Lecture IX

Singular homology

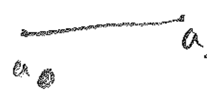
$\pi_n(X, x_0)$ has a natural group structure

$\pi_n(S^k)$ is computable, but not easily

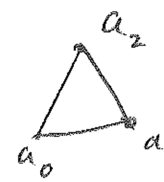
Start with easier invariant (to calculate) called singular homology

Def An <u>n-simplex</u> is the convex hull of $n+1$ affinely independent points $a_0, a_1, \ldots, a_n \in \mathbb{R}^N$
(i.e. $a_1 - a_0, \ldots, a_n - a_0$ - lin indep.)

0-simplex

1-simplex 

Convex hull of $a_0 \ldots a_n$ is $\{\sum_{i=0}^{n} t_i a_i : t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$

2-simplex 

Def. X - topological space. A <u>singular n-simplex</u> is the equivalence class of continuous maps of ordered n-simplices into X.

$\sigma : [a_0, \ldots, a_n] \to X$, $\tau : [b_0 \ldots b_n] \to X$

equivalent if diagram

$$\begin{tikzcd} {[a_0, \ldots, a_n]} \arrow[r, "\text{canonical}"] \arrow[dr, "\sigma"'] & {[b_0, \ldots, b_n]} \arrow[d, "\tau"] \\ & X \end{tikzcd}$$

commutes

Singular chain in X:

$S_n(X)$ - set on n-simplices in X

$C_n(X)$ = free abelian group $\{ \sum_{\sigma \in S_n(X)} r_\sigma \sigma \}$ $r_\sigma = 0$ for all but finite set

Boundary homomorphism:

$\partial = \partial_n : C_n(X) \to C_{n-1}(X)$,

defined by: $\partial \left(\sum_{\sigma \in S_n(X)} r_\sigma \sigma \right) = \sum_{\sigma \in S_n(X)} r_\sigma (\partial \sigma)$

$\sigma : [a_0, \dots, a_n] \to X$

$\partial \sigma = \sum (-1)^i \sigma | [a_0 \dots \hat{a}_i \dots a_n]$,

where $[a_0 \dots \hat{a}_i \dots a_n] = [a_0 \dots a_{i-1} a_{i+1} \dots a_n]$

$\partial \sigma = \sigma | [a_1, a_2] - \sigma | [a_0, a_2] + \sigma | [a_0 a_1]$

Lemma $\partial_{n-1} \partial_n = 0$

Proof $\sigma : [a_0, \dots a_n] \to X$

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma | [a_0 \dots a_n] \to X$$

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma | [a_0, \dots \hat{a}_i \dots a_n]$$

$$\partial \partial \sigma = \sum_{i=0}^{n} (-1)^i \partial \sigma | [a_0 \dots \hat{a}_i \dots a_n] = \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i-1} (-1)^j \sigma | [a_0 \dots \hat{a}_j \dots \hat{a}_i \dots a_n]$$

$$+ \sum_{i=0}^{n} (-1)^i \sum_{j=i+1}^{n} (-1)^{j-1} \sigma | [a_0 \dots \hat{a}_i \dots \hat{a}_j \dots a_n] = 0$$

Homology:

$\partial_n : C_n(X) \to C_{n-1}(X)$, $\partial_{n-1} \circ \partial_n = 0$

Def. $Z_n = Z_n(X) = \ker \partial_n$ = group of (singular) n-cycles in X

$B_n = B_n(X) = \operatorname{Im} \partial_{n+1}$ —— // —— n-boundaries

Def. $H_n(X) = Z_n(X) / B_n(X)$ – n-th singular homology group

Ex. $X = \{pt\}$

$\forall n$ $\exists$ 1 n-simplex : $\sigma : [a_0, \ldots, a_n] \to \{pt\}$

$\partial \sigma_n = \sum_{i=0}^{n} (-1)^n \sigma_n |_{[a_0 \ldots \hat{a}_i \ldots a_n]} = \sum_{i=0}^{n} (-1)^n \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n\text{-even} \\ 0 & \text{if } n\text{-odd} \end{cases}$

$\boxed{C_n(pt) \cong \mathbb{Z} \text{ gen. by } \sigma_n}$

$H_0(X) = Z_0 / B_0 \cong \mathbb{Z}$ $\quad H_n(\{pt\}) = 0$ $\forall n > 0$

Def. A chain complex C is a sequence (C_n, ∂_n) of abelian groups C_n and homomorphisms $\partial_n : C_n \to C_{n-1}$ s.t. $\partial_{n-1} \partial_n = 0$

Def. C, D – chain complexes
$\phi : C \to D$, $\phi_n : C_n \to D_n$ so that diagram is comm.