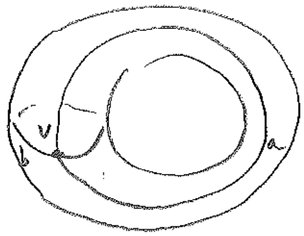
IV CW-complexes

Example Torus



$X^0 = \{v\}$ 0-skeleton

$X^1 = X^0 \cup a \cup b$ 1-skeleton

$X^2 = X$ - 2-skeleton

Observe: each component $X^n \setminus X^{n-1}$ is homeomorphic to $\operatorname{Int} D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

Components of $X^n \setminus X^{n-1}$ are the n-cells

Ex $X = S^2$ $\quad r \in S^2$, set $X^0 = \{v\}$

set $X^1 = X^0$

set $X^2 = S^2 = X$

Observation X is the disjoint union of all cells

$\{v\}$ 0-cell $\quad \{a\}, \{b\}$ 1-cells $\quad X \setminus \{a\}, \{b\}, \{v\}$ 1-2 cells

Topology:

Def Y, Z - spaces, $A \subset Z$, let $\phi : A \to Y$ be a map (cont.)

$Y \cup_\phi Z$ is the quot. of $Y \sqcup Z$ by the eq. rel. $a \sim \phi(a)$

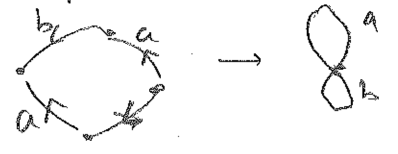
Ex. Attaching cell: $Z = D^n$, $A = S^{n-1}$, $\phi : S^{n-1} \to Y$

$Y \cup_\phi D^n$ is the space obtained from Y by attaching an n-cell with attaching map $\phi : S^{n-1} \to Y$

Ex. Torus $X^0 = \{v\}$ $\quad X^1 = X^0 \cup_{\phi_a^1 \sqcup \phi_b^1} D_a^1 \sqcup D_b^1$

$X^2 = X^1 \cup_\phi D^2$ attaching map.

$\phi : S^1 \to X^1$



Notation: $Y \cup_\phi e^n$ instead of $Y \cup_\phi D^n$

CW-complex:

- X^0 = discr. space (discr. top) [0-skeleton]
- $X^n = X^{n-1} \sqcup_{\phi_\alpha^n} \bigcup \bigsqcup_{\alpha \in A} D_\alpha^n \qquad \phi_\alpha^n : S_\alpha^{n-1} \to X^{n-1}$

$$e_\alpha^n = \operatorname{Int} D_\alpha^n$$

- $X = \bigcup_{n \geq 0} X^n$ (nested) weak top:

  C is closed in X if $C \cap X^n$ closed in X^n $\forall n$

Characteristic map:

$$\begin{tikzcd} D^n \arrow[r, "\text{char. map}"] & X^n \\ S^{n-1} \arrow[u, phantom, "\cup"] \arrow[r, "\text{attach map}"] & X^{n-1} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

on the interior homeomorphism

$$\begin{tikzcd} X^{n-1} \sqcup \bigsqcup D_\alpha^n \arrow[d, "\text{quot via } \phi_\alpha^n"] & D_\alpha^n \cong D^n \arrow[l] \arrow[dl, "\Phi_\alpha^n"] \\ X^{n-1} \cup_{\phi_\alpha^n} \bigsqcup_{\alpha \in A} D_\alpha^n = X^n \end{tikzcd}$$

Ex. (1) $S^n = e^0 \cup_\phi e^n$

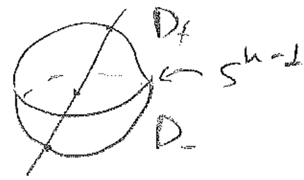
(2) $S^m \times S^n = e^0 \cup e^m \cup e^n \cup_\phi e^{m+n}$

(3) $\mathbb{RP}^n = \{1\text{-dim vect. spaces of } \mathbb{R}^{n+1}\}$

$$\text{"} \; S^n / \{u, -u\} \quad \forall u \in S^n$$

$$\cong D^n / (t \sim -t \text{ for all } t \in S^{n-1})$$

$$\cong \mathbb{RP}^{n-1} \cup_{\phi_n} e^n \qquad \phi_n : S^{n-1} \to \mathbb{RP}^{n-1} = S^{n-1}/(t \sim -t)$$



$$\mathbb{RP}^n \cong e^0 \cup e^1 \cup \ldots \cup_{\phi_{n-1}} e^{n-1} \cup_{\phi_n} e^n$$

Similarly $\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup e^{2n}$ $(\mathbb{C}^n \cong \mathbb{R}^{2n})$

$$\cong e^0 \cup e^2 \cup \ldots \cup e^{2n}$$

A <u>subcomplex</u> of X is $A \subseteq X$ (closed set), such that it is a union of cells of X.

An n-cell of X is a comp of $X^n \setminus X^{n-1}$

<u>Properties of CW complexes</u>

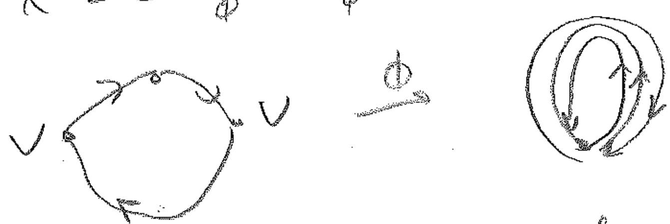
(1) If A is a compact subset of CW complex X, then $A \subseteq$ finite subcomplex of X (App. of Hatcher)

(2) CW complex are Hausdorff

(3) To determine the homotopic type of CW complex we only need attaching maps up to homotopy

<u>Ex</u>. Dunce cap:  is contractible

$X = e^0 \cup_\phi e^1 \cup_\phi e^2 \qquad X = S^1 \cup_\phi D^2$

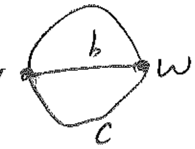


- once in positive direction

<u>Def</u>. A is a subspace of X. $X/A = X / (a_1 \sim a_2 \ \forall a_1, a_2 \in A)$

<u>Ex</u>. $I / \{0,1\} \cong S^1$

<u>Theorem</u> If A is a subcomplex of CW complex X and A is contractible, then quotient map $q : X \to X/A$ is a homotopy equiv.

