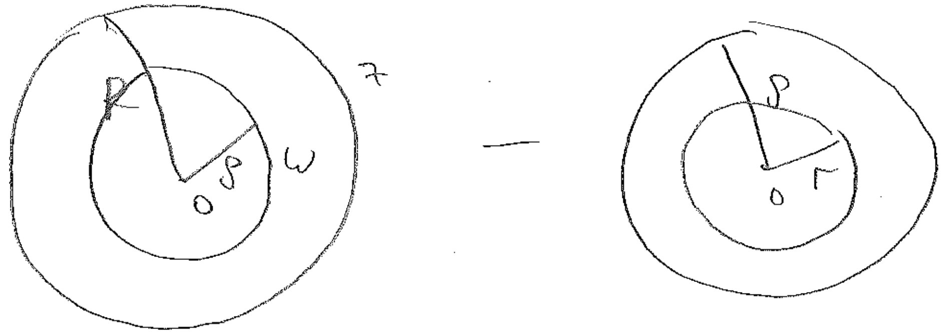
# Lecture X

Borcherds identity

$R > \rho > r > 0$




$$\int_{C_w^\rho} \int_{C_z^R} Y(A,z)Y(B,w)f(z,w)\,dz\,dw -$$

$$- \int_{C_w^\rho} \int_{C_z^r} Y(B,w)Y(A,z)f(z,w)\,dz\,dw =$$

$$= \int_{C_w^\rho} \int_{C_z^R - C_z^r} R\big(Y(A,z)Y(B,w)\big)f(z,w)\,dz\,dw =$$

$$= \int_{C_w^\rho} \int_{C_z^R - C_z^r} R\big(Y(Y(A,z-w)B,w)\big)f(z,w)\,dz\,dw =$$

$$= \int_{C_w^\rho} \int_{C_z^R - C_z^r} \sum_{n\in\mathbb{Z}} Y(A_{(n)}\cdot B,w)(z-w)^{-n-1}f(z,w)\,dz\,dw$$

$$= \int_{C_w^\rho} \int_{C^\delta(w)} Y(A_{(n)}\cdot B,w)(z-w)^{-n-1}f(z,w)\,dz\,dw$$

$$f(z) = z^m w^k = \sum_{n \geq 0} \binom{m}{n} (A_n B)_{m+k-n}$$

In general $f(z,w) = z^m w^k (z-w)^\ell$

$$\sum_{j \geq 0} \binom{\ell}{j} (-1)^j A_{m+\ell-j} B_{k+j} - (-1)^{j+\ell} B_{k+\ell-j} A_{m+j} =$$

$$= \sum_{n \geq 0} \binom{m}{n} (A_{n+\ell} \cdot B)_{m+k-n}$$

Borcherds identity - can be used as one of the axioms.

<u>Examples of OPE</u>

1) $$b(z) b(w) = \frac{1}{(z-w)^2} + \sum_{n \geq 0} \frac{1}{n!} :\partial_w^n b(w) b(w): (z-w)^n$$

$$b(z) Y(b_{-1}, w) = \sum_{n > -1} \frac{Y(b_n b_{-1}|0\rangle, w)}{(z-w)^{n+1}} +$$

$$+ \sum_{n \leq -1} Y(b_n b_{-1}, w) (z-w)^{-n-1} =$$

$$= \frac{1}{(z-w)^2} + \sum_{m \geq 1} \frac{1}{n!} Y(b_m b_{-1}, w) (z-w)^{-m-1} =$$

$$=$$

2) Similarly $$J^a(z) J^b(w) = \frac{\kappa(J^a, J^b)}{(z-w)^2} + \frac{[J^a, J^b](w)}{z-w} + :J^a(z) J^b(w):$$

3) $$T(z) T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + :T(z) T(w):$$

# Derivation of Sugawara construction

<u>Lemma</u>: A $\mathbb{Z}_+$-graded vertex algebra is conformal of central charge $c$ iff $\exists\ \omega \in V_2$ s.t.

