Lecture III Van Kampen Theorem

$X = \bigcup_\alpha A_\alpha$ ← path connected open, each contains the basepoint

$j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ - homomorphism

This extends to $\Phi : \underbrace{*_\alpha}_{\text{free product}} \pi_1(A_\alpha) \to \pi_1(X)$

Let $*_\alpha G_\alpha = \{g_1 \dots g_m\}$ - finite length $\mathbb{Z}_2 * \mathbb{Z}_2$ - good example

Φ is often surjective, expect nontrivial kernel

$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ $\quad j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha} \Rightarrow$

$\Rightarrow$ kernel contains all elements $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ $\forall \omega \in \pi_1(A_\alpha \cap A_\beta)$

<u>Theorem</u> If $X = \bigcup A_\alpha$, each containing $x_0 \in X$ and if $A_\alpha \cap A_\beta$ is path connected $\Rightarrow \Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If, in addition $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected $\Rightarrow \ker \Phi$ is the normal subgroup N generated by $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$

$\Rightarrow \pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$

<u>Examples</u> Wedge sums $\bigvee_\alpha X_\alpha = \bigsqcup_\alpha X_\alpha /$ all basepoints identified

If each of $x_\alpha \in X_\alpha$, x_α - deformation retract of U_α - open neighborhood in X_α

Then X_α is a def. retract of $A_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} U_\beta$

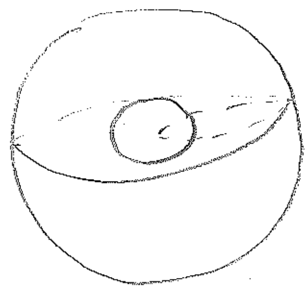
$\Phi : *_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ - isom.

For example $\pi_1(\bigvee_\alpha S^1_\alpha)$ - free group $\mathbb{Z} * \dots * \mathbb{Z}$

Ex. Show that the fundamental group of any connected graph is free.

Ex. 1) Linking of circles

$\mathbb{R}^3 \setminus$ single circle



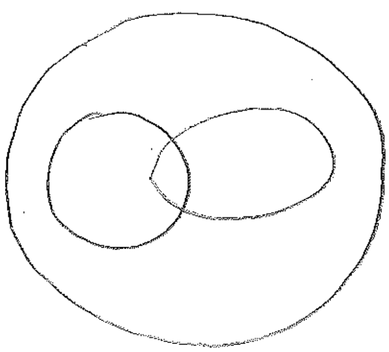
$S^1 \vee S^2 \Rightarrow \pi_1(\mathbb{R}^3 \setminus A) \cong \mathbb{Z}$

since $\pi_1(S^2) = 0$

2) $\mathbb{R}^3 \setminus (A \cup B)$ unlinked circles

$S^1 \vee S^1 \vee S^2 \vee S^2$ $\quad \mathbb{Z} * \mathbb{Z}$

3)



$\mathbb{T}^2 \vee S^2$

$\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$

More van Kampen: Genus g surface Σ_g

$U_0 = \Sigma_g \setminus \{p\}$. U_p - disk neighb. of p

$\Sigma_g = U_0 \cup U_p$

$U_0 \cap U_p \sim S^1$ $\quad S^1 \to U_p$ - trivial

while $S^1 \to U_0$ $\quad 1 \in \mathbb{Z} \to a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$

$\pi_1(\Sigma_g) = ?$

Def. Given groups G, H, K and homomorphisms $K \begin{smallmatrix} \xrightarrow{\kappa} G \\ \xrightarrow[e]{} H \end{smallmatrix}$ a pushout consists of a group P, homomorphis $i : G \to P$, $j : H \to P$

$$\begin{tikzcd} & G \arrow[dr, "i"] \arrow[drr, bend left, "\phi"] & & \\ K \arrow[ur, "\kappa"] \arrow[dr, "e"'] & & P \arrow[r, "\exists!\theta"] & Z \\ & H \arrow[ur, "j"'] \arrow[urr, bend right, "\psi"'] & & \end{tikzcd}$$

$$\boxed{\begin{tikzcd} & \pi_1(U) \arrow[dr, "i_*"] & \\ \pi_1(U \cap V) \arrow[ur, "\kappa_*"] \arrow[dr, "e_*"'] & & \pi_1(X) \\ & \pi_1(V) \arrow[ur, "j_*"'] & \end{tikzcd}}$$

$i\kappa = je : K \to P$ and for any group Z and any homomorphism $\phi : G \to Z$, $\psi : H \to Z$ s.t. $\phi\kappa = \psi e$ $\exists! \theta$ -homomorphism $P \to Z$ s.t. diagram commute

Theorem If $K \begin{smallmatrix} \xrightarrow{\kappa} G \searrow^{i} \\ \xrightarrow[e]{} H \nearrow_{j} \end{smallmatrix} P$ and $K \begin{smallmatrix} \xrightarrow{\kappa} G \searrow^{i'} \\ \xrightarrow[e]{} H \nearrow_{j'} \end{smallmatrix} P'$ both pushout diagrams, then $\exists!$ isomorphism $\theta : P \to P'$ s.t. $\theta i = i'$, $\theta j = j'$

Proof. exercise.

Pushouts in terms of free products

$$\begin{tikzcd} & G \arrow[dr, "i"] \arrow[drr, bend left, "\phi"] & & \\ K \arrow[ur, "\kappa"] \arrow[dr, "e"'] & & \frac{G*H}{N} \arrow[r, "\exists!\theta"] & Z \\ & H \arrow[ur, "j"'] \arrow[urr, bend right, "\psi"'] & & \end{tikzcd}$$

$\phi\kappa = \psi e$

$P = \frac{G*H}{N}$

N-gen. by $\{\kappa(x)(e(x))^{-1} : x \in K\}$