Boson - Fermion correspondence (Lecture XI)

1 - dimensional lattice vertex algebras

Fock modules: π_λ, generated by vector $|\lambda\rangle$

$b_n|\lambda\rangle = 0$, $n>0$ $\quad b_0|\lambda\rangle = \lambda|\lambda\rangle$

Def. $(V, |0\rangle, T, Y)$ - vertex algebra. M is called a V-module if it is equipped $Y_M : V \to \operatorname{End} M[[z^{\pm 1}]]$

$$Y_M(A,z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$$

1) $Y_M(|0\rangle, z) = \mathrm{Id}_M$

2) $\forall\ A, B \in V, C \in M$

$Y_M(A,z)\, Y_M(B,w)\, C \in M((z))((w))$

$Y_M(B,w)\, Y_M(A,z)\, C \in M((w))((z))$

$Y_M(Y(A,z-w)B, w)\, C \in M((w))((z-w))$

are the expansions in their respective domains of the same element $M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$

Proposition 1) $Y_M(TA, z) = \partial_z Y_M(A,z)$, 2) $Y_M(A,z)$ are mut. local

Def. V - conformal VA with c. vector $\omega \Rightarrow M$ is called a conformal V-module if L_0^M of $Y_M(\omega, z)$ acts semisimply on M.

Π - conformal vertex algebra

$\frac{1}{2} b_{-1}^2$ - conformal vecto, $c=1$

$L_n = \frac{1}{2} \sum :b_m b_{n-m}: \quad n \in \mathbb{Z}$

$V_{\sqrt{N}\mathbb{Z}} = \bigoplus_{m \in \mathbb{Z}} \pi_{m\sqrt{N}} \quad N > 0$

$\uparrow$ conformal π-module with grading given by L_0

Vertex algebras vs. superalgebras

If $A(z), B(z)$ are of parity α, β corresp:

$$:A(z) B(w): = \sum_{n \in \mathbb{Z}} \left(\sum_{m<0} A_{(m)} B_{(n)} z^{-m-1} + (-1)^{\alpha\beta} \sum_{m \geq 0} B_n A_m z^{-m-1} \right) w^{-n-1}$$

Proposition $\forall$ even N (resp. odd N) $V_{\sqrt{N}\mathbb{Z}}$ carries a structure of conformal vertex algebra (superalgebra) such that $\pi_0 = \pi$ is a conf. vertex subalgebra of $V_{\sqrt{N}\mathbb{Z}}$ and the structure of $V_{\sqrt{N}\mathbb{Z}}$ as a conformal π-module induced by the vertex algebra coincides with canonical structure.

Proof. $V_{\sqrt{N}\mathbb{Z}}$ generated from vectors $|\lambda\rangle$ and action of b_n $n<0$. Define $V_\lambda(z) = Y(|\lambda\rangle, z)$

Conformal vector $\frac{1}{2} b_{-1}^2 |0\rangle \in \pi \subset V_{\sqrt{N}\mathbb{Z}} \Rightarrow$

$\Rightarrow$ gradation $\deg b_{-n} = n$, $\deg |\lambda\rangle = \frac{\lambda^2}{2}$, $T = \frac{1}{2} \sum_{n \in \mathbb{Z}} b_n b_{-1-n}$

$T|\lambda\rangle = \lambda b_{-1}|\lambda\rangle \Rightarrow \partial_z Y(|\lambda\rangle, z) = Y(T\cdot|\lambda\rangle, z) =$
$= \lambda Y(b_{-1}|\lambda\rangle, z) \Rightarrow \partial_z V_\lambda(z) = \lambda :b(z)V_\lambda(z):$

$$b(z)\,V_\lambda(w) = \frac{Y(b_0|\lambda\rangle, w)}{z-w} + :b(z)V_\lambda(w): =$$
$$= \frac{\lambda V_\lambda(w)}{z-w} + :b(z)V_\lambda(w): \Rightarrow$$

$\Rightarrow [b_n, V_\lambda(z)] = \lambda z^n V_\lambda(z)$, in particular
$[b_0, V_\lambda(z)] = \lambda V_\lambda(z) \Rightarrow$

$\Rightarrow V_\lambda(z)\cdot \underset{\in \Pi_\mu}{A} \in \Pi_{\lambda+\mu}((z))$

Let's write $V_\lambda(z) = \sum_{n\in\mathbb{Z}} V_\lambda(n) z^{-n-\lambda^2/2}$, so that
$V_\lambda(n)$ should have conf. degree $-n$

$\deg|\nu\rangle = \frac{\nu^2}{2}$, $\deg|\lambda+\nu\rangle = \frac{(\lambda+\nu)^2}{2}$

$V_\lambda[s]|\nu\rangle = 0 \quad s > -\lambda^2/2 - \lambda\nu \quad \left(-s + \frac{\nu^2}{2} \geq \frac{(\lambda+\nu)^2}{2}\right)$

