Lecture II Fundamental group and higher homotopy groups

From last time:
proved that $f_0, f_1 : X \to$ concave set are homotopic to each other.

"Fake" proof that S^n is aspherical in dim $k<n$:
Eliminate point from S^n (Peano curve but any map is homotopic to smooth one!)

Also last time <u>homotopy equivalence</u>

$$\begin{tikzcd} X_1 \arrow[r, bend left, "f_1"] & X_2 \arrow[l, bend left, "f_2"] \end{tikzcd} \qquad \begin{matrix} f_1 f_2 \sim id \\ f_2 f_1 \sim id \end{matrix}$$

Retract: $A \subset X$ $\quad r : X \to A$ – retraction, i.e cont map, s.t. $r(a) = a$

r, s.t. $r \circ i = id_A$ wher i is the inclusion

Deformation retract:

$F : X \times [0,1] \to X$ – def. retraction onto A if $\forall x \in X$ and $a \in A$ $\quad F(x,0) = x$, $F(x,1) \in A$ and $F(a,1) = a$

Stong def. retraction leaves all the points in A in place through homotopy

<u>Example</u> S^n, $\mathbb{R}^{n+1} \setminus \{0\}$ – strong def. retract.

$$F(x,t) = \left((1-t) + \frac{t}{\|x\|} \right) x$$

Maps from S^1 to Y, simplest way to understand properties of Y. $\quad \{S^1, Y\}$

In order to costruct a group out of $\{S^1, Y\}$, let us fix a point $* \in S^1$, $y_0 \in Y$

$f: S^1 \to Y$ The set of such homotopy classes

$* \to y_0$ is called $\pi_1(Y, y_0)$

Prove: (assume Y is connected)

i) $\pi_1(Y, y_0)$ is a group with a natural group multiplication

ii) $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$ not canonical, defined up to a path from y_0 and y_1

iii) $\pi_1(Y, y_0)$ – topological invariant of Y (for $Y \simeq Y'$ $\pi_1(Y, y_0) \cong \pi_1(Y', y_0'))$

It is useful to take as S^1, $[0,1]$ s.t. $0,1$ are mapped to one point.

$\Omega(X, x_0)$ - set of continuous maps $(S^1, *) \to (X, x_0)$

$\pi_1(X, x_0)$ - connected components of this space

Properties: 1) If $X \simeq X' \Rightarrow \pi_1(X) \cong \pi_1(X')$

2) Group structure: $\varphi, \psi : [0,1] \to X$

$0, 1 \mapsto x_0$

$\chi = \varphi\psi$ $\chi(t) = \varphi(2t)$ $0 \le t \le \frac{1}{2}$

$\chi(t) = \psi(2t-1)$ $\frac{1}{2} \le t \le 1$

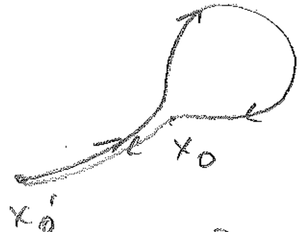
1   $S^1 \to x_0$

inverse - loop in opp. direction

<u>Theorem</u> $\pi_1(S^1) \cong \mathbb{Z}$

follows from $\{S^1, S^1\} \leftrightarrow \mathbb{Z}$

Change of point:



<u>Exercise</u> Prove that in $\mathbb{R}^2 \setminus \{(a,b),(c,d)\}$ is not commutative homotopy group

