Rational CFT

Lecture I

Primary fields:

$$\phi_{h,\bar h}(z,\bar z)\, dz^h d\bar z^{\bar h}$$

$$L_v\, \phi_{h,\bar h} = \left(v(z)\frac{d}{dz} + h\partial_z v\right)\phi_{h,\bar h}$$

(L_v ← holomorphic vector field)

$$T(z)\,\phi_{h,\bar h}(w,\bar w) = \frac{h\,\phi_{h,\bar h}(w,\bar w)}{(z-w)^2} + \frac{\partial_w \phi_{h,\bar h}(w,\bar w)}{(z-w)} + \text{reg.}$$

$\bar T(\bar z)$ . . . .

$$L_v\, \phi_{h,\bar h}(w,\bar w) = \oint_{C_w} v(z)T(z)\, \phi_{h,\bar h}(w,\bar w)$$

From now on, everything is on $\mathbb{CP}^1$

$$|h,\bar h\rangle = \lim_{z,\bar z\to 0} \phi_{h,\bar h}(z,\bar z)|0\rangle$$

$$[L_n, \phi_h(z,\bar z)] = \left(z^{n+1}\frac{d}{dz} + (n+1)z^n h\right)\phi_h(z)$$

$$L_n|h,\bar h\rangle = 0 \quad n>0 \quad L_0|h,\bar h\rangle = h|h,\bar h\rangle$$

$L_{-n_1}\dots L_{-n_k}|h,\bar h\rangle$ → the corresponding fields are call "descendant fields"

On $\mathbb{CP}^1$:

$$\langle T(z)\,\phi_1(z_1)\dots\phi_N(z_N)\rangle = \sum_{i=1}^{N}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\langle \phi_1(z_1)\dots\phi_N(z_N)\rangle$$

Why?

$T_{uu} = z^4 T_{zz} \quad u = \frac{1}{z} \quad \Rightarrow T_{zz}$ should behave like $1/z^4$, $z \to \infty$

holomorphic addition is impossible bc of beh. at ∞.

$\frac{1}{z}, \frac{1}{z^2}, \frac{1}{z^3}$ vanish:

i) $\sum_{i=1}^{N} \frac{\partial}{\partial z_i} \langle \phi_1(z_1) \dots \phi_N(z_N) \rangle = 0$

ii) $\sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} + h_i \right) \langle \phi_1(z_1) \dots \phi_N(z_N) \rangle = 0$

iii) $\sum_{i=1}^{N} \left(z_i^2 \frac{\partial}{\partial z_i} + 2h_i z_i \right) \langle \qquad \rangle = 0$

Invariance under $L_{\pm 1}, L_0$ or under $PSL_2(\mathbb{C})$ transformations.

PSL_2-invariant: $\eta = \frac{w_{12} w_{34}}{w_{14} w_{32}} \quad w_{ij} = w_i - w_j$

$\langle \phi_1(g(z_1)) \dots \phi_N(g(z_N)) \rangle = \prod_{i=1}^{N} (cz_i + d)^{2h_i} \langle \phi_1(z_1) \dots \phi_N(z_N) \rangle$

$z_i - z_j \to g(z_i) - g(z_j) = \frac{z_{ij}}{(cz_i + d)(cz_j + d)}$

$\langle \phi_1(z_1) \dots \phi_N(z_N) \rangle = \prod_{i<j} z_{ij}^{-\gamma_{ij}} f(\eta_a)$ ← harm. ratio

impose on prefactor

$\gamma_{ij} = \gamma_{ji} \qquad \sum_{j \neq i} \gamma_{ij} = 2h_i$

2-point function:

1) $\langle \phi_{h,\bar h}(z,\bar z)\, \phi_{h',\bar h'}(w,\bar w)\rangle = a\, \delta_{h,h'}\, \delta_{\bar h,\bar h'} \frac{1}{(z-w)^{2h}(\bar z-\bar w)^{2\bar h}}$

2) $\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\rangle =$

$= C_{123}\, z_{12}^{-\gamma_{12}}\, z_{13}^{-\gamma_{13}}\, z_{23}^{-\gamma_{23}}$ · antihol.

