Classical Mechanics

Newton's Law:

$$m \frac{d^2\vec{x}}{dt^2} = \vec{F}(\vec{x},t)$$

conservative force: $\vec{F} = -\nabla V$

$$E = \frac{1}{2} m|\vec{v}|^2 + V(\vec{x}) \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

Kinetic — potential energy

$$\frac{dE}{dt} = 0\,!$$

Coordinates $q^1, \dots, q^n$ on M

$\gamma(t) = (q^1(t), \dots, q^n(t))$ $\quad \gamma : \mathbb{R} \to M$

classical trajectory

Principle of least action.

Def. $\mathcal{L} : TM \times \mathbb{R} \to \mathbb{R}$

Lagrangian smooth function.

Parametrized smooth paths:

$$P(M)^{q_1,t_1}_{q_0,t_0} = \{\gamma : [t_0,t_1] \to M : \gamma(t_0) = q_0,\ \gamma(t_1) = q_1\}$$

Infinite-dimensional Frechet manifold

$T_\gamma P(M)$ - smooth vect. fields along γ in M which vanish at endpoints

smooth path Γ in $P(M)$ passing through $\gamma \in P(M)$ is called a variation with fixed ends on $\gamma(t)$ in M

$\gamma_\epsilon(t) = \Gamma(t,\epsilon)$ $\quad \Gamma : [t_0,t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$

$\Gamma(t,0) = \gamma(t)$ $\quad t_0 \le t \le t_1$

$\Gamma(t_0,\varepsilon) = q_0$ $\quad \Gamma(t_1,\varepsilon) = q_1$

$-\varepsilon_1 \le \varepsilon \le \varepsilon_1$ $\qquad -\varepsilon_0 \le \varepsilon \le \varepsilon_0$

tang. vect

$$\delta\gamma = \frac{\partial \Gamma}{\partial \varepsilon}\Big|_{\varepsilon=0} \in T_\gamma P(M)$$

((infinitesimal variation

$$\delta\gamma(t) = \Gamma_*\left(\frac{\partial}{\partial \varepsilon}\right)(t,0) \in T_{\gamma(t)} M$$

Also: $\gamma : [t_0, t_1] \to M$

Tangential lift $\gamma' : [t_0,t_1] \to TM$ defined so that $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)} M$

Def. $S : P(M) \to \mathbb{R}$ Action functional

$$S(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(\gamma'(t), t)\, dt$$

Principle of least action:

γ describes the motion of Lagrangian system iff $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon) = 0$

it is a crit pt.

Coords. on T^*M $(q^1, \dots, q^n, v^1, \dots, v^n)$

Tangential lift: $(\vec{q}(t), \dot{\vec{q}}(t)) =$
of a path $\gamma(t)$

$$= (q^1(t), \dots, q^n(t), \dot{q}^1(t) \dots \dot{q}^n(t))$$

$$L(\gamma'(t), t) = L(\vec{q}(t), \dot{\vec{q}}(t), t)$$

Equations of motion:

Thm Equations of motion for (M, L) given by $\dfrac{\partial L}{\partial \vec{q}} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\vec{q}}} = 0$

Proof: $0 = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon) =$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{t_0}^{t_1} L(\vec{q}(t), \dot{\vec{q}}(t), t)\, dt =$$

$$= \int_{t_0}^{t_1} \left(\left\langle \frac{\partial L}{\partial \vec{q}}, \delta \vec{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\vec{q}}}, \delta \dot{\vec{q}} \right\rangle \right) dt =$$

by parts

$$= \int_{t_0}^{t_1} \left\langle \left(\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} \right), \delta \vec{q} \right\rangle + \underbrace{\left\langle \frac{\partial L}{\partial \dot{q}^i}, \delta q \right\rangle \Big|_{t_0}^{t_1}}_{=0} \Rightarrow$$

statement.

For our simple system

$L = T - V$ kinetic - potential energy.