<u>Higher homotopy groups</u>

$\{A, X\}$ $\quad$ $a_0 \in A,\ x_0 \in X$ fixed

$B \to X$ - fixed

$\Pi(A, X)$ - classes $a_0 \to x_0$

$\Pi(S^n, X) = \pi_n(X)$ - n-th homotopy group

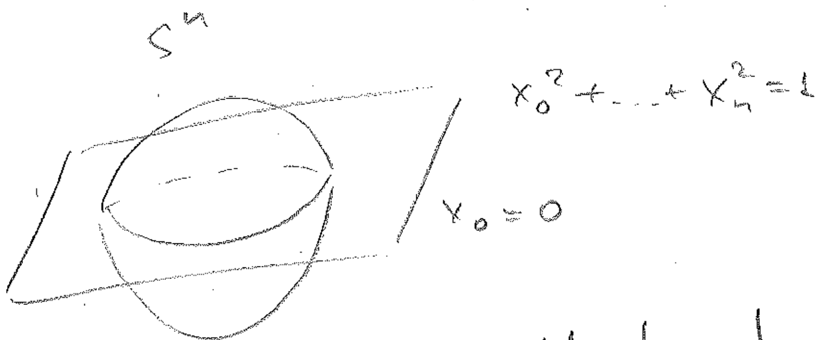
Group structure on $\pi_n(X)$

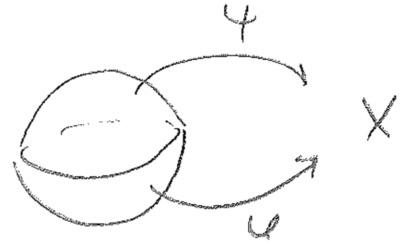
<u>Ex.</u> D^n - n-dim disk with boundary S^{n-1}

Then $D^n / S^{n-1} \cong S^n$

<u>Prop.</u> Space of maps $(S^n, a_0) \to (X, x_0)$ for any topological space $X \leftrightarrow$ space of maps $D^n \to X$ s.t. $S^{n-1} = \partial D^n \to x_0$

Consider $\varphi, \psi : S^n \to X$. Let's construct a composition

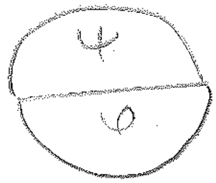


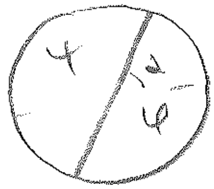


<u>Ex.</u> Show that it is a group

<u>Theorem</u> $\pi_n(X)$ $n>1$ is commutative

<u>Proof</u>





The same argument applies to $x_0 \to x_0'$
$\pi_n(S^n)=\mathbb{Z}$. We will discuss that later

<u>Applications</u>

<u>Theorem</u> $\forall$ nonconst. polynomial with coeff. in $\mathbb{C}$ has a root in $\mathbb{C}$.

<u>Proof</u> $p(z)=z^n+a_1 z^{n-1}+\ldots+a_n$

If $p(z)$ has no roots in $\mathbb{C}$ $\Rightarrow$ $\forall r\geq 0$

$$f_r(s)=\frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$ defines a loop in $\mathbb{C}$ based at 1.

f_0 - trivial loop $\Rightarrow$ $[f_r]\in\pi_1(S^1)$ is trivial

r - large, $r>|a_1|+\ldots+|a_n|$, $r>1$

If $|z|=r$ $\quad |z|^n>(|a_1|+\ldots+|a_n|)\,|z^{n-1}|>$

$> |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|$

Since $|z^n| > |a_1 z^{n-1} + \dots + a_n| \Rightarrow p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$

has no roots on $|z| = r$ when $0 \leq t \leq 1$

Replace $p \to p_t$ in the f_r expression, t goes from 0 to 1 $f_r \to \omega_n(s) = e^{2\pi i n s}$

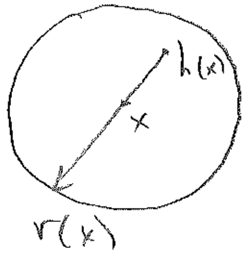
Therefore $n = 0$

Brower fixed point theorem D^2

$\forall$ continuous map $h : D^2 \to D^2$ has a fixed point, i.e. $x \in D^2$ $h(x) = x$

Proof Suppose $h(x) \neq x$ $\forall x \in D^2$

Then we can define a map $r : D^2 \to S^1$



r is continuous. $r(x) = x$ if $x \in S^1$

Therefore r is a retraction of D^2 to S^1

Let's show that $\nexists$

f_0 - any loop in S^1. In D^2 there is a homotopy of f_0 to a const. loop, e.g. the linear homotopy $f_t(s) = (1-t) f_0(s) + t x_0$, where x_0 is the basepoint of f_0.

Since r is identity on S^1 $r f_t$ is then a homotopy in S^1 from $r f_0 = f_0$ to the const loop at x_0, but this contradicts the fact that $\pi_1(S^1)$ is nonzero.