Free fields and path integrals (Lecture XII)

(Σ, γ) $d+1$-dim compact $\quad Map(\Sigma, M)$

$$S(\phi) = \frac{\beta}{4\pi}\int_\Sigma (|d\phi|^2 + m^2\phi^2)\,dv = \frac{1}{2}(\phi, G^{-1}\phi)_{L^2}$$

$\frac{\beta}{2\pi} G = (-\Delta + m^2)^{-1}$ – propagator

$$Z = \int_{Map(\Sigma,\mathbb{R})} e^{-S(\phi)} D\phi = \int_{Map(\Sigma,\mathbb{R})} e^{-\frac{1}{2}(\phi, G^{-1}\phi)} D\phi$$

$$Z = (\det G)^{1/2} \qquad D\phi = \prod_i \frac{d\phi_i}{\sqrt{2\pi}}$$ orthonormal coord.

$$\det G = e^{-\zeta_G'(0)} \qquad \zeta_G(s) = \sum_n \lambda_n^{-s} \quad Res\ c - \frac{d+1}{2}$$

regularization $\qquad \lambda_n \sim O(n^{-2(d+1)})$

Correlation functions:

$$\langle \phi(x_1)\dots\phi(x_n)\rangle = \frac{\int \phi(x_1)\dots\phi(x_n)\, e^{-S(\phi)} D\phi}{\int e^{-S(\phi)} D\phi}$$

$$\langle \phi(x_1)\dots\phi(x_n)\rangle = \begin{cases} 0 & n \text{ is odd} \\ G(x_1,x_2) & \text{for } n=2 \\ \sum\limits_{\substack{\text{pairings} \\ i_+\, i_-}} \prod\limits_{(i_+,i_-)} G(x_{i_+}, x_{i_-}) & n\text{-even}\end{cases}$$

$D'(\Sigma)$ – space of distributions

$$\langle \phi(x_1)\dots\phi(x_n)\rangle = \int_{D'(\Sigma)} \phi'(x_1)\dots\phi'(x_n)\, d\mu_G(\phi)$$

Relation to Hilbert space formalism

Example $d=0$, $\Sigma=[0,L]$

Wiener measure (supported on $C_{per}([0,L])$)
modulo $e^{-\frac{\beta m^2}{2\pi}\int_0^L \phi(x)^2}$. Let $0\le x_1 \ldots \le x_n \le L \Rightarrow$

$$\Rightarrow \int_{C_{per}([0,L])} \phi(x_1)\ldots\phi(x_n)\,d\mu_G(\phi) =$$

$$= \frac{tr\left(e^{-x_1 H}\varphi\, e^{(x_1-x_2)H}\varphi\ldots\varphi\, e^{(x_n-L)H}\right)}{tr\, e^{-LH}}$$

$$H = -\frac{\pi}{\beta}\frac{d^2}{d\varphi^2} + \frac{\beta m^2}{4\pi}\varphi^2 - \frac{m}{2} =$$

$$= m\left(-\sqrt{\frac{\pi}{\beta m}}\frac{d}{d\varphi} + \sqrt{\frac{\beta m}{4\pi}}\varphi\right)\left(\sqrt{\frac{\pi}{\beta m}}\frac{d}{d\varphi} + \sqrt{\frac{\beta m}{4\pi}}\varphi\right) = m a^* a$$

acting on $L^2(\mathbb{R}, d\varphi)$

$[a, a^*] = 1$

Ground state $e^{-\frac{\beta m}{4\pi}\varphi^2} = |0\rangle$

$\mathcal{H} = \text{span}\{a^{*n}|0\rangle\}$ – Hermite polynomials

Note $\varphi = \sqrt{\frac{\pi}{\beta m}}(a + a^*)$

Spectrum: $0, m, 2m, \ldots$

Dimension $d > 0$

$\Sigma = [0, L]^{d+1}$ with periodic identifications.

