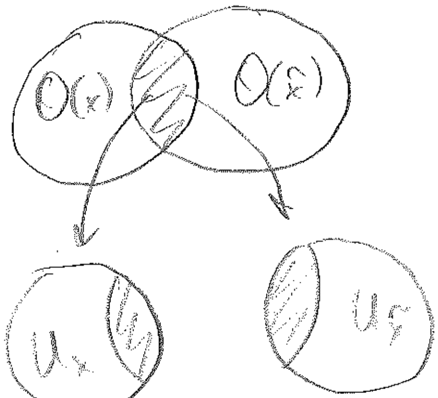
Smooth manifolds (some notions) Lecture XXI

Def. Smooth manifold of C^r type: Hausd. space with count. base
$\forall\, x \in M \ \exists\, O(x)$ and φ_x mapping $O(x)$ on neighb. in $\mathbb{R}^n$



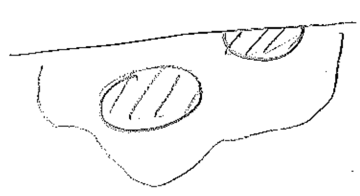
$\varphi_{\tilde{x}}\varphi_x^{-1} : U_{x\tilde{x}} \to U_{\tilde{x},x}$
is a diffeomorphism C^r type

Def. Atlas of C^r-type: covering of M^n
$O(x)$ is called chart

Def. Atlases are called equiv. if their union is also an atlas of C^r type.

Def. C^r-structure on M^n - class of eq. of atlases.

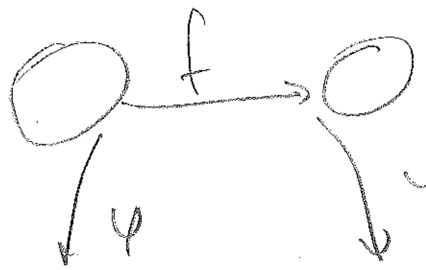
Manifold with boundary:
not $\mathbb{R}^n$, but $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \le 0\}$



$\mathbb{R}^n_- \to \mathbb{R}^n_-$ - smooth of C^r-class if it is the restriction of some map of class C^r

Closed manifold: smooth compact manifold w/o bound.

$M^m_{C^r} \xrightarrow{f} N^n_{C^r}$ of class C^p $p \le r$



$\psi f \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ of class C^p
where $p \le r$

Diffeomanifolds if $\exists\, \varphi : M^n \to N^n$ so that its inverse is of class C^r.

Examples 1) S^n 2) $\mathbb{R}P^n$

3) Grassmann manifolds $G_k(\mathbb{R}^n)$
set of all k-dimensional subspaces in $\mathbb{R}^n$

$G_k^+(\mathbb{R}^n)$ - set of k-dim. subspaces with orientations

$G_1(\mathbb{R}^n) = \mathbb{R}P^{n-1}$ $G_1^+(\mathbb{R}^n) \simeq S^{n-1}$ $G_{n-1}(\mathbb{R}^n) \simeq \mathbb{R}P^{n-1}$

$G_k(\mathbb{R}^n) \approx G_{n-k}(\mathbb{R}^n)$ bilinear form in $\mathbb{R}^n$ $G_k^+(\mathbb{R}^n) \approx G_{n-k}^+(\mathbb{R}^n)$ choice of or. bilinear form.

$G_k^+(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ 2-fold covering space.

4) Lie groups: $GL(\mathbb{R}^n)$ $O(n, \mathbb{R}^n)$ etc.

$\mu : G \times G \to G$ smooth.

Orientation:

Atlas is orientable if $\forall_{\alpha}\ \varphi_\beta\varphi_\alpha^{-1}$ is > 0 at all pts

Orientation: choice of such atlas

Ex. $\mathbb{RP}^n$ is not orientable for n even

Tangent bundle

$\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ of C^r-class $r \geq 1$ $0 \to x$

$|\varphi_1(t) - \varphi_2(t)| = o(t)$ $t \to 0$ - equiv

equiv. class - tangent vector at x.

two classes

$\lambda\varphi_1 + \mu\varphi_2$ Tangent space.

