \in H^{p+q}(X;R)$

Proof. True for $f^{\#}$

Def. Let X be a space, R - a ring

$$H^*(X,R) = \left(\bigoplus_{p=0}^{\infty} H^p(X,R),\ \cup\right)$$

<u>Corollary</u> If $f: X \to Y$ is a homotopy equivalence then $f^*: H^*(Y,R) \to H^*(X,R)$ is a ring isom.

<u>Theorem</u> $H^*(X,R)$ is associative if R is.
$H^*(X;R)$ has unit if R does,
namely $[i] \in H^0(X,R)$ defined by $i(\sigma) = 1 \in R$
for each 0-simplex $\sigma: \Delta^0 \to X$
If R is commutative $\beta \cup \alpha = (-1)^{pq} \alpha \cup \beta$
$\forall \alpha \in H^p(X), \beta \in H^q(X)$

<u>Proof</u> p. 216, Hatcher R comm.
$(\psi \cup \varphi)(\sigma) = \psi(\sigma|[a_0 \dots a_p])\, \phi(\sigma[a_p \dots a_{p+q}]) =$
$= \phi(\sigma|[a_p \dots a_{p+q}])\, \psi(\sigma|[a_0 \dots a_p])$
Let $r_n: [a_0 \dots a_n] \to [a_n \dots a_0]$
be the canonical linear isom. that reverses the ordering.
Define $\rho_n: C_n(X) \to C_n(X)$
$\varepsilon_n = (-1)^{n(n+1)/2}$
$\rho_n(\sigma) = \varepsilon_n r_n \sigma$
Show that ρ is a chain map $(\partial \rho = \rho \partial)$
that is chain homotopic to $1: C_n(X) \circlearrowleft$
Let $\rho^\#, \rho^*$ be induced maps on cochains
$\rho^\#(\psi \cup \phi)(\sigma) = (\psi \cup \phi)\rho_{p+q}(\sigma) =$
$= \varepsilon_{p+q}\, \psi(\sigma|[a_{p+q} \dots a_q])\, \phi(\sigma|[a_q \dots a_0]) =$

$= \varepsilon_{p+q}\varepsilon_p\varepsilon_q((\rho^{\#}\phi)\cup(\rho^{\#}\psi))z$

since $\rho \simeq 1$, $\rho^* = 1$ on cohomology

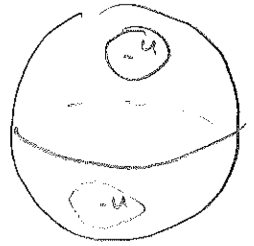
$[\psi]\cup[\phi] = \varepsilon_{p+q}\varepsilon_p\varepsilon_q[\phi]\cup[\psi]$

Poincare duality theorem

Def. An n-dim. manifold is a Hausdorff space M s.t. each pt on M has a neighb. U that is hom. to $\mathbb{R}^n$

Ex. 1) $\mathbb{R}^n$ – n-manifold

2) $\mathbb{RP}^n$ $[z_0, \dots z_n]$ coord. charts



M – m-manifold, N – n-manifold $\Rightarrow M\times N$ $m+n$-manifold

Terminology M is closed manifold if M is compact without boundary.

Poincare Duality ($\mathbb{Z}_2$-coeff.)

Let M be a closed connected n-manifold. Then $H^n(M;\mathbb{Z}_2)\cong\mathbb{Z}_2$ and $H^k(M;\mathbb{Z}_2)$ is a finite-dim space over $\mathbb{Z}_2$, and $\cup: H^k(M;\mathbb{Z}_2)\times H^{n-k}(M,\mathbb{Z}_2)\to H^n(M,\mathbb{Z}_2)\cong\mathbb{Z}_2$ is a nonsingular $\mathbb{Z}_2$-bilinear map, in particular $H^k(M,\mathbb{Z}_2)$ and $H^{n-k}(M,\mathbb{Z}_2)$ have same dim.

