Lecture VIII

Fibrations, fiber bundles

Locally trivial bundles:

Def (E, B, F, p), E, B, F – topological spaces and $p : E \to B$ a map, satisfying the following properties

1) $\forall\, x \in B$ $\exists$ neighborhood U, so that $p^{-1}(U) \cong U \times F$
2) Homeomorphism $U \times F \to p^{-1}(U)$ is compatible with p, i.e.

$$\begin{tikzcd} U \times F \arrow[r] \arrow[dr] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

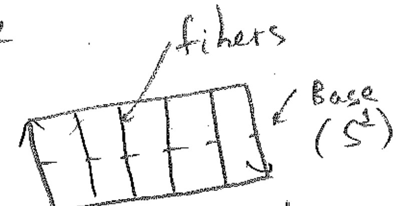
is commutative.

E – total space $\quad$ p: projection

B – base

F – fiber

Examples
1) $B \times F \to B$ – trivial bundle
2) Any covering space
3) Möbius band



4) Klein bottle is a bundle of S^1 over S^1

5) Hopf bundle:

$S^3 \xrightarrow{S^1} S^2$

S^3 – set of vectors of unit length in $\mathbb{C}^2$.

Set of complex lines in $\mathbb{C}^2$, passing through the origin $\mathbb{C}P^1$. $\mathbb{C}P^1 \cong S^2$

Natural map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$
vector $\to$ line

Fiber - intersection of a complex line and S^3, i.e. S^1. Explicitly: $p(z_0, z_1) = \left(2 z_0 z_1^*, |z_0|^2 - |z_1|^2\right)$

 $\underset{z}{\shortparallel}$ $\quad \underset{x}{\shortparallel}$ $\quad |z|^2 + x^2 = 1$

Ex. Show that it is nontrivial

Def $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ are equivalent if and only if $\exists\ \varphi : E_1 \to E_2$ $p_1 = p_2 \varphi$ is homeomorphism

Def. Trivialization of $p : E \to B$ is
$E \to B \times F$, s.t. $e \mapsto (p(e), p_2(e))$, where $p_2(e) \in F$

Homeomorphism of trivialization is not unique

p_1 & p_1' necessarily

Ex. $T^2 = S^1 \times S^1 \to S^1$ $\quad p_1 : S^1 \times S^1 \to S^1$, $(\varphi, \psi) \to \varphi$

$p_1' : S^1 \times S^1 \to S^1$, $(\varphi, \psi) \to (\varphi + \psi)$

Feldbau theorem

$\forall$ locally trivial bundle over D^k is equivalend to the direct product

Proof $D^k \to I^k$ (cube)

1) $I^k = I^{k-1} \times I$ $\quad I_-^k = I^{k-1} \times [0, \frac{1}{2}]$, $I_+^k = I^{k-1} \times [\frac{1}{2}, 1]$

Suppose that there are maps:

$p^{-1}(I_-^k) \overset{\varphi}{\approx} F \times I_-^k$ $\quad$ We need: $p^{-1}(I^k) \approx F \times I^k$

$p^{-1}(I_+^k) \overset{\psi}{\approx} F \times I_+^k$

Let $p^{-1}(I_-^k) \subset p^{-1}(I^k)$ let the coincide on that set

On $p^{-1}(I_+^k)$ we have to construct it in such a way that both parts coincide on $I^{k-1} \times \{\frac{1}{2}\}$

If $a \in I^{k-1} \times \{\frac{1}{2}\} \Rightarrow p^{-1}(a) \to$ two homeomorphisms on F

Therefore we have homeomorphism F to F

Composition of $\psi^{-1}: F \to p^{-1}(a)$ and $\varphi: p^{-1}(a) \to F$

Construction of $p^{-1}(I_+^k) \overset{\tilde{\psi}}{\approx} F \times I_+^k$

One can look at it as a family of homeomorphisms of preimages of sets on F, continuously depending on a point. $\forall$ $pt \in I_+^k$ we take composition of such homeomorphism and another one, which is constructed as follows:

Consider projection $I_+^k \longrightarrow I^{k-1} \times \{\frac{1}{2}\}$. For a given pt on $I^{k-1} \times \{\frac{1}{2}\}$ we have homeomorphism $F \to F$. If we take composition with this homeomorphism, we have "glueing condition"

2) Suppose the locally trivial fiber bundle over I^k is nontrivial. Let us divide it indefinitely (mind). Every point has a neighborhood where it is trivial, therefore we have contradiction ☒

<u>Exercise</u> Is it true that any bundle with fiber D^k is trivial?