Ex. Particle in EM field.

$$L = \frac{m \dot{\vec{r}}^2}{2} + e\left(\frac{\dot{\vec{r}} \cdot \vec{A}}{c} - \varphi\right)$$

$$m\ddot{\vec{r}} = e\left(\vec{E} + \frac{\dot{\vec{r}}}{c} \times \vec{B}\right)$$

$$\vec{E} = -\frac{\partial \varphi}{\partial \vec{r}} \qquad \vec{B} = \operatorname{curl} \vec{A}$$

Relativistic version: $L dt = -mc\, ds - \frac{e}{c} A_\mu dx^\mu$

$$\frac{d^2}{d\tau^2} x^\mu = \frac{e}{mc} F^\mu{}_\nu \frac{dx^\nu}{d\tau} \qquad ds = \sqrt{1 - \frac{v^2}{c^2}}\, c\, dt$$

Ex. Geodesic and Levi-Civita conn.

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$$L(v) = \frac{1}{2}\langle v, v\rangle_g = \frac{1}{2}\|v\|^2$$ Lagrangian of a particle on Riem. manifold; time-like

In GR: $d\tau = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$ - time-like
also length

$\hookrightarrow$ Geodesic equation for these Lagrangians } Exercise

Symmetries: Noether thm

Def. $I : TM \to \mathbb{R}$ – integral of motion if $\frac{d}{dt} I(\gamma'(t)) = 0$ for all extremals of S.

Def. Energy: $E(q,\dot q,t) = \sum_{i=1}^{n} \dot q^i \frac{\partial L}{\partial \dot q^i} - L$

Lemma: E – well-def. function on $TM \times \mathbb{R}$ since $\frac{\partial L}{\partial \dot q^i}$ – comp. of 1-form on M

Prop. $\left.\frac{dE}{dt}\right|_{\gamma = \gamma(t) \text{ - extremal of } S} = 0$ for closed system $\frac{\partial L}{\partial t} = 0$

Def. $L : TM \to \mathbb{R}$ is inv. under $g : M \to M$ if $L(g_*(v)) = L(v)$ $\forall v \in TM$

G – Lie group if $x \in M \to g \cdot x \in M$ is a symmetry

Thm (Noether) Take $\{g_s\}_{s \in \mathbb{R}}$ of diff of M $\Rightarrow$

$$I(q,\dot q) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q^i}(q,\dot q) \left.\left(\frac{d g_s^i(q)}{ds}\right)\right|_{s=0} = \frac{\partial L}{\partial \vec{\dot q}} \vec a$$

where $\vec a = \sum_{i=1}^{n} a^i(\vec q) \frac{\partial}{\partial q^i}$ – generator for $g_s^i(q)$

Proof: $0 = \frac{\partial L}{\partial q} a + \frac{\partial L}{\partial \dot q} \dot a =$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \vec a + \frac{\partial L}{\partial \vec{\dot q}} \frac{d\vec a}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\dot q}} \vec a\right)$$

$\vec a(t) = (\vec a(\gamma(t)) \dots a^n(\gamma(t)))$

Generalization: $dL(X')(\gamma(t)) = \frac{d}{dt} K(\gamma(t))$

X – vector field on M

$$I = \sum_{i=1}^{n} a^i(q) \frac{\partial L}{\partial \dot q^i}(q,\dot q) - K(q,\dot q)$$

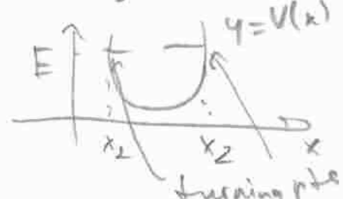
Ex Translation: $g_s(q) = q + sv$

$I = \sum_{i=1}^{n} v^i \frac{\partial L}{\partial \dot q^i}$ – momentum in the direction v.

1-dim Dynamics:

$L = \frac{1}{2} m \dot x^2 - V(x)$ $\quad$ $E = \frac{1}{2} m \dot x^2 + V(x)$ conserved

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x))} \Rightarrow t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}}$$



period $T(E) = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$

If $V(x) < E$ unbounded motion is infinite otherwise periodic.

Ex: Harmonic oscillator: $V = \frac{1}{2} k x^2$ $\quad$ $x(t) = A\cos(\omega t + \alpha)$

$E = \frac{1}{2} m \omega^2 A^2$ $\quad$ $\omega = \sqrt{\frac{k}{m}}$ $\quad$ $T = 2\pi/\omega$