Morse theory (continued) Lecture XIII

<u>Theorem</u> On a compact manifold M with boundary ∂M there exist a Morse function f, so that absolute max is on the boundary, so that near the boundary $\vec{\nabla} f \neq 0$ and transversal to boundary $f^{-1}(\cdot) = \partial M$.

<u>Proof</u> Take function $=$ dist$(pt, \partial M)$ in $\mathbb{R}^n$ (and embed $M \subset \mathbb{R}^n$) deform it slightly and $f = -g$ is what we need.

Again, Morse complex is $C = C(M, f)$ C_i - free ab. group with crit. pts being generators

$H_*(M, \partial M)$ - use $-f$ to start cell dec. from boundary.

Generalization to noncompact — embed in $\mathbb{R}^n$ possibly infinite number of crit. points

<u>Morse inequalities</u>

$b_i = \dim H_i(C) \leq \dim C_i$ in part. $b_i \leq \mu_i$ ← # critical pts

Moreover $b_i + \text{tor}_i \leq \mu_i$ ← max number of generators for tors. gr.

Moreover, $\to C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} C_{i-2} \xrightarrow{\partial}$ and $C_i = \mathbb{Z}^{d_i}$

$\Rightarrow b_i + \text{tor}_i + \text{tor}_{i-1} \leq d_i$

In particular, if $\{\ell_i\}$ is a Morse complex left side gives estimate for lower bound of crit. $\underbrace{\text{points}}_{\text{of index } i}$ of f.

Simplest situatio $0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to 0$

$0 \quad 0 \quad \mathbb{Z}_p \quad 0$

If we know homology and that all C_i are free abelian Second on the left hom. group comes from nonzero chain group $\Rightarrow \mathbb{Z}_p \longrightarrow$ two contrib.

<u>Complicated theorem</u> (Smale)
If M^n is compact, simply connected w/o boundary and $n \geq 6 \Rightarrow M^n$ possesses perfect Morse function so that $\forall i$, number of critical points of index i is exactly $b_i + tor_i + tor_{i-1}$.

Poincare duality revisited

Theorem M^n - smooth compact manifold without boundary. $\forall i$ there is an isomorphism:

$$H_i(M^n) \cong H^{n-i}(M^n)$$

$\hookrightarrow H_i(M) \otimes H_{n-i}(M) \to \mathbb{Z}$
nondegenerate

Proof. Consider perfect Morse function f.

C_i - bottom separatrix disks C^i - top separatrix disks

pairing - orientation

At the same time $-f$. Then C_{n-i} are identified with C^i

Geometric meaning of pairing

$$H_i \otimes H_{n-i} \to \mathbb{Z}$$

realize by lin comb. of bottom sep. cells

by lin. com. of top sep. cells

Pairing is calculated: Take all crit. points of index i and look with what coeff. their separatrix manif. go into corr. cycles. We get two numbers a, b. Multiply them and sum up.

On the singular chain level

$\alpha \in H_i(M)$, $\beta \in H_{n-i}(M)$

realize as sing chains

After small "movement" they will intersect transversally

$\varphi_1 : L_1 \to M$, $\varphi_2 : L_2 \to M$ are transversal at two points a_1, a_2 if $\varphi_1(a_1) \neq \varphi_2(a_2)$ or $\varphi_1(a_1) = \varphi_2(a_2)$ and $\varphi_{1*}(T_{a_1}L_1) + \varphi_{2*}(T_{a_2}L_2) = T_{\varphi_1(a_1)}M$

Small movement of simplices → all intersection pts of their images will be internal points of simplices of max dim i and $n-i$ and they are transversal. Choose ± 1 depending on whether it coincides with or. of M. Show that indep. on representatives.

Case of manifolds with boundary and noncompact manifolds

$$H_i(M^n) \cong H^{n-i}(M^n, \partial M^n)$$ – for oriented manifold w. boundary

$$H^i(M^n) \cong H_{n-i}(M^n, \partial M^n)$$

Take Morse function with min value on the boundary and use it to compute relative homology using bottom sep. manifold. Then $-f$ and compute absolute cohomology using bottom sep. manifold.

Now noncompact case: (oriented, no boundary)

X homeomorphic open subset of $\overline{X}$ - compactification of X. There always is (if X is not compact itself)

1-point compactification: $\mathbb{R}^n \to S^n$

$\overline{H}_i(M^n) = H_i(\overline{M^n}, *)$ – Borel-Moore homology

Poincare Lefschetz: $\overline{H}_i(M^n) \cong H^{n-i}(M^n)$

$\overline{H}^i(M^n) \cong H_{n-i}(M^n)$