<u>Theorem</u> Let $p: E \to B$ - locally trivial fiber bundle (Z, Z') - cell pair. Let $f: Z \to E$, homotopy $\tilde{F}: Z \times I \to B$ of a map pf and homotopy $\Phi': Z' \times I \to E$ of a map $f|_{Z'}$, which is the lift of $\tilde{F}$ on $Z' \times I$ i.e. $p\Phi' \equiv \tilde{F}$ on $Z' \times I$. Then there is a homotopy Φ of a map f, which is a lift of homotopy $\tilde{F}$ and continuation of Φ', i.e. $\Phi \equiv \Phi'$ on $Z' \times I$ and $p\Phi = \tilde{F}$.

$$
\begin{tikzcd}
 & & E \arrow[d, "p"] \\
Z' \subset & Z \arrow[ur, bend left, "\Phi_t"] \arrow[ur, "f"'] \arrow[r, bend right, "\tilde{F}_t"'] & B
\end{tikzcd}
\qquad Z' \xrightarrow{\Phi'_t} E
$$

This is the most general statement of a homotopy lifting theorem

Standard homotopy lifting when $Z' = \emptyset$.

Proof (sketch) 1) Assume $E = B \times F$, p - projection on B $\Rightarrow$ map $Z \to E$ can be considered as a pair of maps $Z \to B$, $Z \to F$.

Homotopy in B is given (this is F_t)

Need to construct the continuation of the homotopy in F. This is the consequence of Borsuk's theorem.

(Borsuk's theorem - ability to continue homotopy from CW-subcomplex to the whole CW-complex)

2) Let now E be any bundle, but $Z = D^k$, a closed disk.

We have $\hat{F} : D^k \times I \to B$. Using this map one can construct bundle over $D^k \times I \simeq D^{k+1}$

This will be a trivial bundle.

By Feldbau's theorem for this induced bundle $E' \to B$ the continuation of homotopy exists because of 1). Composing it with the map $E' \to E$ we obtain the continuation of homotopy

3) Z - CW complex, $Z' \subset Z$ is CW subcomplex

continue from sk^i to sk^{i+1} and use 2) for each of cells which are not parts of Z'.

Exact sequence of a bundle

Let $p: E \to B$ - locally trivial bundle

$b_0 \in B$ - fixed point (reference) $F = p^{-1}(b_0)$ and $f_0 \in F$ - fixed pt (reference)

Consider $\pi_i(E,F)$. Projection $p: E \to B$ induces homomorphism $\pi_i(E,F) \to \pi_i(B)$ since $F \to pt$ under p.

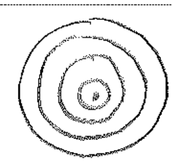
Theorem $\pi_i(E,F) \to \pi_i(B)$ is isomorphism

Proof 1) Monomorphism property

Let $\alpha \in \ker$. It is given by the map $a: D^i \to E$, $a(\partial D^i) \subset F$. At the same time $p \circ a(\partial D^i) = b_0$ and $p \circ a \sim 0$. The corresponding homotopy can be lifted to homotopy of a map from a to E. The resulting homotopy is therefore "ending" in the fiber F (since proj. is mapping to b_0). Such maps are 0 in $\pi_i(E,F)$

2) Epimorphism property

Map $(S^i, s_0) \to (B, b_0)$ can be considered as $f: (D^i, S^{i-1}, s_0) \to (B, b_0, b_0)$. We want to lift it to the map $(D^i, S^{i-1}, s_0) \to (E, F, f_0)$



(D^2 case)

Consider the map $\varphi: S^{i-1} \times I \to D^i$

$\varphi(s, \lambda) = \lambda s$

Instead of $D^i \to E$, we construct map $S^{i-1} \times I \to E$ such that $S^{i-1} \times \{0\}$ maps to f_0. This map can be constructed via covering homotopy theorem. Take S^{i-1} as Z. Consider the map $Z \to f_0 \in E$; composition with p maps it to $b_0 \in B$

$$\begin{array}{ccc} & & f_0 \in E \\ & \nearrow & \downarrow p \\ S^{i-1} & & b_0 \in B \end{array}$$

$f : (D^i, S^{i-1}, s_0) \to (B, b_0, b_0)$ is a homotopy of this map $(Z \to b_0)$

This homotopy has a lift, i.e. map $S^{i-1} \times I \to E$ so that $S^{i-1} \times \{0\}$ maps to f_0. At the same time $S^{i-1} \times \{1\}$ maps into preimage of b_0, i.e. F, therefore QED.

Exact sequence of a pair:

$$\to \pi_i(F) \to \pi_i(E) \to \pi_i(E,F) \to \pi_{i-1}(F) \to \dots$$

$$\to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \dots$$

<u>Corollary</u> If $p : E \to B$ is a covering space

$$\pi_i(B) \cong \pi_i(E) \text{ if } i \geq 2$$

<u>Corollary</u> $\pi_2(S^2) = \mathbb{Z}$

$$\underset{\overset{\shortparallel}{0}}{\pi_2(S^3)} \to \pi_2(S^2) \to \pi_1(S^1) \to \underset{\overset{\shortparallel}{0}}{\pi_1(S^3)}$$

<u>Corollary</u> $\underset{\overset{\shortparallel}{0}}{\pi_3(S^1)} \to \pi_3(S^3) \to \pi_3(S^2) \to \underset{\overset{\shortparallel}{0}}{\pi_2(S^1)}$

$$\pi_3(S^3) \cong \pi_3(S^2)$$

will see later, that $\pi_n(S^n) = \mathbb{Z}$, $n \geq 2$