Lecture VI Covering spaces (continued)

Theorem Suppose $p_1 : \tilde{X}_1 \to X$, $p_2 : \tilde{X}_2 \to X$ are covering maps, with $\tilde{X}_1, \tilde{X}_2$ connected ~~and locally path~~ Let $\tilde{x}_i \in \tilde{X}_i$ s.t. $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0 \in X$. Then

(1) Coverings p_1, p_2 are isomorphic preserving basepoints iff $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

(2) Coverings p_1, p_2 are isomorphic (forgetting basepoints) iff $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1)$, $p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$ are conj. subgroups of $\pi_1(X, x_0)$

(isomorphic covering spaces $f : \tilde{X}_1 \xrightarrow{\text{homeom.}} \tilde{X}_2$ s.t. $p_1 = p_2 f$)
preserving basepoints $f(\tilde{x}_1) = \tilde{x}_2$

Proof: 1) From general lifting theorem

$$\begin{tikzcd} \tilde{X}_1, \tilde{x}_1 \arrow[rr, shift right, "\exists!\, h_1"'] \arrow[dr, "p_1"'] & & \tilde{X}_2, \tilde{x}_2 \arrow[ll, shift right, "\exists!\, h_2"'] \arrow[dl, "p_2"] \\ & X, x_0 & \end{tikzcd}$$

$p_1 = p_2 h_1$

$p_2 = p_1 h_2$ $\quad h_2(\tilde{x}_2) = \tilde{x}_1$

$h_1(\tilde{x}_1) = \tilde{x}_2$

$h = h_2 h_1 : \tilde{X}_1 \to \tilde{X}_1$

by unique lifting property $h_1 h_2 = h_2 h_1 = id$ since they fix basepoints.

(2) follows from the following lemma:

Lemma Let $p : \tilde{X} \to X$ be a covering map, $p(\tilde{x}_0) = x_0$. Then a subgroup H of $G = \pi_1(X, x_0)$ is conjugate to $H_0 = p_*\pi_1(\tilde{X}, \tilde{x}_0)$ iff $H = p_*\pi_1(\tilde{X}, \tilde{x}_1)$ for some $\tilde{x}_1 \in p^{-1}(x_0)$

prove it!

Def. Let $p:\tilde{X}\to X$ be a covering map.
A deck transformation p is a homeomorphism
$g:\tilde{X}\to\tilde{X}$ s.t. $pg=p$

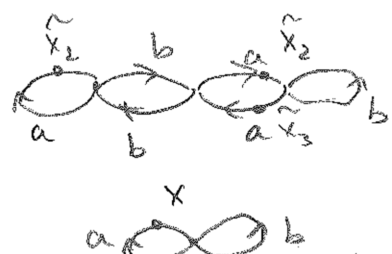
$$\begin{tikzcd} \tilde{X} \arrow[rr, "g"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Ex. The covering transformations form a group.

Def Covering $p:\tilde{X}\to X$ is normal regular if for all $x\in X$ and all $\tilde{x}_1,\tilde{x}_2\in p^{-1}(x)$, $\exists$ covering transformation $g:\tilde{X}\to\tilde{X}$ such that $g(\tilde{x}_1)=\tilde{x}_2$

Ex $p: S^2\to \mathbb{RP}^2=S^2/(u\sim -u)$ $u\in S^2$
$g:S^2\to S^2$ $g(u)=-u$ cov. transf

Ex.



not regular

$\tilde{x}_1$ lies on a loop lifting a
$\hat{x}_2,\hat{x}_3$ on lifts of a with dist. endpoints no transf mapping $\hat{x}_1$ to $\hat{x}_2$ or $\tilde{x}_3$

Lemma Let $p:\tilde{X}\to X$ be a covering map with $X,\tilde{X}$ path connected. Let $p(\tilde{x}_0)=x_0\in X$. Then the following is eq:

1) p is regular
2) $\forall\ \hat{x}_1,\hat{x}_2\in p^{-1}(x_0)$ $\exists$ cov. transf. g s.t. $g(\hat{x}_1)=\hat{x}_2$
3) $\forall\ \hat{x}_1,\hat{x}_2\in p^{-1}(x)$ $\exists!$ —//—

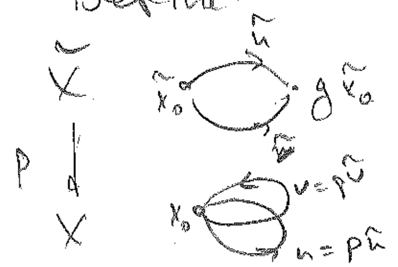
Proof. ex.

