## Noether theorem:

Symmetry group action $G \curvearrowright \{\phi\}$

$\phi_i \to \phi_i + \epsilon\,\delta\phi_i$ – infinitesimal action

implies a conserved current $j$

$j \in \Omega^1(M)$, such that $d * j = 0$

Locally: $\begin{cases} j^\mu = \sum_k \delta\phi_k \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)} \\ \partial_\mu j^\mu = 0 \end{cases}$

More generally: $\delta\mathcal{L} = \partial_\mu K^\mu$

$$\Rightarrow \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - K^\mu$$

is also conserved: $\partial_\mu j^\mu = 0$

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Ex. $O(N)$ scalar field:

$$S = -\frac{1}{2}\int d^4x \left(\sum_{i=1}^{N} \partial_\mu\phi^i\partial^\mu\phi^i + m^2|\phi|^2\right)$$

$$j^\mu = \sum_{i,j}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} t_{ij}\phi^j$$

$\{t_{ij}\}$ generators of $so(N)$

Ex. Energy-momentum tensor:

$x^\mu \to x^\mu + a^\mu \qquad K_\nu^\mu = \partial_\mu\mathcal{L}\,\delta^\mu_\nu$

$$j^\mu = T^\mu{}_\nu a^\nu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\nu\phi_i a^\nu - \partial_\mu\mathcal{L}\,a^\mu$$

$$T^\mu{}_\nu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\nu\phi_i - \delta^\mu{}_\nu\mathcal{L}\right)$$

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Another way of thinking about it:

$$S(\phi_i, \underset{\text{metric}}{g}) = \int \mathcal{L}\sqrt{g}\,d^4x$$

$$T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}(x)}$$ – symmetric tensor

$$\nabla^\mu T_{\mu\nu} = 0$$

Diff invariance:

E-L equations

$$\delta S = \int \frac{\delta S}{\delta\phi_i}\delta\phi_i + \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = 0$$

$$\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \Rightarrow \nabla_\mu T^{\mu\nu} = 0$$

Revisiting Noether Theorem:

$\phi \to \phi + \delta\phi.$

Make parameters dep. on $\underline{\underline{x}}$

$$\phi(x) \to \phi(x) + \underbrace{\epsilon(x)\,\delta\phi(x)}_{} = \delta\tilde{\phi}$$

$$\underset{0}{\overset{\shortparallel}{\delta S}} = \int \frac{\partial\mathcal{L}}{\partial\phi}\delta\tilde{\phi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\tilde{\phi} =$$

vanishes due to symmetry

$$= \int \frac{\partial\mathcal{L}}{\partial\phi}\epsilon(x)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\epsilon(x)\partial_\mu\delta\phi$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\epsilon\,\delta\phi \Rightarrow$$

$$\Rightarrow \int \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\,\partial_\mu\epsilon \Rightarrow$$

$$\Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = 0$$

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Hamiltonian formalism:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\underset{\text{time}}{\partial_0}\phi)}$$

$$H = \int d^3x\,(\pi(x)\partial_0\phi(x) - \mathcal{L}(x))$$



Ex. Scalar field:

$$H = \frac{1}{2}\int d^3x\,(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2)$$

$$\dot{\phi}(\vec{x},t) = \frac{\delta H}{\delta\phi(x)} \qquad \dot{\pi}(\vec{x},t) = -\frac{\delta H}{\delta\pi(\vec{x})}$$

Poisson bracket:

$$\{A, B\} = \int d^3z\left[\frac{\delta A}{\delta\phi(z)}\frac{\delta B}{\delta\pi(z)} - \frac{\delta A}{\delta\pi(z)}\frac{\delta B}{\delta\phi(z)}\right]$$

Functional derivatives:

$$\frac{\delta}{\delta\phi(y)}\int d^4x\,\phi^k(x) = k\,\phi^{k-1}(y)$$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)$$

$$\int dy\,\delta(x-y)f(y) = f(x)$$

Ex. $\frac{\delta}{\delta\phi(y)} e^{\int d^4x\,J(x)\phi(x)} = J(y)\,e^{\int d^4x\,J(x)\phi(x)}$

PB: $[\phi(x), \pi(y)] = \delta^{(3)}(x-y)$

$[q_i, p_j] = \delta_{ij}$ – classical mechanics

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EM: $\mathcal{L} = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}$

$$\pi^0 = 0! \qquad \pi^i = \underset{E^i}{\underset{\shortparallel}{F^{i0}}} \qquad \frac{\delta\mathcal{L}}{\partial A_0} = \partial_i\pi^i$$

Chern classes and Chern-Simons

Reducing structure groups

$GL(n,\mathbb{C}) \to U(n) \quad GL(n,\mathbb{R}) \to O(n)$

$\Omega \to \frac{i}{2\pi}\Omega$ - Hermitean

$\tau_n(A) = \mathrm{Trace}\left[\left(\frac{i}{2\pi}\Omega\right)^n\right]$ - real-valued

Prop. $d\tau_n = 0 \Rightarrow [\tau_n(A)] = H^{2n}(M,\mathbb{R})$

Locally: $\tau_n(A) = d\,\omega^{CS}_{2n-1}$

Calculate: $\delta\tau_n(A) = \frac{in}{2\pi} d\left(\mathrm{tr}\,\delta A\left(\frac{i}{2\pi}\Omega\right)^{n-1}\right)$

Now, take $\delta A_t = A\,dt \quad A_t = tA$

$F_t = t\,dA + t^2A^2$

Integrate:

$\omega^{CS}_{2n-1} = n\left(\frac{i}{2\pi}\right)^n \int_0^1 dt\, t^{n-1}\,\mathrm{tr}\left[A(dA + tA^2)^{n-1}\right]$

$\omega^{CS}_3 = \frac{-1}{2\pi^2}\,\mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right)$

Chern-Simons form.

Corollary: $[\tau_n(A)]$ does not depend on the connection

Def $ch(E) = \mathrm{Trace}\left(e^{\frac{i}{2\pi}\Omega}\right) =$

$= \mathrm{rank}\,E + \tau_1(E) + \frac{1}{2!}\tau_2(E) + \ldots$

Prop. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$

$ch(E_1 \otimes E_2) = ch(E_1)\,ch(E_2)$

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Transformation of CS form:

$\omega^{CS}_{2n-1}(A^g) = \omega^{CS}_{2n-1}(A) +$

$+ (-1)^{n-1}\frac{n!(n-1)!}{(2n-1)!}\,\mathrm{tr}(g^{-1}dg)^{2n-1}\frac{i^n}{(2\pi)^n} + d\alpha$

$n = 2 \qquad \frac{2}{3!(2\pi)^2}\,\mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$

$\theta = -\frac{1}{24\pi^2}\,\mathrm{tr}(g^{-1}dg^{\wedge 3})$ - generator of $H^3(G,\mathbb{Z}) \cong \mathbb{Z}$

simple connected, compact simply-conn

$e^{\frac{i}{4\pi^2}\int_M \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right)}$

well-defined for closed 3-manifold