$d\mu_G$ is carried by genuinely distributional ϕ's

$(x_i = (x_i^0, \vec{x}))_{i=1}^n$ be s.t. $0 < x_1^0 < x_2^0 \ldots < x_n^0 < L$

$$\int_{D'([0,L]^{d+1})} \phi(x_1) \ldots \phi(x_n)\, d\mu_G(\phi) = \frac{\operatorname{tr} e^{-x_1^0 H} \varphi(\vec{x}_1) \ldots \varphi(\vec{x}_n) e^{(x_n^0 - L)H}}{\operatorname{tr} e^{-LH}}$$

H - positive, self-adj.

$$\varphi_{\vec{k}} = \int_{[0,L]^d} e^{i\vec{k}\vec{x}} \phi(0, \vec{x})\, d\vec{x}$$

$$\mathcal{H} = \bigotimes_{\pm\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^d} L^2(\mathbb{C}, d^2\varphi_{\vec{k}})$$

$$a_{\vec{k}} = \sqrt{\frac{\pi L^d}{\beta k_0}} \frac{\partial}{\partial \varphi_{\vec{k}}} + \sqrt{\frac{\beta k_0}{4\pi L^d}}\, \varphi_{-\vec{k}}$$

where $k_0 = \sqrt{\vec{k}^2 + m^2}$

$$[a_{\vec{k}}, a^*_{\vec{k}'}] = \delta_{\vec{k}, \vec{k}'}$$

$$\varphi(0, \vec{x}) = \sum_{\vec{k}} \sqrt{\frac{\pi}{\beta k_0 L^d}}\, e^{-i\vec{k}\vec{x}} (a_{\vec{k}} + a^*_{\vec{k}})$$

$$|0\rangle \sim e^{\sum_{\vec{k}} \frac{\beta k_0}{4\pi L^d} |\varphi_k|^2}$$

$$H = \sum_{\vec{k}} k_0\, a^*_{\vec{k}} a_{\vec{k}}$$

Spectrum: $\sum_{\vec{k}} k_0 \mathbb{Z}_+$

Scalar field with values in S^1

$$S(\phi) = \frac{\beta}{4\pi} \int_\Sigma |d\phi|^2 dv$$

$$\mathrm{Map}(\Sigma, \mathbb{R}/2\pi\mathbb{Z}) = \bigcup_{\chi \in \mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z})} \mathrm{Map}(\tilde{\Sigma}, \mathbb{R})_\chi / 2\pi\mathbb{Z}$$

where $\phi_\chi \in \mathrm{Map}(\tilde{\Sigma}, \mathbb{R})$ is a function on a universal cover $\tilde{\Sigma}$ of Σ eq. w.r.t. action of the fundamental group:

$$\phi_\chi(ax) = \phi_\chi(x) + \chi(a) \quad \forall a \in \pi_1(\Sigma)$$

$\mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z}) \cong H^1(\Sigma, 2\pi\mathbb{Z})$ with χ given by the periods of α

Note: $\mathrm{Hom}(\pi_1(\Sigma), 2\pi\mathbb{Z}) \cong H^1(\Sigma, 2\pi\mathbb{Z})$ where χ is given by the periods of $\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})$

$$\underset{\mathrm{Map}_\chi}{\phi_\chi} = \int_{x_0}^x \alpha_h + \psi \equiv \phi_h + \psi$$

ψ ← univalued function on Σ, x_0 - basepoint, α_h ← harmonic repr. of α

$$S(\phi_\chi) = \frac{\beta}{4\pi} \|\alpha_h\|_{L^2}^2 + \frac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2}$$

$$Z = \int_{\mathrm{Map}(\Sigma, S^1)} e^{-S(\phi)} D\phi = \sum_{\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\frac{\beta}{4\pi}\|\alpha_h\|_{L^2}^2} \left(\frac{2\pi \,\mathrm{vol}_\Sigma}{\det'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2}$$

0-mode $\Rightarrow \sqrt{2\pi\,\mathrm{vol}_\Sigma}$

Exercise $d=0$, $\Sigma=[0,L]_{per}$ $\quad d_n=\frac{2\pi}{L}n\,dx$

$$\det{}'\left(-\frac{\beta}{2\pi}\frac{d^2}{dx^2}\right)=2\pi L^2/\beta$$

$$Z=\sum_{n\in\mathbb{Z}} e^{-\pi\beta L^{-1}n^2}\left(\frac{2\pi L}{\det{}'\left(-\frac{\beta}{2\pi}\frac{d^2}{dx^2}\right)}\right)^{1/2} \underset{\substack{\text{poisson}\\ \text{resummation}}}{=} \sum_{n\in\mathbb{Z}} e^{-\pi\beta^{-1}Ln^2} = \operatorname{tr} e^{-LH}$$

$H=-\frac{\pi}{\beta}\frac{d^2}{d\varphi^2}$ $\quad (e^{in\varphi})$ - eigenvalues.