Ex. S^n, $\mathbb{RP}^n$ closed n-manifolds
$\mathbb{CP}^n$ - closed $2n$-manifold

Ex. $M = \mathbb{RP}^n = e^0 \cup e^1 \cup \ldots \cup e^n$

$$C^k(\mathbb{RP}^n, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Coboundary maps $\delta : C^k(\mathbb{RP}^n; \mathbb{Z}_2) \to C^{k+1}(\mathbb{RP}^n; \mathbb{Z}_2)$

$$H^k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

By Poincaré duality

$$\cup : \underbrace{H^k(\mathbb{RP}^n, \mathbb{Z}_2)}_{\mathbb{Z}_2} \times \underbrace{H^{n-k}(\mathbb{RP}^n; \mathbb{Z}_2)}_{\mathbb{Z}_2} \longrightarrow \underbrace{H^n}_{\mathbb{Z}_2} \cong \mathbb{Z}_2$$

is nonsing.

Cor. Let $\alpha_k \in H^k(\mathbb{RP}^n, \mathbb{Z}_2)$ be nonzero el-t. $k \le n$
Then $\alpha_k = \alpha_1 \cup \alpha_1 \cup \ldots \cup \alpha_1$

Cor. $H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha_1]/\alpha_1^{n+1}$ where $\alpha_1 \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$

Proof By induction on n:
Suppose true for $\mathbb{RP}^{n-1}$. Let $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$
as $n-1$-ske
$i^* : H^k(\mathbb{RP}^n; \mathbb{Z}_2) \to H^k(\mathbb{RP}^{n-1}, \mathbb{Z}_2)$ isom for $k < n$
For $k < n$, $i^*(\alpha_k) \neq 0 \in H^k(\mathbb{RP}^{n-1}, \mathbb{Z}_2)$
By induction $i^*(\alpha_k) = i^*(\alpha_1) \cup \ldots \cup i^*(\alpha_1) =$
$= i^*(\alpha_1 \cup \ldots \cup \alpha_1)$ $\quad \alpha_k = \alpha_1 \cup \ldots \cup \alpha_1$ $\quad k < n$
P-duality $\alpha_{n-1} \cup \alpha_1$ is nonzero $\Rightarrow \alpha_n = \alpha_1^n$

<u>Corollary</u> $\mathbb{RP}^3$, $\mathbb{RP}^2 \vee S^3$ have isomorphic homology groups, but diff. cup product structure (exercise)

<u>Ex.</u> $M = \mathbb{CP}^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$

$H^{2k}(\mathbb{CP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \forall 0 \le k \le n$

$H^{2k}(\mathbb{CP}^n; \mathbb{Z}_2) \times H^{2(n-k)}(\mathbb{CP}^n; \mathbb{Z}_2) \to H^{n}(\mathbb{CP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$

$\alpha_k \in H^k(\mathbb{CP}^n, \mathbb{Z}_2) \quad 0 \le k \le n$

$\alpha_k = \underbrace{\alpha_1 \cup \dots \cup \alpha_1}_{k \text{ times}}$ $\quad H^*(\mathbb{CP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha_1]/(\alpha_1^{n+1})$

$\alpha_1 \in H^2(\mathbb{CP}^n; \mathbb{Z}_2)$

What about $H^{2k}(\mathbb{CP}^n; \mathbb{Z})$ if $0 \le k \le n$

Let α_k generate $H^{2k}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$

$\alpha_1 \cup \dots \cup \alpha_1 = m_k \alpha_k \in H^{2k}(\mathbb{CP}^n; \mathbb{Z})$

Use coeff. sequence for cohomology groups

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$

$0 \to C^n(X; A) \xrightarrow{\alpha_*} C^n(X; B) \to C^n(X; C) \to 0$

Long exact seq:

$H^n(X, A) \xrightarrow{\alpha^*} H^n(X, B) \xrightarrow{\beta^*} H^n(X, C)$

$\to H^{n+1}(X, A) \to H^{n+1}(X, B) \to \dots$

Take $X = \mathbb{C}P^n$ $\quad 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

$$H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \xrightarrow{\times 2} H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \to H^{2k}(\mathbb{C}P^n; \mathbb{Z}_2) \to 0$$
$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow$$

$m_k \alpha_k$
$\shortparallel$
$\alpha_1 \cup \dots \cup \alpha_1$
m_k - odd

Poincare duality for any field F

$C^n(X; F) = \mathrm{Hom}(C_n(X), F)$ - vector space over F

<u>Theorem</u> Let M be a closed connected n-manifold

Then $H^k(M, F)$ is a finite-dim vect. space over F

If $H^n(M; F) \neq 0$ (i.e. M orientable over field F)
M is or. or char $F = 2$

then $H^n(M; F) \cong F$ and $\cup : H^k(M; F) \times H^{n-k}(M; F) \to H^n(M; F) \cong F$ is nondeg.