$V_\lambda(-\frac{\lambda^2}{2} - \lambda\nu)|\nu\rangle = c_{\lambda,\nu}|\lambda+\nu\rangle$

$c_{\lambda,\mu}$ - some complex numbers ($\underset{\uparrow \text{vac. axiom}}{c_{\lambda,0} = 1}$)

The formulas we derived determine $V_\lambda(z)$ entirely. Define $S_\lambda|\nu\rangle = c_{\lambda,\nu}|\nu+\lambda\rangle$, $[S_\lambda, b_n] = 0 \; n\neq 0$

Then $V_\lambda(z) = S_\lambda z^{\lambda b_0} \exp\left(-\lambda \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\lambda \sum_{n>0} \frac{b_n}{n} z^{-n}\right)$

Really: $[T, V_\lambda(z)] = \lambda :b(z)V_\lambda(z):$
$[b(z), V_\lambda(w)] = \lambda V_\lambda(w)\,\delta(z-w)$

Locality:

$$V_\lambda(z)V_\mu(w) = (z-w)^{\lambda\mu} :V_\lambda(z)V_\mu(w):,$$

where $:V_\lambda(z)V_\mu(w): = S_\lambda S_\mu z^{(\lambda+\mu)b_0} \exp(\ldots\ldots)$

$(z-w)^{\lambda\mu}$ we understand $z^{\lambda\mu}(1-\frac{w}{z})^{\lambda\mu}$ if $\lambda\mu<0$

$$(z-w)^M V_\lambda(z)V_\mu(w) = (-1)^{p(\lambda)p(\mu)}(z-w)^M V_\mu(w)V_\lambda(z)$$

where $p(\lambda)$ means parity of $|\lambda\rangle$

$$S_\lambda S_\mu = (-1)^{p(\lambda)p(\mu)+\lambda\mu} S_\mu S_\lambda$$ - only if it is satisfied

$$\Updownarrow$$

$$c_{\lambda,\nu+\mu}\, c_{\mu,\nu} = (-1)^{p(\lambda)p(\mu)+\lambda\mu} c_{\mu,\nu+\lambda}\, c_{\lambda,\nu}$$

$\forall \lambda,\mu,\nu \in \sqrt{N}\mathbb{Z}$

Setting $c_{0,\nu}=1 \ \forall \nu$

$\nu=0$ $\mu=\lambda \Rightarrow p(m\sqrt{N}) = m^2 N \bmod 2$

If N is even $\Rightarrow V_{\sqrt{N}\mathbb{Z}}$ is even, if N-odd $\Rightarrow$

$\Rightarrow$ parity of $\pi_{m\sqrt{N}}$ is eq. $m \bmod 2$

Solution of the equations: assume $c_{\sqrt{N},m\sqrt{N}}$ are nonzero $\forall m\neq 0 \Rightarrow$ normalize $c_{\sqrt{N},m\sqrt{N}}=1 \ \forall m\in\mathbb{Z}$

$\Rightarrow c_{\lambda,\nu}=1 \ \forall \lambda,\nu\in\sqrt{N}\mathbb{Z}$

Other conformal vectors:

$\omega_\lambda = \frac{1}{2}b_{-1}^2 + \lambda b_{-2}$ $\lambda\in\mathbb{C}$ $L_0|\mu\rangle = \frac{1}{2}\mu(\mu-2\lambda)$

If we want L_0 to be integers or half-integers if N is odd $\Rightarrow \lambda\in\frac{\sqrt{N}}{2}\mathbb{Z}$.

Free fermions

$\mathbb{C}((t)) \oplus \mathbb{C}((t))dt \qquad \psi_n = t^n, \ \psi_n^* = t^{n-1}dt$

$$[\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0, \quad [\psi_n, \psi_m^*]_+ = \delta_{n,-m}$$

Λ - fermionic Fock representation

$\psi_n|0\rangle = 0, \ n \geq 0 \qquad \psi_n^*|0\rangle = 0, \ n > 0$

Therefore $\quad \Lambda \simeq \Lambda(\psi_n)_{n<0} \otimes \Lambda(\psi_n^*)_{n\leq 0}|0\rangle$

$$Y(\psi_{-1}|0\rangle, z) = \psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}$$

$$Y(\psi_0^*|0\rangle, z) = \psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n}$$

$\mathbb{Z}_+$-gradation $\quad \psi_{n_1} \dots \psi_{n_k} \psi^*_{m_1} \dots \psi^*_{m_\ell}|0\rangle =$

$$= -\sum_{i=1}^{k} n_i - \sum_{j=1}^{\ell} m_j$$

$V_{\sqrt{N}\mathbb{Z}}$ set $N=1 \Rightarrow V_{\mathbb{Z}}$ $\quad \mathbb{Z}_+$-gradation defined by $\omega_{-1/2}$

$$\deg|m\rangle = \tfrac{1}{2}m(m-1), \ \deg b_n = -n$$

<u>Theorem</u> There is an isomorphism of vertex superalgebras $\Lambda \simeq V_{\mathbb{Z}}$.

