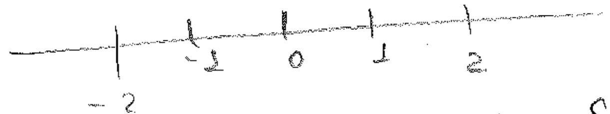
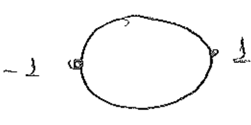
# Lecture V Covering spaces

**Ex.** 1) $p: \mathbb{R} \to S^1 \quad p(x) = e^{2\pi i x}$

**Def.** A continuous map $p: \tilde{X} \to X$ is a covering map if each point $x \in X$ belongs to an open set $U \subset X$ s.t. $p^{-1}(U)$ is a disjoint union $\bigsqcup \tilde{U}_\alpha$ of open sets in $\tilde{X}$ such that $p$ maps each $\tilde{U}_\alpha$ homeomorphically onto $U$.




$U = S^1 \setminus \{-1\}$

$p^{-1}(U) = \mathbb{R} \setminus \mathbb{Z} + \frac{1}{2}$

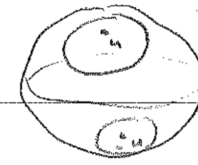
**Def.** A chart for $p: \tilde{X} \to X$ is an open set $U$ in $X$ s.t. $p^{-1}(U) = \bigsqcup \tilde{U}_\alpha$, $\tilde{U}_\alpha$ - open in $\tilde{X}$ and $\forall \alpha$ $p|\tilde{U}_\alpha \to U$ is a homeomorphism.

**Examples** 2) $p_n: S^1 \to S^1 \quad p_n(z) = z^n$

3) $p \times p: \mathbb{R}^2 \to S^1 \times S^1 =$ torus

4) $p \times p_n: \mathbb{R} \times S^1 \to S^1 \times S^1$

5) $p_n \times p_m: S^1 \times S^1 \to S^1 \times S^1$



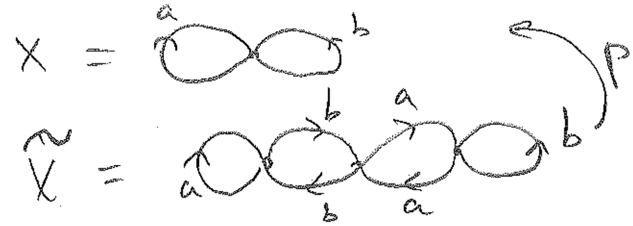
6) $p: S^2 \to \mathbb{RP}^2 = S^2/(u \sim -u)$

$p: S^n \to \mathbb{RP}^n = S^n/\sim$

7) $X =$ 

$S^1 \vee S^1$ $\quad \tilde{X} =$ 

8)




<u>Path-lifting theorem</u>

Let $p: \tilde{X} \to X$ be a covering map, let $u: I \to X$ be continuous, let $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = u(0)$.
Then $\exists!$ continuous $\hat{u}: I \to \tilde{X}$ such that $p\hat{u} = u$, $\hat{u}(0) = \tilde{x}_0$

<u>Homotopy lifting theorem</u>

Let $p: \hat{X} \to X$ be a covering map. Let $H: Y \times I \to X$ be continuous, let $\tilde{g}: Y \to \tilde{X}$ be continuous such that $p\tilde{g} = H_0$
Then $\exists!$ continuous $\hat{H}: Y \times I \to \tilde{X}$ such that $p\hat{H} = H$ (lift) and $\hat{H}_0 = \tilde{g}$

<u>Corollary 1</u> Let $p: \tilde{X} \to X$ be a covering map with $X$ - path-connected. Then $|p^{-1}(x)|$ is independent of $x \in X$

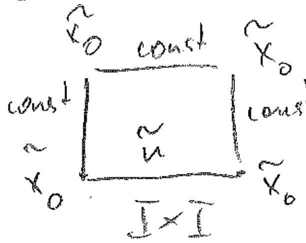
<u>Proof</u> Let $u$ be a path in $X$ from $x$ to $y$
$\beta_u: p^{-1}(x) \to p^{-1}(y)$ is defined by:
$\exists$ unique lift $\hat{u}$ of $u$, starting at $\tilde{x}$. Define
$\beta_u(\tilde{x}) = \tilde{u}(1)$. $\beta_u$ is bijective with inverse $\beta_{\bar{u}}$
from ! of the lift

