$8\frac{1}{2}$

Summary:

Connection A on E:

1) $d_A : \Omega^p(E) \to \Omega^{p+1}(E)$

$$d_A s|_u = d s|_u + \omega_u \wedge s|_u$$

$$\{\omega^\alpha_{i\beta} dx^i\} = \omega_u \in \Omega^1_u(\mathrm{End}(E))$$

2) $\nabla_X : T^{r,s}(E) \to T^{r,s}(E)$

$X \in \Gamma(TM)$

On $\Gamma(E)$ $\quad \nabla_X s = d_A s(X)$

3) $\omega_u = g_{uv}\, d g_{uv}^{-1} + g_{uv}\, \omega_v\, g_{uv}^{-1}$

Gauge transf: $h_u^{-1} \tilde{g}_{uv} h_v = g_{uv}$

equivalent bundle

$$\tilde{\omega}_v = h_v d h_v^{-1} + h_v \omega_v h_v^{-1}$$

Parallel transport along the curve:

$$\frac{d z^\alpha(x)}{dt} + \sum_{i,\beta} \omega^\alpha_{i\beta}(x) \frac{dx^i}{dt} z^\beta(x) = 0$$

$\nabla_{\dot\gamma} z = 0$ - equivalent definition

$$z|_{\gamma(t_2)} = \mathcal{U}(t_2, t_1)\, z|_{\gamma(t_1)}$$

$$\mathcal{U} = P\exp \int_{t_1}^{t_2} \omega_i \frac{dx^i}{dt}\, dt$$

viewed as a matrix

$d_A^2 \overset{?}{=} 0 \qquad d_A \circ d_A z = d_A(dz + \omega \wedge z)$

$= d(dz) + d\omega \wedge z - \omega \wedge dz + \omega \wedge dz + \omega \wedge \omega \wedge z = \Omega \wedge z$

Ω - curvature.

$$\Omega = d\omega + \omega\wedge\omega$$

$$\Omega_u = g_{uv}\,\Omega_v\, g_{uv}^{-1} \Rightarrow \Omega \in \Omega^2(\mathrm{End}(E))$$

<u>Bianchi identity:</u>

$$d\Omega + \underset{\omega\wedge\Omega - \Omega\wedge\omega}{\underset{\shortparallel}{[\omega, \Omega]}} = 0$$

<u>Another way to define curvature:</u>

$$\Omega(X,Y)s = \nabla_X^A(\nabla_Y^A s) - \nabla_Y^A(\nabla_X^A s) - \nabla^A_{[X,Y]} s$$

<u>Principal bundles</u>

- P - total space
- $P\times G \to P$ - right action (free)

  $(p,g) \to pg = R_g p$
- $M = P/G$ - manifold $\quad \pi: P \to M$ surjection

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi_u"] \arrow[dr, "\pi"'] & U\times G \arrow[d, "pr_1"] \\ & U \end{tikzcd} \Bigg\}\ \text{local triviality}$$

ψ_u - G-equivariant: $\psi_u(p) = (\pi(p), g_u(p))$

$g_u(pg) = g_u(p)g$

$\psi_u(p) = (m, g_u(p))$, $\psi_v(p) = (m, g_v(p))$ $\big|\, g$

$$\psi_u \circ \psi_v^{-1}(m, g_v(p)) = (m, \underset{g_u(p)}{\underset{\shortparallel}{g_{uv}(p)g_v(p)}})$$

<u>Def</u> Connection on P:

$\{A|_u \in \Omega^1(U, \mathfrak{g})\}$

$$A_u = g_{uv}\, d g_{uv}^{-1} + g_{uv} A_v g_{uv}^{-1}$$

on $U\cap V$

<u>Other definitions:</u>

① a) $T_pP = G_p + Q_p$ — Q_p: subspace smoothly depending on p; G_p: space tangent to fiber

b) $Q_{pg} = (R_g)_* Q_p$

② $\omega \in \Omega^1(P, \mathfrak{g})$: a) $\omega(X_\xi) = \xi$ — X_ξ: generator of $\xi \in \mathfrak{g}^*$

b) $R_g^*\omega = Ad_{g^{-1}}\omega$

<u>Relation:</u> ① $Q_p = \mathrm{Ker}\,\omega_p$

② $A_u = \sigma_u^*\omega$

$$\omega_{\pi^{-1}(U)} = Ad_{g_u^{-1}} \circ \pi^* A_u + g_u^*(\theta)$$

$$\theta = g^{-1}dg = (L_{g^{-1}})_*$$

Connections on a tangent bundle and Riemannian geometry

$\pi: TM \to M$

<u>Def</u> (pseudo) Riemannian metric:

$x \in M \longrightarrow g_x$ - inner product (nondegenerate, smooth symmetric bilinear map)

signature: (p, q) — negative, positive eigenvalues of quadr. form.

Smooth section of $\underline{T^*M \otimes T^*M}$

$\{g_{ij}(x)\}$ $\langle v, w\rangle = \sum_{ij} g_{ij} v^i w^j$ - function.

$$\Delta s^2 = \sum_{ij} g_{ij}(x) \Delta x^i \Delta x^j$$

$$\frac{ds}{dt} = \sqrt{g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt}} = |\gamma'(t)|$$

$$\text{length}(t_1, t_2) = \int_a^b |\gamma'(t)|\, dt$$

Tensors: g^{ij}, g_{ij} - raising and lowering indices.

$\langle \omega, \rho \rangle$ - pairing on k-forms

$$\omega \wedge * \rho = (\omega, \rho) \cdot \text{volume form}$$

$\sqrt{g}\, dx^1 \dots dx^n$ - volume form.

$g = \det g$

$$F_{j_1 \dots j_{n-k}} = \frac{1}{k!} \sqrt{g}\, \epsilon_{j_1 \dots j_{n-k} i_1 \dots i_k} F^{i_1 \dots i_k}$$

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

1) $* * \omega = (-1)^{k(n-k)} \omega$

2) $\langle * \omega, * \varphi \rangle = \langle \omega, \varphi \rangle$

On compact manifold:

$$\langle \omega, \omega' \rangle = \int_M \omega \wedge * \omega'$$

$$\langle \delta \omega, \omega' \rangle = \langle \omega, d \omega' \rangle$$

divergence $= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, \alpha^{i i_1 \dots i_{k-1}})$

$$\delta = (-1)^{nk+n+1} * d * : \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\Delta = (d + \delta)^2 = d\delta + \delta d$$

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j} f \right)$$

<u>Hodge's theorem</u>: Every closed form is cohomologous a unique harmonic form.

Levi-Civita connection

$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

Thm: There is a unique connection:

1) Torsion free:

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

2) $\nabla_X g = 0$ or

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Cristoffel symbols: symmetric bc 1)

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$$

$$\Rightarrow \partial_i g_{jk} = g_{rk} \Gamma^r_{ji} + g_{rj} \Gamma^r_{ik}$$

$$2 \sum_e \Gamma^r_{ij} g_{rk} = \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}$$

$$\Downarrow$$

$$\Gamma^r_{ij} = \frac{1}{2} \sum_k g^{rk} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

Other useful formulas:

$$\frac{d}{dt} \det A(t) = (\det A(t)) \, \mathrm{tr}\left(A(t)^{-1} \frac{dA}{dt}\right)$$

Thus: $dg = g \, g^{ij} dg_{ij}$

Notice: $g^{ki} dg_{is} = - dg^{ki} g_{is}$