Lecture XX

Hurewicz map, further technique

Ex. Moore spaces:

construct CW complex X $H_n(X) \cong G$ and $H_i(X) = 0$, $i \neq n$, simply connected

$M(G, n)$

$G = \mathbb{Z}_m$ $X = S^n$ with e^{n+1} attached $S^n \to S^n$ of deg m.

More generally, $\vee_\alpha S^n_\alpha$ attach cells to give the relations.

How unique they are?

We have to show that for two $X, Y \in M(G, n)$ constructed as above

there is an isom. on H_n level

It is true on the level of π_n

For mapping cylinder M_f $\pi_i(M_f, X) = 0$ for $i \leq n$

If that is true for $i = n+1 \Rightarrow$ we get what we wanted by Hurewicz theorem ($H_{n+1}(M_f, X) = 0$)

Enlarge Y by attaching $(n+2)$-cells to make $\pi_{n+1} = 0$. The new M_f is such that $\pi_{n+1}(M_f, X) = 0$. But attaching $n+2$ cells has no effect on $H_n \Rightarrow$ orig. f has to be isom on H_n

Ex. $X = (S^1 \vee S^n) \cup e^{n+1}$ for any $n > 1$, such that $S^1 \hookrightarrow X$ induces isomorphism on all homology groups and on π_i for $i < n$.

First of all: $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$

Why? basis for π_n of universal cover is repr. by $S^n \hookrightarrow S^1 \vee S^n$ acted upon $\pi_1(S^1 \vee S^n) \cong \mathbb{Z}$

gen of $\pi_1(S^1 \vee S^n) = t$, group ring $\mathbb{Z}[\pi_1(S^1 \vee S^n)]$ is $\mathbb{Z}[t, t^{-1}]$

$\Rightarrow \pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$

Let X be such that e^{n+1} is attached to $2t-1$.

$\pi_n(X) \cong \mathbb{Z}[t, t^{-1}]/2t-1$. Set $t = 1/2$

embeds $\mathbb{Z}[1/2] \subset \mathbb{Q}$

Show that we have isomorphism for H_n:

$H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$ is an isom.

since comp. $S^n \xrightarrow{\text{att. map}} S^1 \vee S^n \xrightarrow{\text{collapse}} S^n$

$2 - 1 = 1$

This example shows that we can have isom on homology level but not on homotopy groups level

Hurewicz map

$\pi_n(X,A,x_0)$ for $n>0$ $\quad f:(D^n,\partial D^n,s_0)\to(X,A,x_0)$

Hurewicz map: $h:\pi_n(X,A,x_0)\to H_n(X,A)$ is defined by $h([f])=f_*(\alpha)$ where α is a fixed generator of $H_n(D^n,\partial D^n)\simeq\mathbb{Z}$ and $f_*:H_n(D^n,\partial D^n)\to H_n(X,A)$ induced by f.

$f\simeq g\Rightarrow f_*=g_*$

Proposition Hurewicz map $h:\pi_n(X,A,x_0)\to H_n(X,A)$ is a homomorphism, assuming $n>1$ so that $\pi_n(X,A,x_0)$ is a group

Proof Enough to show that for $f,g:(D^n,\partial D^n)\to(X,A)$ the induced maps on homology satisfy $(f+g)_*=f_*+g_*$

If that is true, then $h([f+g])=(f+g)_*(\alpha)=$
$=f_*(\alpha)+g_*(\alpha)=h([f])+h([g])$

Let $c:D^n\to D^n\vee D^n$ (D^{n-1}-equatorial $\to$ pt)

and $q_1,q_2:D^n\vee D^n\to D^n$ are quot. maps

$$\begin{tikzcd} H_n(D^n,\partial D^n) \arrow[r, "c_*"] & H_n(D^n\vee D^n,\partial D^n\vee\partial D^n) \arrow[r, "(f\vee g)_*"] \arrow[d, shift left, "q_{1*}\oplus q_{2*}", "\cong"'] & H_n(X,A) \\ & H_n(D^n,\partial D^n)\oplus H_n(D^n,\partial D^n) \arrow[u, shift left, "i_{1*}+i_{2*}"] & \end{tikzcd}$$