<u>Ex</u>. $X =$ "theta-graph" 

$A = \{v, w\} \cup a \qquad X \simeq X/A$

 $\cong S^1 \vee S^1$

Theorem Homotopical sequence of (X, A) exact

Proof Check all six homomorphism combinations
Check 3, all others are an exercise

Ex. Let $(X, A) \to (Y, B)$ induce isomorphism of $\pi_i(X) \to \pi_i(Y)$ and $\pi_i(A) \to \pi_i(B)$
Show that this map induces isomorphism all relative homotopy groups. (use 5-lemma)

Ex. $\underset{\text{cone}}{C}X$, X

More on cell complexes

Borsuk property X - topological space, $A \subset X$

For $f : X \to Y$ we denote $f_A : A \to Y$ (the restriction)

Let $F_A : A \times I \to Y$ be the homotopy s.t. $F_A(a,0) = f_A(a)$

If for any Y and any map $f : X \to Y$ any homotopy F_A can be continued to the homotopy $F : X \to Y$ of f, then (X, A) is called Borsuk's pair.

Lemma If X - cell complex, $A \subset X$ - cell subcomplex
Then (X, A) - Borsuk's pair

Theorem about cell approximation:
Any map of the pair of cell complexes is homotopic to a cell map, i.e. $f(sk_n X) \subset sk_n Y$.

Corollary For any cell comlex X

$$\pi_i(X) \cong \pi_i(sk_{i+1}X)$$

Theorem Let (X, Y) be the cell pair, i.e. $Y \subset X$ and is a cell subcomplex. Assume that Y is contractible. Then $X/Y \sim X$

Proof Let $p: X \to X/Y$ be a projection we need to construct $X/Y \to X$, which is homotopically inverse. Use Borsuk's theorem let $f: X \to X$ identity map, f_Y is a restriction There is a homotopy between f_Y and a zero map. By Borsuk's theorem you can continue to f.

Therefore $\exists F: X \times I \to X$

$$F: X \times 0 \xrightarrow{id} X$$

$$F: X \times 1 \to X \quad \sim \quad F: X/Y \to X.$$

Theorem Let X be k-connected (i.e. $\pi_i(X) = 0$ $i = 1, \dots, k$)

Then $X \sim X'$, where there is only one 0-dim cell and no cells of dim $1, 2, 3 \dots k$.

Fundamental groups of CW complexes

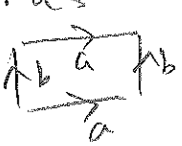
<u>Theorem</u> Let X be a CW complex with just one 0-cell vertex e^0, 1-cells e^1_α $(\alpha \in A)$, 2-cells e^2_β $(\beta \in B)$ and perhaps cells of other dim.

Then 1) $\pi_1(X^1, x_0)$ is a free group with free basis $\{[e^1_\alpha] : \alpha \in A\}$ where $[e^1_\alpha]$ is represented by char. map $\Phi^1_\alpha : D^1 \to X^1$ for e^1_α

2) $\pi_1(X, x_0) \cong \pi_1(X^1, x_0)/N$ where N is the smallest normal subgroup containing $\{\phi^2_{\beta *}[\gamma] : \beta \in B\}$ where $\phi^2_\beta : S^1 \to X^1$ is attaching map for 2-cell e^2_β & γ generates $\pi_1(S^1, s_0)$

If more than 1 vertex X^1 is a connected graph which one can contract.

<u>Ex.</u> Torus $\langle a,b \mid aba^{-1}b^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}$



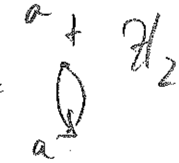
$aba^{-1}b^{-1}$

Ex. Proj. space

$\gamma =$

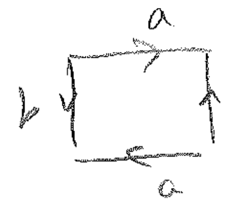
$\gamma^1 =$

$\gamma^1/T =$



$\mathbb{Z}_2$

<u>Ex.</u> Klein bottle



$\langle a, b \mid abab^{-1} \rangle$ $\mathbb{Z} * \mathbb{Z}_2$

$(ab)^2 = b^2$

$abab \neq b^2$

$ab\,a^{-1}b\,b^{-1}a\,ab$ $\quad b^{-1}a\,ab = b^{-1}a\,ab\,a^{-1}b\,b^{-1}a = b^{-1}a\,b^{-1}a$ $\quad (ab)^2 = (b^{-1}a)^2$

$ab \quad b^{-1}a \quad b^{-1}a^{-1}b$

Fundamental group of the orientable surface of genus g (20

0-cell $2g$-1-cells , 1-2cell

1-skeleton $\bigvee_{2g} S^1$ $\pi_1(M_g) \cong \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1] \dots [a_g,b_g] \rangle$

<u>Corollary</u> $M_h \not\cong M_g$

Nonorientable surfaces similarly

<u>Corollary</u> For every group G $\exists$ 2-dim cell complex X_G with $\pi_1(X_G) \cong G$