$\gamma_{ij} = h_i + h_j - h_k$

Conformal blocks and duality

Definition, which looks reasonable now:

$\langle h,\bar h| = \lim_{z,\bar z\to\infty} \langle 0|\, \phi_{h,\bar h}(z,\bar z)\, z^{2h_0}\, \bar z^{2\bar h_0}$

Then $\langle n|\, \phi_m(z,\bar z)\, |e\rangle = C^{e}_{nm}\, z^{h_n - h_m - h_e}\, \bar z^{\bar h_n - \bar h_m - \bar h_e}$

OPE of primary fields: (assumption)

$\phi_n(z,\bar z)\, \phi_m(0,0) = \sum_p \sum_{\{k,\bar k\}} C^{p\{k,\bar k\}}_{nm}\, z^{h_p - h_n - h_m + \sum k}$
$\bar z^{\bar h_p - \bar h_n - \bar h_m + \sum \bar k}\, \phi_p^{\{k,\bar k\}}(0,0)$

p – runs over primary fields

$k,\bar k$ – descendants

$\phi_p^{\{k,\bar k\}}(0) = L_{-k_1} \dots L_{-k_N} \dots L_{-\bar k_M}\, \phi_p(0,0)$

__Exercise:__ Show that $C^{p\{k,\bar k\}}_{n,m} = C^{p}_{nm}\, \beta^{p\{k\}}_{nm}\, \beta^{\bar p\{\bar k\}}_{nm}$

β – coeff. dep. on conformal dim and central charge.

4-point function:

$\langle k | \phi_e(1,1)\, \phi_n(x,\bar{x}) | m \rangle =$

$= \langle k | \phi_e(1,1) \sum_p C^p_{nm}\, x^{h_p - h_n - h_m}\, \bar{x}^{\bar{h}_p - \bar{h}_n - \bar{h}_m}$

$\sum \beta^{p\{k\}}_{nm}\, \beta^{p\{\bar{k}\}}_{nm}\, x^{\sum k}\, \bar{x}^{\sum \bar{k}}\, \phi_p^{\{k\},\{\bar{k}\}}(0,0)|0\rangle$

<u>Definition:</u>

$$\mathcal{F}^{ek}_{nm}(p|x) = x^{h_p - h_n - h_m} \sum_{\{k\}} \beta^{p\{k\}}_{nm} \frac{\langle k | \phi_e(1,1) L_{-k_1} \cdots L_{-k_N} | p\rangle}{\langle k | \phi_e(1,1) | p \rangle}\, x^{\sum k}$$

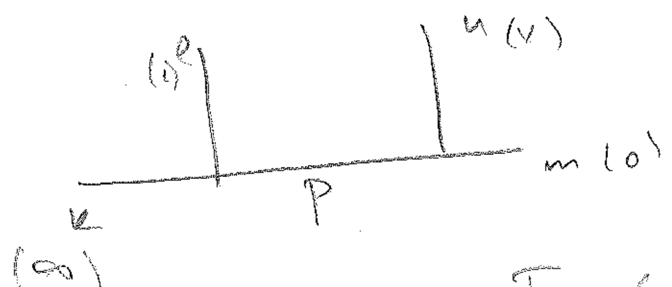
$$G^{ek}_{nm}(x,\bar{x}) = \sum_p C^p_{nm}\, C_{ke}^{p}\, \mathcal{F}_{kenm}(p|x)\, \overline{\mathcal{F}_{kenm}(p,\bar{x})}$$

Notice: $\mathcal{F}^{ek}_{nm}(p|x) \simeq x^{h_p - h_n - h_m}(1 + \ldots)$

$x \to 1 - x \qquad x \to \frac{1}{x}$

$\langle k | \phi_m(1,1)\, \phi_n(1-x, 1-\bar{x}) | e \rangle \Rightarrow G^{ek}_{nm}(x,x) = G^{mk}_{ne}(1-x, 1-\bar{x}) =$

$= x^{-2h_n}\, \bar{x}^{-2\bar{h}_n}\, G^{em}_{nk}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right)$



$$\sum_p C^p_{nm}\, C_{kep}\, \mathcal{F}_{kenm}(p|x)\, \overline{\mathcal{F}_{kenm}(p,x)} =$$

$$= \sum_q C^q_{ne}\, C^q_{km}\, \mathcal{F}_{kmne}(q|1-x)\, \overline{\mathcal{F}_{kmne}(q|1-x)}$$

Conformal blocks are in general multivalued leading to "braiding" structure.

Introducing intertwining operators:

For every triplet of $h_i \, h_j \, h_k$ of Vir repres.