Let us look at the 2d case

(Σ,γ) - Riemann surface of genus g

$(a_i,b_j)_{i,j=1}^{h_\Sigma}$ - basis of $H_1(\Sigma,\mathbb{Z})$ $\quad (\omega^i)_{i=1}^{g}$ - hol. (1,0)-forms

$$\int_{a_i}\omega^j=\delta^{ij} \quad \int_{b_i}\omega^j=\tau^{ij}, \quad \operatorname{Im}\tau=\tau_2$$

$$d_n=\frac{\pi}{i}(\bar\tau\vec m+\vec n)^t\tau_2^{-1}\omega+c.c.$$

for $\vec m,\vec n\in\mathbb{Z}^g$ gives harm. forms in $H^1(\Sigma,2\pi\mathbb{Z})$

(with a_i-periods $-2\pi m_i$ and b_j-periods $2\pi n_j$)

Notice: $\quad \|d_n\|^2_{L^2}=(2\pi)^2(\bar\tau m+n)^t\tau_2^{-1}(\tau m+n)$

$$\sum_{d\in H^1(\Sigma,2\pi\mathbb{Z})} e^{-\frac{\beta}{4\pi}\|d_n\|^2_{L^2}}=\beta^{-g/2}(\det\tau_2)^{1/2}\,\Theta_{Q_\beta}(\tau,\bar\tau)$$

theta-function

Q - lattice in $E_{s_+,s_-}=E_{s_+}\oplus E_{s_-}$

$|\cdot|^2_{E_{s+}}-|\cdot|^2_{E_{s-}}$ - pairing

$$\Theta_Q(\tau,\bar\tau)=\sum_{(q_+,q_-)\in Q^g} e^{\pi i(q_+,\tau q_+)-\pi i(q_-,\bar\tau q_-)}$$

Massless fermions

(Σ, γ) – R. surface $\quad K = T^{*(1,0)}\Sigma \quad K^{1/2} = L$

$\Psi = (\psi, \tilde{\psi}) \in \Gamma(L \oplus \bar{L})$

$\bar{\Psi} = (\tilde{\chi}, \chi) \in \Gamma(\bar{L} \oplus L)$

$\bar{\partial}_L$ is $\bar{\partial}$-op. on L and $\partial_{\bar{L}}$ – c.c.

$$S(\psi, \bar{\psi}) = -\frac{1}{\pi}\int (\chi \bar{\partial}_L \psi + \tilde{\chi}\partial_{\bar{L}}\tilde{\psi})$$

$$Z_L = \int e^{-S(\bar{\Psi}, \Psi)} D\bar{\Psi} D\Psi = \det(\partial_{\bar{L}})\det(\bar{\partial}_L) =$$

$$= \det(\bar{\partial}_L^* \bar{\partial}_L)$$

Spin structure: P or AP boundary cond. $\quad z \to z+1$, $z \to z+\tau$

$$Z_{pa}(\tau) = \operatorname{tr}_{\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R} q^{L_0 - 1/24}\, \bar{q}^{\tilde{L}_0 - 1/24}$$

$$\mathcal{H}_R = \mathbb{C}^2 \otimes \Big(\Lambda\big(\bigoplus_{n=1}^{\infty} \mathbb{C}\big)\Big)^{\otimes 2}$$

$$Z_{pp}(\tau) = \underbrace{\operatorname{tr}}_{\text{str.}}{}_{\mathcal{H}_R \otimes \tilde{\mathcal{H}}_R} (-1)^{F + \tilde{F}} q^{L_0 - 1/24}\, \bar{q}^{\tilde{L}_0 - 1/24}$$

vanishes b.c. of alt. signs.

$$\mathcal{H}_{NS} = \Big(\Lambda\big(\bigoplus_{n=0}^{\infty} \mathbb{C}\big)\Big)^{\otimes 2}$$

$$Z_{ap}(\tau) = \operatorname{str}_{\mathcal{H}_{NS} \otimes \bar{\mathcal{H}}_{NS}} q^{L_0 - 1/24}\, \bar{q}^{\bar{L}_0 - 1/24}$$

$$Z_{aa}(\tau) = \operatorname{tr}_{\mathcal{H}_{NS} \otimes \bar{\mathcal{H}}_{NS}} q^{L_0 - 1/24}\, \bar{q}^{\bar{L}_0 - 1/24}$$

