Lecture XII Elements of Morse theory

Goal: construct cell decomposition of M based on function $f : M \to \mathbb{R}$

M - closed manifold without boundary

$f \in C^\infty(M;\mathbb{R})$. Critical point of f is a point where $\nabla f = 0$. In local coordinates

$$f(x) = f(0) + \sum_{i,j} a_{ij} x_i x_j + O(|x|^3) \quad a_{ij} = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}$$

Morse sing: a_{ij} - nondegenerate

Lemma (Morse) The function above can be reduced to canonical form

$$x_1^2 + x_2^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$$

via choice of coordinates.

Def. Function is of Morse type or Morse f-n if all critical pts are Morse type.

Function is a strongly Morse function if it is of Morse type and every crit. value it takes only at 1 pt.

Theorem In the space of all smooth functions on M strongly Morse f-ns form a dense set.

Ex. $f(x) = x^3$. One can construct function, coinciding with near crit. pt. that near any other pt, and near that pt $x^3 + \tau x$ τ - close to 0 $\tau > 0$ no crit. pts near 0

$\tau < 0$ two Morse crit. pts

Cell structure of M, for $f: M \to \mathbb{R}$
where f is Morse

Let $M_t = f^{-1}((-\infty, t])$
and let's look how it changes with $t \nearrow$

Assume that it is strongly Morse

First $M_t = \emptyset$, then min, slightly above min $\simeq D^n$

<u>Proposition</u> If $a < b$ and there is no crit. point of f on $[a, b] \Rightarrow M_a \underset{\text{homeom}}{\simeq} M_b$

$\boxed{\text{Index} = \text{negative index of inertia of quad. form}}$

This homeom is given by the action of $\underline{\text{grad} f} = \vec{v}$

Index of a crit pt - negative inertia index of quadr form

Consider crit. pt of index i; t_0 - crit. value

Consider $M_{t_0 - \varepsilon}$ and $M_{t_0 + \varepsilon}$ so that no crit. value on $[t_0 - \varepsilon, t_0 + \varepsilon]$

<u>Theorem</u> $M_{t_0 + \varepsilon}$ is hom. eq. to $M_{t_0 - \varepsilon}$ with attached disk of dim i, i.e.

$M_{t_0 + \varepsilon} \sim D^i \cup_{\varphi} M_{t_0 - \varepsilon}$, where $\varphi: \partial D^i \to M_{t_0 - \varepsilon}$ some continuous map

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$\forall$ critical pt. of f
One can define top and bottom separatrix manifolds

Closure of $y \in M$ for which $\frac{dx}{dt} = \mp \vec{\nabla} f$ $\quad y \searrow$ crit. pt. $+\infty$

$f(x, y) = x^2 - y^2$ $\quad$ x - top, y - bottom sep. manifold

$\frac{dx}{dt} = 2x$ $\quad$ ~~$x = e^{2t} a$~~ $\quad$ $t \to +\infty$, $y(+\infty) = 0$

$\frac{dy}{dt} = -2y$ $\quad$ $y = e^{-2t} b$ $\quad$ $y(0) = b$

In general near Morse sing points
Separatrix manifolds are transversal disks, such that the sum of dim = dim M

The disk from the theorem ... part of the bottom separatrix manifold where f takes values $f_0-\varepsilon$ to f_0

<u>Theorem</u> Any compact manifold is hom. eq. to finite CW complex.

Gluing handles

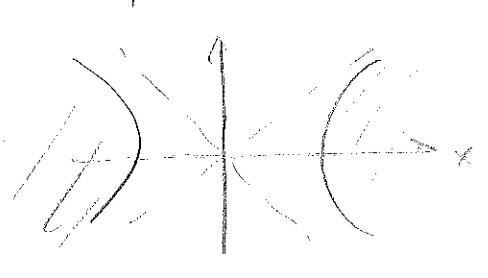
$f: M \to \mathbb{R}$ - Morse function $M_a = f^{-1}((-\infty, a])$

$a < b$ One critical value α with index i.

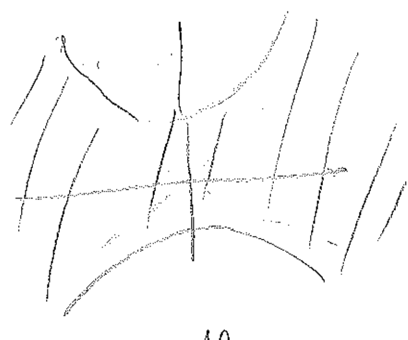
Then $M_b \sim D^i \cup_\varphi M_a$, where φ is an embedding

$\partial D^i \to \partial M_a = f^{-1}(a)$

$n=2$, $i=1$, $f = -x^2+y^2$ $\quad a<0 \quad b>0$



M_a



M_b

<u>Handle</u>: $(D^i \times D^{n-i}, S^{i-1}\times D^{n-i})$

Attachment of handle to M_a $D^i \times D^{n-i} \cup_\varphi M$.

$\varphi: S^{i-1}\times D^{n-i} \to \partial M_a$

Perfect Morse function

Morse function, whose critical values at the points of index i are below all critical values of index $i+1$ for all i.

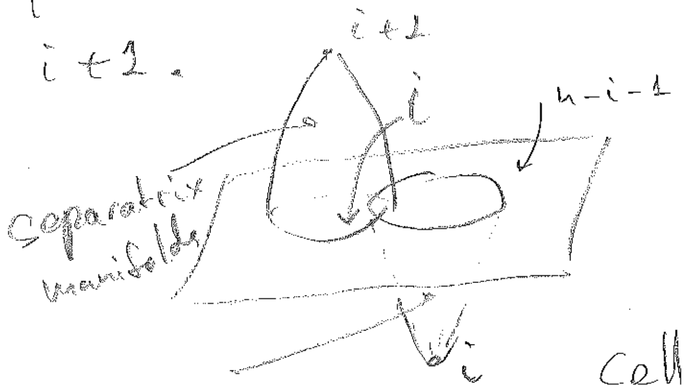
Theorem (Smale) On every compact manifold there is a perfect Morse function.

Torus and height function are examples.

Boundary operator in Morse complex

Assume, we have M^n and perfect Morse function M^n has a cell partition, so that crit. points corr. to cells. Have to compute incidence coeff.

Assume M^n - orientable. Consider intermediate piece $f^{-1}(\xi)$, wher ξ is above i and below $i+1$.



$n-i-1+i = n-1$
that's transversal inters. (if they intersect)

Cells - bottom separatrix disks with orientations

Orient. of bottom $\to$ orient. of top since manifold is orient.

$f^{-1}(\xi)$ is also oriented. Its orientation, together with any tangent vector along which f is increasing gives a fixed orient of all M. (5

Second pair of two disks is also oriented $\Rightarrow$
$\Rightarrow$ both spheres are oriented.

At any pt. of intersection of these spheres we can have a frame of one sphere and another and attach ± 1 dependingly whether we have the same orient. as the section

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Theorem Homology of closed orientable manifold coincide with homology of finegen. chain complex $\{C_i\}$ - generated by critical pts of index i and diff. are given by the sum of flows.