Corollary 2 Let $p: \tilde{X} \to X$ be a covering map, suppose $p(\tilde{x}_0) = x_0$. Then $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof. Let $\tilde{u}: I \to \tilde{X}$ be a loop based at $x_0$, such that $p_*[\tilde{u}] = 1$. Then $p\tilde{u} \simeq$ const by homotopy $H: I \times I \to X$

$H_0 = p\tilde{u}$, $H_1 =$ const. Homotopy lifting theorem:

$\exists (!)$ homotopy $\tilde{H}: I \times I \to \tilde{X}$ such that $p\tilde{H} = H$ and $\tilde{H}_0 = \tilde{u}$

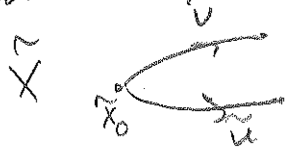



$\xrightarrow{\tilde{H}} \tilde{X} \Rightarrow [\tilde{u}] = 1$ in $\pi_1(\tilde{X}, \tilde{x}_0)$

Corollary 3 Let $p: \tilde{X} \to X$ be a covering map. Let $p(\tilde{x}_0) = x_0$. Let $u, v: I \to X$ be paths from $x_0$ to $x_1$

Let $\tilde{u}, \tilde{v}$ be the (unique) lifts of $u, v$ starting at $\tilde{x}_0$

Then $\tilde{u}(1) = \tilde{v}(1)$ iff $[u\bar{v}] \in p_* \pi_1(\tilde{X}, \tilde{x}_0)$



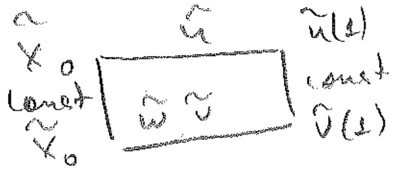


Proof: If $\tilde{u}(1) = \tilde{v}(1) \Rightarrow \tilde{u}\bar{\tilde{v}}$ - loop, easy

Suppose $[u\bar{v}] = p_*[\tilde{w}]$ for some loop $\tilde{w}$ in $\tilde{X}$, based at $\tilde{x}_0$. Set $w = p\tilde{w}$  $u \simeq wv \, (\mathrm{rel}\, \partial I): I \to X$

$H : I \times I \to X$ such that $H_0 = \omega v$, $H_1 = u$

$\exists!$ lift $\hat{H} : I \times I \to \tilde{X}$ s.t. $\hat{H}_0 = \tilde{\omega}\tilde{v}$




$\xrightarrow{\hat{H}} \tilde{X}$ $\qquad \hat{u}(1) = \hat{v}(1)$

<u>Cor. 4</u> Suppose $p : \tilde{X} \to X$ covering map with $\tilde{X}, X$ - path connected and $p(\tilde{x}_0) = x_0$. Then $p^{-1}(x_0) = |\pi_1(X, x_0) : p_* \pi_1(\tilde{X}, \tilde{x}_0)|$
index of subgroup

<u>Proof.</u> Exercise.

Locally path connected: $\forall y \in Y$, $\forall U \in Y$ $\exists$ path conn. $V$ in $Y$ s.t. $y \in V \subset U$

Ex.

<u>General lifting theorem</u>

Let $p : \tilde{X} \to X$ be a covering map, $f : Y \to X$ continuous

pet $p(\tilde{x}_0) = x_0 = f(y_0)$. If $Y$ is path-connected and locally path connected

then $\exists!$ continuous $\hat{f} : Y \to \tilde{X}$ such that

$p\hat{f} = f$, $\hat{f}(y_0) = \tilde{x}_0$ iff $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$

<u>Ex.</u> Every map $f : S^2 \to T^2$ is null-homotopic

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d] \\ S^2, s_0 \arrow[ur, "\exists?\ \hat{f}"] \arrow[r] & T^2, f(s_0) \end{tikzcd}$$

$\pi_1 = \{1\}$ $\qquad \pi_1 = \mathbb{Z}^2$

$\exists \hat{f} : S^2 \to \mathbb{R}^2$ s.t. $p\hat{f} = f$

$\hat{f} \underset{\hat{H}}{\simeq}$ const

$f \underset{p\hat{H}}{\simeq}$ const

<u>Ex.</u> Is every map $T^2 \to S^2$ null homotopic?