Bosonization

$$\frac{1}{2}\sum_L (-1)^{\delta(L)} Z_L = C^{g-1} Z_{1/2}$$

bosonic part. function $r^2 = \frac{1}{2}$

$$Q_\beta = \left\{ \left(\frac{\sqrt{\beta}\, m + (\sqrt{\beta})^{-1} n}{\sqrt{2}}, \frac{\sqrt{\beta}\, m - \sqrt{\beta}^{-1} n}{\sqrt{2}} \right) \Big| \, m, n \in \mathbb{Z} \right\} \subset \mathbb{R} \oplus \mathbb{R}$$

$$Z = Z_\beta = \underbrace{e^{(-6 \ln 2\pi + 1 + \ln \beta/2)(g-1)}}_{} \Theta_{Q_\beta}(\tau, \bar\tau) \left(\frac{\mathrm{vol}_\Sigma \det \tau_2}{\det'(-\Delta)} \right)^{1/2}$$

discard this term $\left(+\lambda \int_\Sigma r\, dv\right)$ - to the action $\sim \lambda 4\pi (1-g)$

Then $Z_\beta = Z_{1/\beta}$ — T-duality

β — has a meaning of radius of S^1

On genus 1. $\det'(-\Delta) = \tau_2^2 \, |\eta(\tau)|^4$

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

The marking indep. $\Rightarrow Z_\beta(\tau) = Z_\beta\left(\frac{a\tau + b}{c\tau + d}\right)$

Compactifications (toroidal)

$T^N = (\mathbb{R}/2\pi\mathbb{Z})^N \quad g = \sum_{i,j} g_{ij}\, d\phi^i d\phi^j \quad \omega = \sum_{i,j} b_{ij}\, d\phi^i \wedge d\phi^j$

$\phi : \Sigma \to T^N$

$$S(\phi) = \frac{1}{4\pi} \left(\| d\phi \|^2_{L^2} + i \int_\Sigma \phi^* \omega \right)$$

$$Z = Z_d = \Theta_{Q_d}(\tau, \bar\tau) \left(\frac{\mathrm{vol}_\Sigma \det \tau_2}{\det'(-\Delta)} \right)^{N/2} = Z_{d^{-1}}$$

where $d = \{ d_{ij} = g_{ij} + b_{ij} \}$

$$Q_d = Q_{d^{-1}} = \left\{ \left(\frac{d\, m + n}{\sqrt{2}}, \frac{d^t m - n}{\sqrt{2}} \right) \Big| \, m, n \in \mathbb{Z}^N \right\} \subset \mathbb{R}^N \oplus \mathbb{R}^N$$

is an even self-dual lattice with $|(x,y)|^2 = (x, g^{-1} x) - (y, g^{-1} y)$

Mirror symmetry (toy story)

Toroidal compactification to T^2 $\quad \Psi = \phi^1 + T\phi^2$

complex coord. $\quad$ in $\mathbb{H}_+$

$$g = \left(R_2/T_2\right) d\Psi d\bar{\Psi} \quad \text{with } R_2 > 0$$

$$\omega = i\left(R_2/T_2\right) d\Psi \wedge d\bar{\Psi} \quad \text{set } R = R_1 + iR_2$$

$$Z = Z_{R,T} = \theta_{Q_{R,T}}(\tau,\bar{\tau}) \left(\frac{\mathrm{vol}_\Sigma \det \tau_2}{\det'(-\Delta)}\right)$$

$$Q_{R,T} = \left\{\left(\frac{Rm^1 + TRm^2 + Tn^1 - n^2}{\sqrt{2R_2T_2}}, \frac{\bar{R}m^1 + T\bar{R}m^2 + Tn^1 - n^2}{\sqrt{2R_2T_2}}\right) \;\; m^i, n^i \in \mathbb{Z}\right\} \subset \mathbb{C} \oplus \mathbb{C}$$

$$|(z_1, z_2)|^2 = |z_1|^2 - |z_2|^2$$

quadratic form

$$Z_{R,T} = Z_{T,R}$$

Mirror symmetry: identity of CFTs in two different CY manifolds with the role of modular parameters of complex and Kahler structures interchanged.