<u>Ex.</u> $M = \mathbb{C}P^n$, $F = \mathbb{Z}_p$

α_k gener $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$

$\alpha_1 \cup \dots \cup \alpha_1 = m_k \alpha_k$, m_k-odd $\quad$ Find that $m_k = 1$

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}(\alpha_1)/\alpha_1^{n+1}$$

<u>Cor.</u> $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isom hom groups

$H^2(S^2 \vee S^4) \times H^2(S^2 \cup S^4) \to H^4(S^2 \vee S^4)$ zero

$$S^2 \rightleftarrows S^2 \vee S^4$$

Ex. $M = S^2 \times S^2$ closed 4-manifold

$H^0(M, \mathbb{Z}_p) \cong \mathbb{Z}_p$

$H^2(M; \mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$

$H^4(M; \mathbb{Z}_p) \cong \mathbb{Z}_p$

$\cup : H^2(M; \mathbb{Z}_p) \times H^2(M, \mathbb{Z}_p) \to H^4(M, \mathbb{Z}_p) = \mathbb{Z}_p$

nonsing, symm.

$\alpha\beta = \beta\alpha$

Claim $\alpha \cup \alpha = 0 = \beta \cup \beta$

$M = S^2 \times S^2 \xrightarrow{\pi} S^2$ $\quad \pi^* : H^2(S^2, \mathbb{Z}) \to H^2(S^2 \times S^2, \mathbb{Z})$

$\alpha = \pi^* \xi, \ \xi \in H^2(S^2, \mathbb{Z})$

$\alpha \cup \alpha = \pi^* \xi \cup \pi^* \xi = \pi^*(\xi \cup \xi)$

$H^4(S^2; \mathbb{Z}) = 0$

Matrix $A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$ some $c \in \mathbb{Z}$

$\det A \neq 0 \mod p \ \ \forall p \Rightarrow c = \pm 1$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Lefschetz fixed point theorem

Let X be a finite CW complex

$f : X \to X$ continuous

$H^n(X; \mathbb{Q})$ - finite dim vector space / $\mathbb{Q}$

$f^{*n} : H^n(X; \mathbb{Q}) \to H^n(X; \mathbb{Q})$ $\quad \mathbb{Q}$-linear

Def. The Lefschetz number of $f : X \to X$

is $\sum_n (-1)^n \operatorname{tr}(f^{*n} : H^n(X; \mathbb{Q}) \to H^n(X; \mathbb{Q})) = \Lambda(f)$

Rmk $f \simeq g : X \to X \Rightarrow \Lambda(f) = \Lambda(g)$

Lefschetz fixed point theorem

Let $f: X \to X$ be a cont. self map on a finite CW complex. If $\Lambda(f) \neq 0 \Rightarrow f$ has fixed pt.

Ex. $X = \mathbb{C}P^2$, $f: \mathbb{C}P^2 \to \mathbb{C}P^2$

Recall: $H^*(\mathbb{C}P^2, \mathbb{Q}) = \mathbb{Q}[\alpha]/\alpha^3$

$\alpha \in H^2(\mathbb{C}P^2; \mathbb{Q})$

$\alpha \cup \alpha \in H^4$

$f^* : \alpha \to r\alpha$

$f^*(\alpha \cup \alpha) = f^*(\alpha) \cup f^*(\alpha) = r^2 \alpha \cup \alpha$

$\operatorname{tr} f^{*2} = r \qquad \operatorname{tr} f^{*4} = r^2$

$\Lambda(f) = r^2 + r + 1$ Always fixed pt.