<u>Proof.</u> $\quad V_{\pm 1}(z)V_{\mp 1}(w) = \dfrac{1}{z-w} \ : V_1(z)V_{-1}(w): = \dfrac{1}{z-w} + \text{reg}$

$$V_{\pm 1}(z)V_{\pm 1}(w) = (z-w) \ : V_{\pm 1}(z)V_{\pm 1}(w): = \text{reg}$$

$$V_1(z) = \sum_{n\in\mathbb{Z}} \phi_n^* z^{-n}, \quad V_{-1}(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-1}$$

$$[\phi_n, \phi_m]_+ = [\phi_n^*, \phi_m^*]_+ = 0, \quad [\phi_n, \phi_m^*]_+ = \delta_{n,-m}$$

We find that $\phi_n|0\rangle = 0, n\geq 0 \quad \phi_n^*|0\rangle = 0, n>0$

$\sigma : \Lambda \longrightarrow V_{\mathbb{Z}}$ seding $|0\rangle$ to $|0\rangle$

σ is injective and surjective (exercise)

$\sigma(\psi_n \psi_{n+1} \cdots \psi_{-1}|0\rangle) = |n\rangle \quad n<0$

$\sigma(\psi^*_{-n+1} \psi^*_{-n+2} \cdots \psi^*_0|0\rangle) = |n\rangle \quad n>0$

moreover, $:V_1(z)V_{-1}(z): = b(z)$

Remark about conformal vectors

In $V_{\mathbb{Z}}$ $\quad \omega_\lambda = \frac{1}{2} b_{-1}^2 + \lambda b_{-2} \quad \lambda \in \mathbb{Z}/2$

$$\sigma^{-1}(b(z)) = h(z) = :\psi^*(z)\psi(z):$$

$$h_{-1}^2|0\rangle = \sum_{n\in\mathbb{Z}} :\psi_n^* \psi_{-n-1}: \cdot \psi_0^* \psi_{-1}|0\rangle = (\psi^*_{-1}\psi_{-1} + \psi_{-2}\psi_0^*)|0\rangle$$

$$h_{-2} = (\psi^*_{-1}\psi_{-1} - \psi_{-2}\psi_0^*)|0\rangle$$

$$\Rightarrow \sigma^{-1}(\omega_\lambda) = (\mu \psi^*_{-1}\psi_{-1} + (1-\mu)\psi_{-2}\psi_0^*)|0\rangle$$

$\mu = \lambda + 1/2 \Rightarrow$

$$T_\mu = \mu :\partial_z \psi^* \psi(z): + (1-\mu) :\partial_z \psi(z)\psi(z)^*:$$

$$c = 1 - 12\lambda^2 = -2(6\mu^2 - 6\mu + 1)$$

$$\deg \psi_n = -\mu - n, \quad \deg \psi_n^* = \mu - n$$

Jacobi triple product identity

characters Λ and $V_{\mathbb{Z}}$

charge: $\psi_n^* = -\text{charge } \psi_n = 1$, charge $|0\rangle = 0$

$$\operatorname{ch}\Lambda = \sum_{n,m} \dim \Lambda_{n,m}\, q^n u^m =$$

grad. charge

$$= \prod_{n>0} (1 + u q^{n-1})(1 + u^{-1} q^n)$$

$\deg b_{-n} = -n$ $\quad$ $\deg |m\rangle = m(m-1)/2$

charge $|m\rangle = m$ $\Rightarrow$

$$\Rightarrow \operatorname{ch} V_{\mathbb{Z}} = \sum_{m \in \mathbb{Z}} u^m q^{m(m-1)/2} \prod_{n>0} (1-q^n)^{-1}$$

$$\prod_{n>0} (1-q^n)(1-u q^{n-1})(1-u^{-1} q^n) = $$

$$= \sum_{m \in \mathbb{Z}} (-1)^m u^m q^{m(m-1)/2}$$

$V_{\sqrt{2}\mathbb{Z}}$ $\quad$ $L_1(sl_2)$ - irr. repr. of $V_1(sl_2)$

$e_{-1}v_1,\ h_{-1}v_1,\ f_{-1}v_1$

$|\sqrt{2}\rangle,\ \sqrt{2}\, b_{-1}|0\rangle,\ |-\sqrt{2}\rangle$ - generate $\widehat{sl}(2)$ representation

It is called basic representation

$V_{\sqrt{3}\mathbb{Z}}$ - superconformal algebra $(N=2)$