$L_n^V$ of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2}$ satisfy

$L_{-1}^V = T$ and $L_2^V \cdot \omega = \frac{c}{2}|0\rangle$

Moreover, there is a(!) homomorphism of VOAs:

$Vir_c \to V$ s.t. $v_c \to |0\rangle$, $L_{-2} v_c \mapsto \omega$

<u>Proof</u>. Suppose $L_{-1}^V = T$, $L_0^V$ is a grad. operator

$$Y(\omega,z)Y(\omega,w) = \sum_{n\in\mathbb{Z}} \frac{Y(L_n^V\omega, w)}{(z-w)^{n+2}} =$$

$$= \frac{c}{2(z-w)^4} + \boxed{\frac{Y(L_1^V\omega, w)}{(z-w)^3}} + \frac{2Y(\omega,w)}{(z-w)^2} + \frac{Y(T\omega,w)}{z-w} + \ldots$$

vanish (exercise)

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$$\boxed{\begin{aligned} Y(S,z) &= \tfrac{1}{2} :J^a(z) J_a(z): \\ \omega &= \frac{1}{k+h^\vee} S \end{aligned}}$$

Have to show i) $L_{-1} = T$

(ii) $L_2 \cdot \omega = \frac{c}{2}|0\rangle$

$S_{-1} \cdot v_k = 0$, $S_0 \cdot v_k = 0$

and $\left[\frac{1}{k+h^\vee} S_n, J_m^b\right] = -m J_{n+m}^b$ for $n = 0, 1$

Compute OPE $S(z)J^b(w)$

$$Y(S,z) = \sum S_n z^{-n-2}$$

$$S_1 \cdot J^b_{-1} v_k = \frac{1}{2}\sum_a J^a_0 J_{a,1} \cdot J^b_{-1} v_k + \frac{1}{2}\sum_a J^a_1 J_{a,0} \cdot J^b_{-1} v_k$$

$$\frac{k}{2}\sum_a (J^a_1 [J_a, J^b]) v_k = 0$$

$$S_0 J^b_{-1} v_k = \sum_a \left(\frac{1}{2} J^a_0 J_{a,0} + J^a_{-1} J_{a,1}\right) \cdot J^b_{-1} v_k \to k\text{-term}$$

$2h^\vee \quad \mathrm{cas}(\mathfrak{g}) \to$ acts as identity

$$S_0 J^b_{-1} v_k = (k + h^\vee) J^b_{-1} v_k$$

$$S_{-1} J^b_{-1} v_k = (k + h^\vee) J^b_{-2} v_k$$

$$S(z) J^a(w) = (k + h^\vee)\left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w}\right) + \text{reg.}$$

Note: $\quad k = -h^\vee \quad [S_n, J^a_m] = 0$

central elements.

Finishing the proof that $\frac{1}{k+h^\vee} S(z)$ is a conf. vector.

$$S_2 S = S_2 \cdot \frac{1}{2}\sum J_{a,-1} J^a_{-1} v_k = \frac{k+h^\vee}{2}\sum_a J_{a,1} J^a_{-1} v_k =$$

$$= \frac{k \dim \mathfrak{g} (k + h^\vee)}{2} v_k \Rightarrow c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}$$

Similarly, one can prove $\omega_\lambda = \frac{1}{2} b_{-1}^2 + \lambda b_{-2}$ is a conformal vector.

Note: $Y^a(z)$, $b(w)$ (for $\lambda = 0$) are primary fields!

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Theorem (Strong reconstruction theorem)

Let $V$ be a vector space sat 1)-3) conditions of mut. local $a^d(z)$ s.t. $a^d(z)|0\rangle = a^d + z(\ldots)$ $T|0\rangle = 0$ $[T, a^d(z)] = \partial_z a^d(z)$

$V$ is spanned by the vectors $a^{d_1}_{j_1} \ldots a^{d_m}_{j_m} |0\rangle$ $j_i < 0$

$\Rightarrow$ these structures define VOA and it is (!)

Proof: exercise. Hint: Goddard's uniqueness theorem.

Application: Let $k \in \mathbb{Z}_+$ $V_k(\mathfrak{g})$

$(e^{d_m}_{-1})^{k+1} v_k$ — sing. vector

$d_m$ - max root.