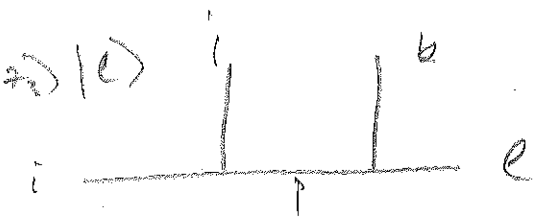
$$\Phi^{j,\beta}_{i,k}(z) : \mathcal{H}_{h_k} \longrightarrow \mathcal{H}_{h_i} \qquad \beta \in \mathcal{H}_j$$

primary

$$\langle i | \Phi^{j,\beta}_{ik}(z) | k \rangle = \| \Phi^{j}_{ik} \| \, z^{-(h_j + h_k - h_i)}$$

$$[L_n, \Phi^{j,\beta}_{ik}(z)] = \left(z^{n+1} \frac{d}{dz} + (n+1) z^n \Delta(\beta) \right) \Phi$$

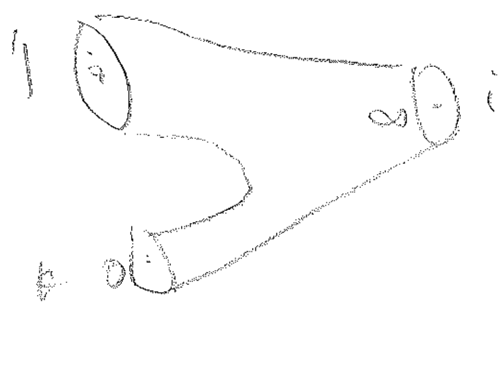
$$\langle i | \Phi^{j,\beta}_{ip}(z_1) \, \Phi^{k,\beta}_{pe}(z_2) | e \rangle$$



$$\mathcal{F}^{ijke}_{p}(z_1, z_2)$$

$$\langle i | \phi_j(z_1) \, \phi_k(z_2) | e \rangle = \sum_p d_p |\mathcal{F}_p|^2$$

$$\begin{pmatrix} i \\ j \, k \end{pmatrix} : \mathcal{H}_j \otimes \mathcal{H}_k \longrightarrow \mathcal{H}_i$$



$$L_n^{(\infty)} = \oint_{C_\infty} \zeta^{n+1} T(\zeta) \, d\zeta$$

$$L_n^{(\infty)} = \oint_0 \zeta^{n+1} T(\zeta) + \int_z \zeta^{n+1} T(\zeta)$$

$$= L_n \big|_0 + \sum_{k=0}^{\infty} \binom{n+1}{k} z^{n+1-k} L_{k-1}^{(z)}$$

Therefore

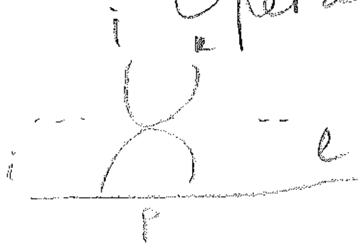
$$\lambda_n \begin{pmatrix} i \\ j\,k \end{pmatrix}_{(z)} (\beta \otimes \gamma) = \begin{pmatrix} i \\ j\,k \end{pmatrix}_{(z)} \Big(\sum \binom{n+1}{k} z^{n+1-k} \lambda_{k-1} \beta \Big) \otimes \gamma + \beta \otimes \lambda_n \gamma \Big)$$

$$\Delta_z(\lambda_n) = 1 \otimes \lambda_n + \sum_{k=0}^{\infty} \binom{n+1}{k} z^{n+1-k} \lambda_{k-1} \otimes 1$$

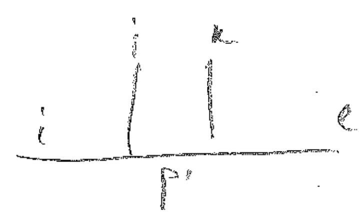
$$\lambda_n \begin{pmatrix} i \\ j\,k \end{pmatrix}_{(z)} (\beta \otimes \gamma) = \begin{pmatrix} i \\ j\,k \end{pmatrix}_{(z)} \Delta_z(\lambda_n)\, (\beta \otimes \gamma)$$

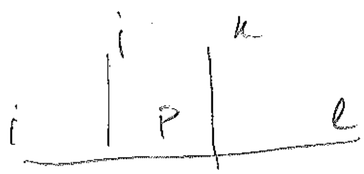
$$\begin{pmatrix} i \\ j\,k \end{pmatrix}_{z} (\beta \otimes \cdot\,) = \Phi^{i,\beta}_{ik}(z)$$

Operations on conformal blocks

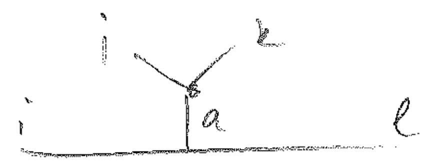


$$= \sum_{p'} B_{pp'} \begin{bmatrix} j & k \\ i & e \end{bmatrix}$$





$$= \sum_{q} F_{p,q} \begin{bmatrix} j & k \\ i & e \end{bmatrix}$$



Braiding and fusion

Pentagon identity and hexagon identity.
(not indep.)

$$[\phi_i] \times [\phi_j] = N_{ij}^{\;k} [\phi_k]$$

↗ positive integers, which we have to identify

$$[N_i, N_j] = 0$$ b.c. of assoc.