Remark on Liouv. cocycle:

$$\frac{\delta}{\delta\sigma}\Big|_{\sigma=0} \ln\left(\frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\right) = -\frac{1}{12\pi} r(x)$$

Prove!

$$\frac{\delta}{\delta\sigma}\Big|_{\sigma=0} Z_{e^{\sigma}\gamma} = \frac{N}{24\pi} r(x) Z_\gamma$$

Ex: derive full Liouville cocycle formula

Ex. $\zeta_{-\Delta}(s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1} \operatorname{tr} e^{t\Delta}$, $\; e^{t\Delta} = \frac{1}{4\pi t} + \frac{1}{12\pi} r(x) + O(t)$

Toroidal compactifications and correlation functions

$$\int \prod_{i=1}^{n} e^{i q_i \phi(x_i)} \, e^{-\frac{\beta}{4\pi}\int_\Sigma |d\phi|^2 d\upsilon} \, D\phi$$

$\underline{q_i \in \mathbb{Z}}$

theta-function

$$\sum_{\alpha \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\frac{\beta}{4\pi}\|d_h\|^2_{L^2} + i\sum_i q_i \phi_h(x_i)} \; *$$

$$\int e^{-\frac{\beta}{4\pi}(\psi, -\Delta\psi)_{L^2} + i\sum q_i \psi_i(x_i)} D\psi$$

$$\hookrightarrow \delta_{\sum_i q_i, 0} \, e^{\frac{\pi}{\beta}\sum_{i,j=1}^{n} q_i q_j G(x_i, x_j)} \left(\frac{2\pi \, \mathrm{vol}_\Sigma}{\det'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2}$$

$$G(x,y) = \frac{1}{2\pi}\ln \mathrm{dist}(x,y) + \text{finite}.$$

$\uparrow$ singularity

Renormalize: $\tilde{G}(x,x) = \lim_{y\to x} G(x,y) - \frac{1}{2\pi}\ln \mathrm{dist}(x,y)$

$$\langle :e^{iq_1\phi(x_1)}: \ldots :e^{iq_n\phi(x_n)}: \rangle_\gamma =$$

On $\mathbb{CP}^1$ $H^1 = 0$, no contrib. from harm. forms

$$= \delta_{\sum_i q_i, 0} \, e^{-\sum_i \frac{q_i^2}{4\beta}\sigma(x_i)} \prod_{i<j} |z_i - z_j|^{\frac{q_i q_j}{\beta}}$$

in $\gamma = e^{\sigma} dz d\bar{z}$ and $G(x,y) = \frac{1}{2\pi}\ln|z(x) - z(y)|$

Operator picture

Fock spaces

$$\mathcal{H} = L^2(S^1, d\varphi_0)^{\mathbb{Z}} \otimes \mathcal{F} \otimes \tilde{\mathcal{F}} \qquad [\alpha_n, \alpha_m] = n\delta_{n,-m} \quad \alpha_n^* = \alpha_{-n}$$

infinite sum, lapeled by the winding number

$$p_0 = \frac{1}{i}\frac{d}{d\varphi_0} \qquad p_0|p,w\rangle = p|p,w\rangle$$

$$\varphi(\tau,x) = \varphi_0 - i\beta^{-1}p_0\tau + wx + \frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-n(\tau-ix)} - \sum\frac{\tilde{\alpha}_n}{n}e^{-n(\tau+ix)}$$

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_m\alpha_{n-m}: \qquad \alpha_0 = \frac{1}{\sqrt{2}}(\sqrt{\beta}w + \sqrt{\beta}^{-1}p_0)$$

$$\tilde{\alpha}_0 = \frac{1}{\sqrt{2}}(\sqrt{\beta}w - \sqrt{\beta}^{-1}p_0)$$

$$H = L_0 + \tilde{L}_0$$

$$P = L_0 - \tilde{L}_0 \qquad \boxed{c = 1}$$

Vertex operators

$$V_q(\tau,x) = :e^{iq\varphi(\tau,x)}:$$

Ex. $(\Omega, V_{q_1}(\tau_1,x_1)\ldots V_{q_n}(\tau_n,x_n)\Omega\rangle =$

$$= \delta_{\sum_i q_i,0}\prod_{i<j}|z_i - z_j|^{\frac{q_iq_j}{\beta}}$$

$$\Delta = \frac{q^2}{4\beta} \qquad [L_n, V_q(z,\bar{z})] = (n+1)\Delta z^n V_q(z,\bar{z}) + z^{n+1}\partial_z V_q$$

$$[\tilde{L}_n, V_q(z,\bar{z})] = (n+1)\Delta\bar{z}^n V_q \ldots$$

T-duality – a unitary transf.

$$U_T|n,w\rangle = (-1)^{nw}|w,n\rangle \qquad U_T\alpha_n = \alpha_n U_T$$

$$U_T\tilde{\alpha}_n = -\tilde{\alpha}_n U_T$$