Lecture I Basic topological notions

Algebraic topology:
Properties of topological spaces via abstract algebra

Top. spaces $\longrightarrow$ invariants
up to homeomorphism
homotopy - to be discussed later.

Notations: $\mathbb{R}^n = \{(x_1, \dots, x_n), x_i \in \mathbb{R}\}$ $\quad \|\vec{x}\| = \sqrt{\sum x_i^2}$

$D^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$ - closed n-disk

$S^n = \{u \in \mathbb{R}^{n+1}, \|u\| = 1\}$

$X \cong Y$ means X homeomorphic to Y

Motivating problems $\overset{1920}{\rightsquigarrow}$ Classical theorems

1) Does $\mathbb{R}^m \cong \mathbb{R}^n \Rightarrow m = n$? (yes)

$S^m \cong S^n \Rightarrow m = n$? (yes)

Ex $S^n \cong \mathbb{R}^n \cup \{\infty\}$
1-point compactif. of $\mathbb{R}^n$

2) Brower Fixed point theorem
If $f: D^n \to D^n$ is continuous, then f has a fixed point, i.e. $\exists\, x \in D^n$, s.t. $f(x) = x$

All proofs involve "counting"

Also, the proofs consider spaces and maps modulo continuous deformation (homotopy)

Examples of homeomorphisms

i) All open intervals are homeomorphic to each other

ii) Ex. Show explicitly in coordinates that sphere without the north pole is homeomorphic to $\mathbb{R}^n$

iii) Show that  and  are not homeomorphic

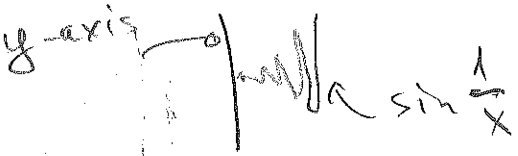
Connectedness

Path connectedness: $\forall\, x_1, x_2 \in X$ $\exists$ continuous map $f: [0,1] \to X$ such that $f(0) = x_1, f(1) = x_2$

In general, connected space is such that you cannot divide it into open sets with zero intersection.

Path connected $\Rightarrow$ connected

Connected $\not\Rightarrow$ Path connected



$\leftarrow$ example

Ex. Show that $GL(n, \mathbb{R})$ is not connected (space of invertible matrices)

In this course, connected = path connected

Compactness

In euclidean space it means that the set is closed and bounded.

Ex. i) S^n is compact

ii) Matrix groups:

$GL(n,\mathbb{R})$ not connected, not compact

$SO(n,\mathbb{R})$ $\det = 1$ – compact, connected

$U(n,\mathbb{R})$ – compact, connected

$U(1), SO(2) \cong S^1$

Infinite-dimensional objects in topology:

$C(X,Y)$ – space of continuous maps from $X \to Y$

Connected components – homotopy classes

If $\exists f_t$, $t \in [0,1]$ $f_0, f_1 : X \longrightarrow Y$ are homotopically equivalent

Denote $\{X, Y\}$ the set of all homotopy classes of maps

Ex. $X = pt$. What is the set of all homotopy classes?

Ex. $S^1 \to \mathbb{R}^2 \setminus \{0,0\}$????

We call f to be homotopic to 0, if it is homotopic to the map $X \to pt$ in Y

<u>Ex.</u> $S^{n-1} \to Y$ is homotopic to 0 iff one can continuously extend it to D^n

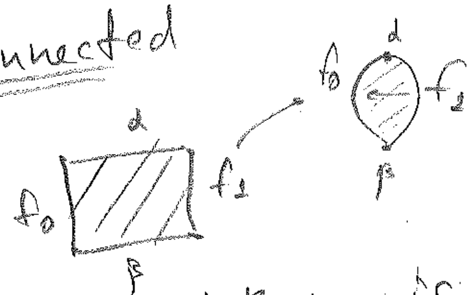
[Really, iff $g(x^1, \dots, x^n)$ is a map of $(x^1)^2 + \dots + (x^n)^2 \le 1$ and $f_1(x^1, \dots, x^n)$ is a map g, considered on $(x^1)^2 + \dots + (x^n)^2 = 1 \Rightarrow f_t(x^1, \dots, x^n) = g(tx^1, \dots, tx^n)$ is the necessary homotopy. Inverse can be proved the same way.

For future: $S^{n-1} \xrightarrow{id} S^{n-1}$ - is not homotopic to 0.

<u>Simply connected spaces</u>

If every map of $S^1 \to X$ is homotopic to 0 then X is called <u>simply connected</u>

In a simply connected space



We call X aspherical in dim k if $\forall f: S^k \to X$ is homotopic to 0.

<u>Ex.</u> Two maps $f_0, f_1 : X \to$ concave set are homotopic to each other.

This is why S^n is aspherical in dim $k<n$ (eliminate point from S^n) (Peano curve-exception! any map homotopic to smooth!)

Finally let us study $\{S^1, S^1\}$

$\Omega(S^1)$ - maps of $S^1 \to S^1$ (loop space of S^1)

Let us show that there is 1-to-1 corr. between $\{S^1, S^1\}$ and $\mathbb{Z}$

First, we learn that $\Omega^1(S^1)$ is homeomorphic to the space of maps $\varphi : [0,1] \to \mathbb{R}$, s.t. $\varphi(0)=0$, $\varphi(1)\in\mathbb{Z}$

Really $S^1 \to S^1 \longleftrightarrow [0,1] \to \mathbb{R}$

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow & \downarrow \\ S^1 & \rightarrow & S^1 \end{array}$$

$\alpha \to e^{2\pi i \alpha}$ locally inverse

In other words $\varphi(1) = \varphi(0) + m$ $m \in \mathbb{Z}$

Let $\varphi(1) = k$. Therefore for each homotopic loop we will have k. The space of all loops with the same k is path-connected:

$\varphi, \varphi_1 \to \lambda\varphi + (1-\lambda)\varphi_1$. ▨

Later will prove that $\{S^n, S^n\}$ is also $\mathbb{Z}$

What about $\{S^1, \mathbb{R}^2 \setminus \{0\}\}$?

$S^1(\varphi) \longrightarrow \mathbb{R}^2(r,\varphi)$ $r = g_f(\varphi)$, $\theta = h_f(\varphi)$

map from $S^1 \to \mathbb{R}_+$

again 1-to-1 corr. with $\mathbb{Z}$.

<u>Generalization</u> $f : X \to \underset{g_f}{Y} \times \underset{h_f}{Z}$ $\{X, Y\times Z\} = \{X,Y\}\times\{X,Z\}$

In particular $\{X, S^{n-1}\} \cong \{X, \mathbb{R}^n \setminus \{0\}\}$

<u>Ex.</u> $\{S^1, \underbrace{S^1\times S^1}_{=T^2}\}$ is labeled by a pair of integers

Y_1, Y_2 — are called homotopically equivalent if

$h_1 : Y_1 \to Y_2$ $h_1 \circ h_2 \sim Id$

$h_2 : Y_2 \to Y_1$ $h_2 \circ h_1 \sim Id$

Then $\{X, Y_1\} \sim \{X, Y_2\}$