$\lambda v_1 + \mu v_2$

Coordinates $\varphi : t \to (\varphi_1(t), \dots, \varphi_n(t))$

$\left(\frac{d\varphi_1}{dt}(0), \dots, \frac{d\varphi_n}{dt}(0)\right)$

Orientability $\to$ orientability of tangent spaces

TM^n (x, v) $x \in M^n$, $v \in T_x M^n$

Topological space:

If atlas for M^n is $(\varphi_\alpha, V_\alpha)$ $\varphi_\alpha : V_\alpha \to U_\alpha \times \mathbb{R}^n$

gives 1-1 correspondence between TV_α and TU_α $\overset{\shortparallel}{U_\alpha \times \mathbb{R}^n}$

Ex. $\forall$ M^n TM^n is orientable

Prop. Every cont map $M^m \to N^n$ can be approx. by smooth map and it is homotopic to smooth map

Degree of a map

$f: M^m \to N^n$, choose local coord. $x_1, \dots, x_m$ in the neighb. of $x \in M^m$ and local coord. $y_1 \dots y_n$ in the neighb. of $f(x) \in N^n$

$$y_i = \varphi_i(x_1 \dots x_m)$$

Def. $x \in M^m$ - regular for f if $\operatorname{rank}\left(\frac{\partial \varphi_i}{\partial x_j}\right) = \max = \min(m,n)$

does not dep. on coord. choice

For regular x, implicit function theorem says that one can choose $x_1, \dots, x_m$ around x and $y_1 \dots y_n$ near $f(x)$ s.t. f is as follows:

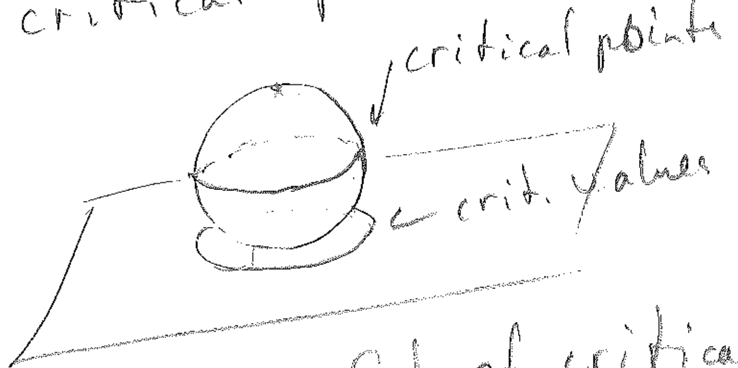
$m \geq n$ $\quad y_1 = x_1, \dots\ y_n = x_n$

$m \leq n$ $\quad y_1 = x_1 \dots\ y_m = x_m,\ y_{m+1} = \dots = y_n = 0$

Def. $m \leq n$ $M^m \to N^n$ so that all points are reg. called imm. and $m \geq n$ submersion

If x is not regular it is called critical

critical points $\longrightarrow$ critical values



projecting sphere on a plane

Sard's lemma Set of critical pts has measure 0.

Degree of a map

$\forall$ smooth map has noncrit. values

$f: M^n \to N^n$. Consider y - noncrit. value $\Rightarrow$ all points of $f^{-1}(y)$ are regular. Index of x_i s.t. $x_i \to y$ is sgn(det)

<u>Theorem</u> $\sum$ indices of $f^{-1}(y)$ does not depend on y. Index doesn't change under homotop. eq. map.

$\sum$ - degree of a map.

Classification of maps $M^n \to S^n$

where M^n is connected compact orientable manifold without boundary.

<u>Theorem</u> Two maps $M^n \to S^n$ of the same degree are homotopic to each other.

We show that $\forall k \in \mathbb{Z}$ $\exists$ map $M^n \to S^n$ of degree k.

Construct such maps:

M^n balls $U_1, \dots, U_k$ nonintersecting. Map each of them on $S^n \setminus S$ (south pole) so that Jac. is of given sign. The rest of a manifold map to a given point S.

Index of a vector field

Def. $p: E \to B$ $s: B \to E$ (section) s.t. $p \circ s = id_B$

section of a TM bundle is called vector field on M.

Any pt $\to$ vector

Singular pt. of vector field $\to$ 0-vector

$\hookrightarrow$ Isolated if its punctured neighb. no other sing. pts.

On the compact manifold M^n vector fields with isolated singular points form an open everywhere dense set in the set of all vector fields, i.e. almost all vector fields have isolated singular points

Index of a singular point in $\mathbb{R}^n$:

Take S_ε^{n-1}, ε-small with center at isolated sing. pt.

At any pt. in this sphere take a vector of $\vec{v}(x)$ and move it to sing. pt. This produces a map a map $S_\varepsilon^{n-1} \to S_1^{n-1}$. The degree of this map is the index ind

Theorem $\exists\, U \underset{\text{diff.}}{\cong} D \subset \mathbb{R}^n$ and v has only isolated sing pt. in $\bar{U}$, not lying on $\partial\bar{U} \Rightarrow$ index of v w.r.t. $\partial\bar{U}$, i.e. deg of $\partial\bar{U} \to S^{n-1}$ is the sum of indices of $v \in U$