$q_2c\simeq q_1c\simeq id\Rightarrow$
$\Rightarrow(q_{1*}\oplus q_{2*})c_*$ is diag map $x\mapsto(x,x)$

Let i_{1*},i_{2*} induced by incl. $D^n\hookrightarrow D^n\vee D^n$

$(f\vee g)i_1=f \qquad (f\vee g)i_2=g\Rightarrow(f\vee g)_*(i_{1*}+i_{2*})$

$(f \vee g)_*(i_{1*} + i_{2*})(x,x) = f_*(x) + g_*(x) \Rightarrow$

$\Rightarrow$ composition $(f \vee g)_* c_* : x \to f_*(x) + g_*(x)$

On the other hand $f + g = (f \vee g)c \Rightarrow (f+g)_*$ ■

Absolute Hurewicz map: $h : \pi_n(X, x_0) \to H_n(X)$

$h([f]) = f_*(\alpha)$ for $f : (S^n, s_0) \to (X, x_0)$ and α chosen generator of $H_n(S^n)$. If $X = S^n \Rightarrow f_*(\alpha) = (\deg f)\alpha$

so h is a degree map $\pi_n(S^n) \to \mathbb{Z}$

homomorphism for $n \geq 1$.

Comm. diagram:

$$\begin{tikzcd}
\to \pi_n(A, x_0) \arrow[r] \arrow[d, "h"] & \pi_n(X, x_0) \arrow[r] \arrow[d, "h"] & \pi_n(X, A, x_0) \arrow[r] \arrow[d, "h"] & \pi_{n-1}(A, x_0) \to \arrow[d] \\
\to H_n(A) \arrow[r] & H_n(X, x_0) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \to
\end{tikzcd}$$

Elements of the kernel:

When $\pi_1(X, x_0)$ acts nontrivially on $\pi_n(X, x_0)$

$[\gamma][f] - [f]$ – in the kernel, since they are homotopic

if we do not req. basepoint to be fixed

$(\gamma f)_*(\alpha) = f_*(\alpha)$

In the rel. case $h : \pi_n(X, A, x_0) \to H_n(X, A)$

contains elements of the form $[\gamma][f] - [f]$

for $[\gamma] \in \pi_1(A, x_0)$

$\pi_n(S^1 \vee S^n, S^1) \to H_n(S^1 \vee S^n, S^1)$ is the hom

$\pi_1(A, x_0) = 0$ – important!! $\mathbb{Z}[t, t^{-1}] \xrightarrow[t \to 1]{} \mathbb{Z}$

Consider $\pi_n'(X,A,x_0)$ - quotient by $([\gamma][f]-[f])$

$h': \pi_n'(X,A,x_0) \to H_n(X,A)$

<u>Theorem</u> If (X,A) is $(n-1)$-conn. pair of path-conn. spaces with $n \geq 2$ and $A \neq \emptyset \Rightarrow h': \pi_n'(X,A,x_0) \to H_n(X,A)$ is an isomorphism and $H_i(X,A)=0$ for $i<n$.

Stable homotopy groups:

<u>Suspension theorem</u>:

$\pi_i(X) \to \pi_{i+1}(SX)$ is isom for $i<2n+1$ and a surj for $i=2n+1$ if X - n-connected.

This holds for $i \leq n \Rightarrow SX$ is is $n+1$-conn.

Therefore $\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X)$ all maps are isom. eventually. This is called $\pi_i^S(X)$

$\pi_i^S(S^0) = \pi_{i+n}(S^n)$ for $n > i+1$

π_i^S - stable i-stem, always finite.

Graded ring structure on $\pi_i^S \times \pi_j^S \to \pi_{i+j}^S$

$S^{i+j+k} \to S^{j+k} \to S^k$