<u>Theorem</u> Let $p : \tilde{X} \to X$ be a covering with $\tilde{X}, X$ connected & locally path connected. Let $p(\tilde{x}_0) = x_0$. Then p is regular iff $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$

<u>Proof</u> Suppose $\tilde{x}_1 \in p^{-1}(x_0)$, p-regular iff $\forall \tilde{x}_1 \in p^{-1}(x_0)$ $\exists$ isom g from p to itself s.t. $g(\tilde{x}_0) = \tilde{x}_1$ i.e. iff $p_* \pi_1(\tilde{X}, \tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_1)$ i.e. iff $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is normal in $\pi_1(X, x_0)$

<u>Theorem</u> Let $p : \tilde{X} \to X$ be a regular covering with $\tilde{X}, X$ connected and locally path connected. Then G - group of covering transformations of p is isomorphic to $\pi_1(X, x_0) / p_* \pi_1(\tilde{X}, \tilde{x}_0)$

<u>Proof</u> Define $\theta : G \to \pi_1(X, x_0) / p_* \pi_1(\tilde{X}, \tilde{x}_0)$ by



$g \longmapsto [u] \in \pi_1(X, x_0) / p_* \pi_1(\tilde{X}, \tilde{x}_0)$

where $\tilde{u}$ path in $\tilde{X}$ from $\tilde{x}_0$ to $g\tilde{x}_0$

$u = p\tilde{u}$

Another path $\tilde{v}$ from $\tilde{x}_0$ to $g\tilde{x}_0$

$v = p\tilde{v}$

since $\tilde{u}(1) = \tilde{v}(1)$

$u\bar{v}$ is an element of $p_* \pi_1(\tilde{X}, \tilde{x}_0)$

$[u] = [v]$ in $\pi_1(X, x_0) / p_* \pi_1(\tilde{X}, \tilde{x}_0)$

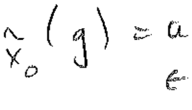
<u>Ex.</u> Prove that θ is 1to1 and that it is a group homomorphism.

Def. Covering $p: \tilde{X} \to X$ is universal if $X, \tilde{X}$ are path connected and $\tilde{X}$ is simply connected

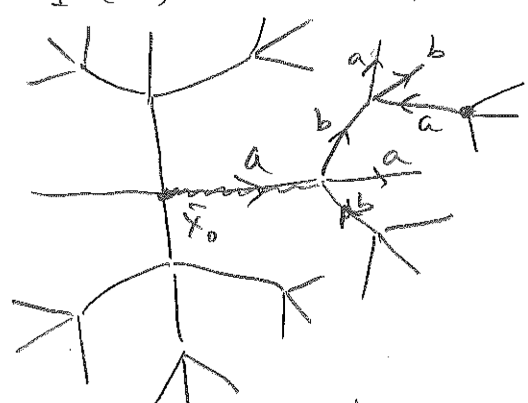
Ex. $p: S^2 \to \mathbb{RP}^2$, $p: \mathbb{R}^2 \to T^2$ universal
$\mathbb{Z}^2$ acts on $\mathbb{R}^2$ as translations

Cor. If $p: \tilde{X} \to X$ is a universal covering then p is regular and group of covering transf. is isomorphic to $\pi_1(X)$.

Ex. Universal covering of $S^1 \vee S^1 = X$



$\pi_1(X, x_0) \cong \langle a, b \rangle$



$\theta_{\tilde{x}_0}(g) = a b a^{-1} \in \langle a, b \rangle$

Def: X - called semilocally simply connected if every point $x \in X$ has an open neighborhood U s.t. $\pi_1(U, x) \to \pi_1(X, x)$ is null is trivial, i.e. every loop in U (based at x) is null homotopic in X

Theorem Suppose X is connected, semi-locally simply connected and $x_0 \in X$. $\forall$ subgroup H of $\pi_1(X, x_0)$ $\exists$ covering map $p: \tilde{X} \to X$, point $\tilde{x}_0 \in p^{-1}(x_0)$ s.t.

$p_* \pi_1(\tilde{X}, \tilde{x}_0) = H \subseteq \pi_1(X, x_0)$

<u>Lemma</u> If X has a universal covering $\Rightarrow$ X - semi-locally simply connected

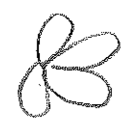
<u>Theorem</u> Any subgroup of a free group is free

If H is a subgroup of F_k with index m $\Rightarrow$ $H \cong F_{km-m+1}$

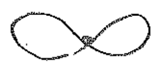
<u>Proof.</u> $F_k = \pi_1(X, x_0)$, $X = \underbrace{S^1 \vee \dots \vee S^1}_{k}$

$H \subseteq F_k$. There exists a covering $p: \hat{X} \to X$

s.t. $p_* \pi_1(\hat{X}, \hat{x}_0) = H \subseteq \pi_1(X, x_0) = F_k$

$\hat{X}$  $\hat{X}/T$ 

max tree

$X =$ 

$\hat{X}$ is a 1-dim CW complex (i.e. graph)

path connected $\pi_1(\hat{X}) = \pi_1(\hat{X}/T)$ is free

If $|F_k : H| = m \Rightarrow \hat{X}$ has m vertices,

T has $m-1$ edges

$\hat{X}/T$: 1 vertex, $km - (m-1)$ edges