The quotient is irreducible

Still has VOA on it - it is just a factor of VOA by its ideal

Similarly, $Vir_c$ is reducible iff.

$$c = C(p,q) = 1 - \frac{6(p-q)^2}{pq} \quad p, q > 1 \quad (p,q) = 1$$

$L_{C(p,q)}$ is an irred. quotient, also VOA

# Correlation functions



$\langle \varphi, Y(A_1, z_1) \ldots Y(A_n, z_n) v \rangle$

n-point functions

$\langle \varphi, Y(A_1, z_1) \ldots Y(A_n, z_n) v \rangle =$

$= \langle \varphi, Y(A_1, z_1) \ldots Y(A_n, z_n) Y(v, z_{n+1}) |0\rangle_{z_{n+1}=0}$

makes sense $v = |0\rangle$

<u>Theorem</u> $A_1, \ldots, A_n \in V$. $\forall \varphi \in V^*$ and any perm of $n$ elements $\langle \varphi, Y(A_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(A_{\sigma(n)}, z_{\sigma(n)}) |0\rangle$ is the expansion in $\mathbb{C}((z_{\sigma(1)})) \ldots ((z_{\sigma(n)}))$ of an element $f^{\sigma}_{A_1 \ldots A_n}(z_1 \ldots z_n) \in \mathbb{C}[[z_1 \ldots z_n]][(z_i - z_j)^{-1}]_{i \neq j}$

1) $f^{\sigma}$ does not dep. on $\sigma$

2) $\forall\ i \neq j$

$f_{Y(A_i, z_i - z_j) A_j, A_1 \ldots \hat{A}_i \ldots \hat{A}_j \ldots A_n}(z_j, z_1 \ldots, \hat{z}_i \ldots \hat{z}_j \ldots z_n)$ power series is considered as formal Laurent power series with coeff. in $\mathbb{C}[[z_1, \ldots \hat{z}_i \ldots \hat{z}_j \ldots z_n]][(z_k - z_l)^{-1}]_{k \neq l,\ l \neq i,j}$

is the expansion of $f_{A_1 \ldots A_n}(z_1 \ldots z_n)$

$\Rightarrow \partial_{z_i} f_{A_1 \ldots A_n}(z_1 \ldots z_n) = f_{A_1 \ldots TA_i \ldots A_n}(z_1 \ldots z_n)$

Finally, let $F_n$ be the space of all collections $\{ f_{A_1 \dots A_m}(z_1 \dots z_m) \in \mathbb{C}[[z_1, \dots, z_m]][(z_i - z_j)^{-1}]_{i \neq j}$, $1 \leq m \leq n$, $A_i \in V \}$ satisfying 1), 2), 3)

$\kappa_n : V^* \to F_n$ and $\mu_n : F_n \to V^*$

$\kappa_n : \varphi \to$ collection of n-point functions $\langle \varphi, Y \dots Y |0\rangle$

$\mu_n$: collection $\to$ functional $\varphi$ on $V$ $\langle \varphi, A \rangle = f_A(0)$

<u>Theorem</u> $\kappa_n, \mu_n$ are mutually inv. isom of $V^*$ and the space of n-point functions

<u>Proof.</u> $\mu_n \circ \kappa_n = id$. - obvious

We need to show that <u>$\mu_n$ is injective.</u>

Suppose we have a collection s.t. $f_A(0) = 0$ $\forall A \in V$ $\Rightarrow$

$\Rightarrow \partial_z f_A(z)|_{z=0} = 0$ $\forall A \in V \Rightarrow f_A(z) \equiv 0$ $\forall A \in V$

Moreover $f_{A_1 A_2}(z_1, z_2)$ can be written as power series $(z_1 - z_2)^{\pm 1}$ with coeff. $f_A(z_2)$ $\Longleftrightarrow f_{A_1, A_2}(z_1, z_2) = 0$

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For Heisenberg algebra $F$

$$\omega_n(z_1 \dots, z_n) = \langle \varphi, b(z_1) \dots b(z_n) |0\rangle$$

Bootstrap: $\omega_n(z_1 \dots z_n) = \dfrac{\omega_{n-2}(z_1 \dots \hat{z}_i \dots \hat{z}_j \dots z_n)}{(z_i - z_j)^2} + reg.$

$\Omega_\infty$ - all functions satisfy these conditions

Isomorphism $\Omega_\infty \cong F^*$